Rapidly rotating Bose-Einstein condensates

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*for general references, see [1, 2, 3, 4]
1 Physics of one vortex line in harmonic trap

Assume general axisymmetric trap potential

\[ V_{tr}(\mathbf{r}) = V_{tr}(r, z) = \frac{1}{2} M \left( \omega_{\perp}^2 r^2 + \omega_z^2 z^2 \right) \]

Basic idea (Bogoliubov): for weak interparticle potentials, nearly all particles remain in condensate for \( T \ll T_c \)

- for ideal gas at \( T = 0 \), all particles are in condensate.
- hence treat occupation of non-condensate as small
- dilute: \( s \)-wave scattering length \( a_s \ll \) interparticle spacing \( n^{-1/3} \)
- equivalently, require \( n a_s^3 \ll 1 \)
- assume self-consistent condensate wave function \( \Psi(\mathbf{r}) \)
- gives nonuniform condensate density \( n(\mathbf{r}) = |\Psi(\mathbf{r})|^2 \)
- for \( T \ll T_c \), normalization requires \( N = \int dV |\Psi(\mathbf{r})|^2 \)
• assume an energy functional

\[
E[\Psi] = \int dV \left[ \Psi^* (T + V_{tr}) \Psi + \frac{1}{2} g |\Psi|^4 \right],
\]

where \( T = -\hbar^2 \nabla^2 / 2M \) is kinetic energy operator and \( g = 4\pi a_s \hbar^2 / M \) is interaction coupling parameter

• balance of kinetic energy \( \langle T \rangle \) and trap energy \( \langle V_{tr} \rangle \) gives mean oscillator length \( d_0 = \sqrt{\hbar / M \omega_0} \) where \( \omega_0 = (\omega^2_\perp \omega_z)^{1/3} \) is geometric mean

• balance of kinetic energy \( \langle T \rangle \) and interaction energy \( \langle gn \rangle \) gives healing length

\[
\xi = \frac{\hbar}{\sqrt{2M gn}} = \frac{1}{\sqrt{8\pi a_s n}}
\]

• with fixed normalization and \( \mu \) the chemical potential, variation of \( E[\Psi] \) gives Gross-Pitaevskii (GP) eqn

\[
(T + V_{tr} + g|\Psi|^2) \Psi = \mu \Psi
\]

• can interpret nonlinear term as a Hartree potential \( V_H(\mathbf{r}) = gn(\mathbf{r}) \), giving interaction with nonuniform condensate density
• generalize to time-dependent GP equation

\[ i\hbar \frac{\partial \Psi}{\partial t} = (T + V_{\text{tr}} + V_H) \Psi \]

• this result implies that stationary solutions have time dependence \( \exp(-i\mu t/\hbar) \)

*Introduce hydrodynamic variables*

• write \( \Psi(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)| \exp[iS(\mathbf{r}, t)] \) with phase \( S \)
• condensate density is \( n(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|^2 \)
• current is

\[ \mathbf{j} = \frac{\hbar}{2Mi} [\Psi^* \nabla \Psi - \Psi \nabla \Psi^*] = |\Psi|^2 \frac{\hbar \nabla S}{M} \]

• identify last factor as velocity \( \mathbf{v} = \frac{\hbar \nabla S}{M} \)
• note that \( \mathbf{v} \) is irrotational so \( \nabla \times \mathbf{v} = 0 \)
• general property: circulation around contour $C$ is

$$\oint_{C} dl \cdot v = \frac{\hbar}{M} \oint_{C} dl \cdot \nabla S = \frac{\hbar}{M} \Delta S|_{C}$$

since $v = \hbar \nabla S/M$

• change of phase $\Delta S|_{C}$ around $C$ must be integer times $2\pi$ since $\Psi$ is single-valued

• hence circulation in BEC is \textit{quantized} in units of $\kappa \equiv 2\pi \hbar/M$

• rewrite time-dependent GP equation in terms of $|\Psi|$ and $S$

  – imaginary part gives conservation of particles

$$\frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0$$

  – real part gives generalized Bernoulli equation
Presence of harmonic trap yields much richer system than a uniform interacting Bose gas [5]

- trap gives new energy scale $\hbar \omega_0$ and new length scale $d_0 = \sqrt{\hbar/M \omega_0}$
- assume repulsive interactions with $a_s > 0$
- trap leads to new dimensionless parameter $Na_s/d_0$
- typical value of ratio of lengths: $a_s/d_0 \sim 10^{-3}$
- but $Na_s/d_0$ is large for typical $N \sim 10^6$
- in strong repulsive limit ($Na_s/d_0 \gg 1$), condensate expands to mean radius $R_0 \gg d_0$
- neglect radial gradient of $\Psi$ when $R_0 \gg d_0$
- GP equation then simplifies and gives local density

$$\frac{4\pi a_s \hbar^2}{M} |\Psi(r, z)|^2 = \mu - V_{tr}(r, z)$$

[called Thomas-Fermi (TF) limit]

- harmonic trap produces quadratic density variation with condensate dimensions $R_j^2 = 2\mu/(M \omega_j^2)$
One vortex line in trapped BEC

First assume bulk condensate with uniform density $n$ and a single straight vortex line along $z$ axis

- Gross and Pitaevskii [6, 7]: take condensate wave function
  \[ \Psi(r) = \sqrt{n} e^{i\phi} f\left(\frac{r}{\xi}\right) \]
  where $r$ and $\phi$ are two-dimensional polar coordinates
- speed of sound is $s = \sqrt{gn/M}$
- assume $f(0) = 0$ and $f\left(\frac{r}{\xi}\right) \to 1$ for $r \gg \xi$
- velocity has circular streamlines with $\mathbf{v} = \left(\frac{\hbar}{Mr}\right) \hat{\phi}$
- this is a quantized vortex line with $\oint d\mathbf{l} \cdot \mathbf{v} = 2\pi \hbar/M$
- $v \sim s$ when $r \sim \xi$, so vortex core forms by cavitation
- equivalently, centrifugal barrier gives vortex core of radius $\xi$
**Static behavior of a vortex line in axisymmetric trap**

\[ V_{\text{tr}}(r, z) = \frac{1}{2} M \left( \omega_\perp^2 r^2 + \omega_z^2 z^2 \right) \]

- If \( \omega_z \gg \omega_\perp \), strong axial confinement gives disk-shaped condensate
- If \( \omega_\perp \gg \omega_z \), strong radial confinement gives cigar-shaped condensate
- for vortex on axis, condensate wave function is
  \[ \Psi(r, z) = e^{i\phi} |\Psi(r, z)| \]
- velocity is \( \mathbf{v} = (\hbar/Mr) \hat{\phi} \), like uniform condensate
- centrifugal energy again forces wave function to vanish for \( r \lesssim \xi \)
- density is now toroidal; hole along symmetry axis
- TF limit: separated length scales with
  \[ \xi \text{ (vortex core)} \ll d_0 \text{ (mean oscillator length)} \]
  \[ d_0 \text{ (mean oscillator length)} \ll R_0 \text{ (mean condensate radius)} \]
- hence TF density is essentially unchanged by vortex apart from small hole along the vortex core
Energy of rotating TF condensate with one vortex

• use density of vortex-free TF condensate; cut off the logarithmic divergence at core radius $\xi$

• if condensate is in rotational equilibrium at angular velocity $\Omega$, the appropriate energy functional is \[ E'[\Psi] = E[\Psi] - \Omega \cdot L[\Psi] \] where $L$ is the angular momentum

• let $E'_0$ be energy of rotating vortex-free condensate

• let $E'_1(r_0, \Omega)$ be energy of a rotating condensate with straight vortex that is displaced laterally by distance $r_0$ from symmetry axis

• approximation of straight vortex works best for disk-shaped condensate ($\omega_z \gtrsim \omega_\perp$)

• Difference of these two energies is energy associated with formation of vortex \[ \Delta E'(r_0, \Omega) = E'_1(r_0, \Omega) - E'_0 \]

• $\Delta E'(r_0, \Omega)$ depends on position $r_0$ of vortex and on $\Omega$
Plot $\Delta E'(r_0, \Omega)$ as function of $\zeta_0$ for various fixed $\Omega$ [9], where $\zeta_0 = r_0/R_0$ is scaled displacement from center.
curve (a) is $\Delta E'(r_0, \Omega)$ for $\Omega = 0$

- $\Delta E'(r_0, 0)$ decreases monotonically with increasing $\zeta_0$
- curvature is negative at $\zeta_0 = 0$
- for no dissipation, fixed energy means constant $\zeta_0$
- only allowed motion is uniform precession at fixed $r_0$
- angular velocity is given by variational Lagrangian method [10, 11, 3] $\dot{\phi}_0 \propto -\partial E(r_0)/\partial r_0$
- precession arises from nonuniform trap potential (not image vortex) and nonuniform condensate density
- in presence of weak dissipation, vortex moves to lower energy and slowly spirals outward
As $\Omega$ increases, curvature near $\zeta_0 = 0$ decreases

- curve (b) is when curvature near $\zeta_0 = 0$ vanishes
- it corresponds to angular velocity

$$\Omega_m = \frac{3}{2} \frac{\hbar}{MR^2} \ln \left( \frac{R_\perp}{\xi} \right)$$

- for $\Omega \gtrsim \Omega_m$, energy $\Delta E'(\zeta_0, \Omega)$ has local \textit{minimum} near $\zeta_0 = 0$
- dissipation would now drive vortex back \textit{toward} the symmetry axis
- $\Omega_m$ is angular velocity for onset of \textit{metastability}
- vortex at center is \textit{locally} stable for $\Omega > \Omega_m$, but not \textit{globally} stable, since $\Delta E'(0, \Omega_m)$ is positive
Experimental creation/detection of vortices in dilute trapped BEC

- first vortex made at JILA (1999) [12]
- use nearly spherical $^{87}\text{Rb}$ condensate containing two different hyperfine components
- use coherent (Rabi) process to control interconversion between two components
- spin up condensate by coupling the two components with a stirring perturbation
- turn off coupling, leaving one component with trapped quantized vortex surrounding nonrotating core of other component
- use selective tuning to make nondestructive image of either component
• this vortex with large filled core precesses around trap center

• can also create vortex with (small) empty core [13] that also precesses
  – theory predicts $\dot{\phi}/2\pi \approx 1.58 \pm 0.16$ Hz, and
  – experiment finds $\dot{\phi}/2\pi \approx 1.8 \pm 0.1$ Hz

• see no outward radial motion for $\sim 1$ s, so dissipation is small on this time scale
École Normale Supérieure (ENS) in Paris studied vortex creation in elongated rotating cigar-shaped condensate with one component [14, 15]

- used off-center toggled rotating laser beam to deform the transverse trap potential and stir the condensate at an applied frequency $\Omega/2\pi \lesssim 200$ Hz

- find vortex appears at a critical frequency $\Omega_c \approx 0.7\omega_\perp$ (detected by expanding the condensate, which now has a disk shape, with vortex core as expanded hole)

- vortex nucleation is dynamical process associated with surface instability (quadrupole oscillation)
• ENS group observed small vortex arrays of up to 11 vortices (arranged in two concentric rings)

• like patterns predicted and seen in superfluid $^4$He [16]
• MIT group has prepared considerably larger rotating condensates in less elongated trap
• they have observed triangular vortex lattices with up to 130 vortices [17]

• like Abrikosov lattice of quantized flux lines (which are charged vortices) in type-II superconductors
• JILA group has now made large rotating condensates with several hundred vortices and angular velocity \( \Omega/\omega_{\perp} \approx 0.995 \) [18]
• these rapidly rotating systems open many exciting new possibilities (discussed below)
3 Vortex arrays in mean-field Thomas-Fermi regime

Feynman’s mean vortex density in a rotating superfluid

• solid-body rotation has \( \mathbf{v}_{sb} = \Omega \times \mathbf{r} \)
• \( \mathbf{v}_{sb} \) has constant vorticity \( \nabla \times \mathbf{v}_{sb} = 2\Omega \)
• each quantized vortex at \( \mathbf{r}_j \) has localized vorticity
  \[
  \nabla \times \mathbf{v} = \frac{2\pi \hbar}{M} \delta^{(2)}(\mathbf{r} - \mathbf{r}_j) \hat{z}
  \]
• assume \( \mathcal{N}_v \) vortices uniformly distributed in area \( \mathcal{A} \) bounded by contour \( \mathcal{C} \)
• circulation around \( \mathcal{C} \) is \( \mathcal{N}_v \times \frac{2\pi \hbar}{M} \)
• but circulation in \( \mathcal{A} \) is also \( 2\Omega \mathcal{A} \)
• hence vortex density is \( n_v = \mathcal{N}_v / \mathcal{A} = M \Omega / \pi \hbar \)
• area per vortex \( 1/n_v \) is \( \pi \hbar / M \Omega \equiv \pi l^2 \) which defines radius \( l = \sqrt{\hbar / M \Omega} \) of circular cell
• intervortex spacing \( \sim 2l \) decreases like \( 1/\sqrt{\Omega} \)
• analogous to quantized flux lines (charged vortices) in type-II superconductors
As $\Omega$ increases, the mean vortex density $n_v = M\Omega/\pi\hbar$ increases linearly following the Feynman relation

- in addition, centrifugal forces expand the condensate radially, so that the area $\pi R_{\perp}^2$ also increases
- hence the number of vortices $N_v = n_v\pi R_{\perp}^2 = M\Omega R_{\perp}^2/\hbar$ increases faster than linearly with $\Omega$
- conservation of particles implies that the condensate also shrinks axially
- TF approximation assumes that interaction energy $\langle g|\Psi|^4 \rangle$ and trap energy $\langle V_{tr}|\Psi|^2 \rangle$ are large relative to kinetic energy for density variations $(\hbar^2/M)\langle (\nabla|\Psi|)^2 \rangle$
- radial expansion of rotating condensate means that central density eventually becomes small
Quantitative description of rotating TF condensate

Kinetic energy of condensate involves

$$\frac{\hbar^2}{2M} \int dV |\nabla \Psi|^2 = \int dV \frac{1}{2} \frac{Mv^2}{2} |\Psi|^2 + \frac{\hbar^2}{2M} \int dV (|\nabla |\Psi|)^2$$

where $\Psi = \exp(iS)|\Psi|$ and $v = \hbar \nabla S/M$ is flow velocity

- generalized TF approximation: retain the energy of superflow but ignore the energy from density variation
- this approximation will fail eventually when vortex lattice becomes dense and cores start to overlap
- in rotating frame, generalized TF energy functional is

$$E'[\Psi] = \int dV \left[ \left( \frac{1}{2} M v^2 + V_{tr} - M\Omega \cdot \mathbf{r} \times v \right) |\Psi|^2 + \frac{1}{2} g |\Psi|^4 \right]$$

- here, $v$ is flow velocity generated by all the vortices
For $\Omega$ along $z$, can complete square and rewrite $E'[\Psi]$ as

$$E'[\Psi] = \int dV \left[ \frac{1}{2} M \left( \frac{v - \Omega \times r}{v - v_{sb}} \right)^2 |\Psi|^2 + \frac{1}{2} M \omega^2 z^2 |\Psi|^2 
+ \frac{1}{2} M \left( \omega^2_\perp - \Omega^2 \right) r^2 |\Psi|^2 + \frac{1}{2} g |\Psi|^4 \right]$$

- in the rotating frame, the dominant effect of the dense vortex array is that spatially averaged flow velocity $\mathbf{v}$ is close to $\Omega \times \mathbf{r} = \mathbf{v}_{sb}$
- hence can ignore first term in $E'[\Psi]$, giving

$$E'[\Psi] \approx \int dV \left[ \frac{1}{2} M \omega^2 z^2 |\Psi|^2 + \frac{1}{2} M \left( \omega^2_\perp - \Omega^2 \right) r^2 |\Psi|^2 
+ \frac{1}{2} g |\Psi|^4 \right]$$

- $E'$ now looks exactly like TF energy for nonrotating condensate but with a reduced radial trap frequency $\omega^2_\perp \rightarrow \omega^2_\perp - \Omega^2$
Hence TF wave function depends explicitly on $\Omega$ through the altered radial trap frequency $\omega^2 \rightarrow \omega^2 - \Omega^2$

$$|\Psi(r, z)|^2 = n(0) \left( 1 - \frac{r^2}{R^2} - \frac{z^2}{R^2_z} \right)$$

where

$$R^2 = \frac{2\mu}{M(\omega^2 - \Omega^2)} \quad \text{and} \quad R^2_z = \frac{2\mu}{M\omega^2_z}$$

- for pure harmonic trap, must have $\Omega < \omega$ to retain radial confinement
- normalization $\int dV |\Psi|^2 = N$ shows that

$$\frac{\mu(\Omega)}{\mu(0)} = \left( 1 - \frac{\Omega^2}{\omega^2} \right)^{2/5}$$

in three dimensions
- central density given by $n(0) = \mu(\Omega)/g$
- $n(0)$ decreases with increasing $\Omega$ because of reduced radial confinement
• TF formulas for condensate radii show that

\[
\frac{R_z(\Omega)}{R_z(0)} = \left(1 - \frac{\Omega^2}{\omega^2}\right)^{1/5}, \quad \frac{R_\perp(\Omega)}{R_\perp(0)} = \left(1 - \frac{\Omega^2}{\omega^2}\right)^{-3/10}
\]

confirming axial shrinkage and radial expansion

• for nonzero \( \Omega \), aspect ratio changes

\[
\frac{R_z(\Omega)}{R_\perp(\Omega)} = \frac{\sqrt{\omega^2 - \Omega^2}}{\omega_z}
\]

• this last effect provides an important diagnostic tool to determine actual angular velocity \( \Omega \) [19, 18]

• measured aspect ratio [18] indicates that \( \Omega/\omega_\perp \) can become as large as \( \approx 0.993 \)
How uniform is the vortex array?

The analysis of the TF density profile $|\Psi_{TF}|^2 = n_{TF}$ in the rotating condensate assumed that the flow velocity $\mathbf{v}$ was precisely the solid-body value $\mathbf{v}_{sb} = \Omega \times \mathbf{r}$

- this led to the cancellation of the contribution
  $$\int dV \ (\mathbf{v} - \Omega \times \mathbf{r})^2 n_{TF}$$
  in the TF energy functional
- a more careful study [20] shows that there is a small nonuniformity in the vortex lattice
- specifically, each regular vortex lattice position vector $\mathbf{r}_j$ experiences a small displacement field $\mathbf{u}(\mathbf{r})$, so that $\mathbf{r}_j \rightarrow \mathbf{r}_j + \mathbf{u}(\mathbf{r}_j)$
- as a result, the two-dimensional vortex density changes to
  $$n_v(\mathbf{r}) \approx \bar{n}_v (1 - \nabla \cdot \mathbf{u})$$
  where $\bar{n}_v = M\Omega/\pi\hbar$ is the uniform Feynman value
• variation with respect to $u$ yields an Euler-Lagrange equation that can be solved to give
\[ u(r) \approx \frac{\bar{l}^2}{4R_\perp^2} \ln \left( \frac{\bar{l}^2}{\xi^2} \right) \frac{r}{1 - r^2/R_\perp^2} \]

where $\pi \bar{l}^2 = 1/n_v$ can be taken as the area of a circular vortex cell inside the slowly varying logarithm

• the deformation of the regular vortex lattice is purely radial (as expected from symmetry)

• $R_\perp^2/\bar{l}^2$ is the number of vortices $N_v$ in the rotating condensate, so that the nonuniform distortion is small, of order $1/N_v$ (at most a few %), even though the TF number density $n_{TF}$ changes dramatically near edge

• correspondingly, the vortex density becomes
\[ n_v(r) \approx n_v - \frac{1}{2\pi R_\perp^2} \ln \left( \frac{\bar{l}^2}{\xi^2} \right) \frac{1}{(1 - r^2/R_\perp^2)^2} \]

(the correction is again of order $1/N_v$)
recent JILA experiments [21] confirm these predicted small distortions for relatively dense vortex lattices [note suppressed zero for Figs. (a)-(d); Fig. (e) shows that effect is small]
Tkachenko oscillations of the vortex lattice

Tkachenko (1966) [22] studied equilibrium arrangement of a rotating vortex array as model for superfluid $^4$He

- assumed two-dimensional incompressible fluid with straight vortices
- showed that a triangular lattice has lowest energy in rotating frame
- small perturbations about equilibrium positions had unusual collective motion in which vortices undergo nearly transverse wave of lattice distortions (like two-dimensional transverse “phonons” in vortex lattice, but with no change in fluid density)
- for long wavelengths (small $k$), Tkachenko found a linear dispersion relation $\omega_k \approx c_T k$

- speed of Tkachenko wave $c_T = \sqrt{\frac{1}{4} \hbar \Omega / M} = \frac{1}{2} \hbar / M \bar{l}$, where $\bar{l} = \sqrt{\hbar / M \Omega}$ is radius of circular vortex cell
In a rotating two-dimensional gas, the compressibility becomes important, as shown by Sonin [23, 24] and Baym [25]

- let the speed of sound in the compressible gas be $c_s$
- coupling between the vortices and the compressible fluid leads to generalized dispersion relation

$$\omega^2 = c_T^2 \frac{c_s^2 k^4}{4\Omega^2 + c_s^2 k^2}$$

- if $c_s k \gg \Omega$, recover Tkachenko’s result $\omega = c_T k$ (short-wavelength incompressible limit)
- but if $c_s k \ll \Omega$ (long wavelength), mode becomes soft with $\omega \propto k^2$
- Sonin [24] obtains dynamical equations for waves in a nonuniform condensate, along with appropriate boundary conditions at the outer surface
- Baym [25] uses theory for uniform condensate plus approximate boundary conditions from Anglin and Crescimanno [26]
• rough agreement with JILA experiments [27] on low-lying Tkachenko modes in rapidly rotating BEC (up to $\Omega/\omega_\perp \approx 0.975$)

• A is $1/4$ period after weak perturbation (note deformation of lines of vortices)

• B is $3/4$ period after weak perturbation (note reversed deformation of lines of vortices)
4 Vortex arrays in mean-field quantum-Hall regime

Lowest-Landau-Level (quantum-Hall) behavior

When the vortex cores overlap, kinetic energy associated with density variation around each vortex core becomes important

- hence the TF approximation breaks down (it ignores this kinetic energy from density variations)
- return to full GP energy $E'[^\Psi]$ in the rotating frame.
- in this limit of rapid rotations ($\Omega \lesssim \omega_\perp$), Ho [28] incorporated kinetic energy exactly
- condensate expands and is effectively two dimensional
- for simplicity, treat a two-dimensional condensate that is uniform in the $z$ direction over a length $Z$
- condensate wave function $\Psi(\mathbf{r}, z)$ can be written as $\sqrt{N/Z} \psi(\mathbf{r})$, where $\psi(\mathbf{r})$ is a two-dimensional wave function with unit normalization $\int d^2r \, |\psi|^2 = 1$
General two-dimensional energy functional in rotating frame becomes

$$E'[\psi] = \int d^2r \psi^* \left( \frac{p^2}{2M} + \frac{1}{2} M \omega_\perp^2 r^2 - \Omega L_z + \frac{1}{2} g_{2D} |\psi|^2 \right) \psi,$$

where \( p = -i\hbar \nabla \), \( L_z = \hat{z} \cdot r \times p \), and \( g_{2D} = N g / Z \)

One-body oscillator hamiltonian in rotating frame \( \mathcal{H}_0 \) is exactly soluble and has eigenvalues [29]

$$\epsilon_{nm} = \hbar [\omega_\perp + n (\omega_\perp + \Omega) + m (\omega_\perp - \Omega)]$$

where \( n \) and \( m \) are non-negative integers

- in limit \( \Omega \rightarrow \omega_\perp \), these eigenvalues are essentially independent of \( m \) (massive degeneracy)
- \( n \) becomes the Landau level index
- lowest Landau level with \( n = 0 \) is separated from higher states by gap \( \sim 2\hbar \omega_\perp \)
Large radial expansion means small central density $n(0)$, so that interaction energy $gn(0)$ eventually becomes small compared to gap $2\hbar \omega_\perp$

Hence focus on “lowest Landau level” (LLL), with $n = 0$ and general non-negative $m \geq 0$

- ground-state wave function is Gaussian $\psi_{00} \propto e^{-r^2/2d_\perp^2}$
- general LLL eigenfunctions have a very simple form

$$\psi_{0m}(r) \propto r^m e^{im\phi} e^{-r^2/2d_\perp^2}$$

- here, $d_\perp = \sqrt{\hbar/M \omega_\perp}$ is analogous to the “magnetic length” in the Landau problem
- in terms of a complex variable $\zeta \equiv x + iy$, these LLL eigenfunctions become

$$\psi_{0m} \propto \zeta^m e^{-r^2/2d_\perp^2} \propto \zeta^m \psi_{00}$$

with $m \geq 0$ (note that $\zeta = re^{i\phi}$ when expressed in two-dimensional polar coordinates)

- apart from ground-state Gaussian $\psi_{00}$, this is just $\zeta^m$ (a non-negative power of the complex variable)
• assume that the GP wave function is a finite linear combination of these LLL eigenfunctions

\[
\psi_{LLL}(\mathbf{r}) = \sum_{m \geq 0} c_m \psi_0^m(\mathbf{r}) = f(\zeta) e^{-r^2/2d^2_\perp}
\]

where \(f(\zeta) = \sum_{m \geq 0} c_m \zeta^m\) is an \textit{analytic function} of the complex variable \(\zeta\)

• specifically, \(f(\zeta)\) is a complex polynomial and thus can be factorized as \(f(\zeta) = \prod_j (\zeta - \zeta_j)\) apart from overall constant

• \(f(\zeta)\) vanishes at each of the points \(\{\zeta_j\}\), which are the positions of the zeros of \(\psi_{LLL}\)

• in addition, phase of wave function increases by \(2\pi\) whenever \(\zeta\) moves around any of these zeros \(\{\zeta_j\}\)

• we conclude that the LLL trial solution has singly quantized vortices located at positions of zeros \(\{\zeta_j\}\)

• basic conclusion: vortices are \textit{nodes} in the condensate wave function
• spatial variation of number density $n(r) = |\psi_{LLL}(r)|^2$ is determined by spacing of the vortices

• core size is comparable with the intervortex spacing $\bar{\ell} = \sqrt{\hbar/M\Omega}$ which is simply $d_\perp$ in the limit $\Omega \approx \omega_\perp$

• unlike TF approximation at lower $\Omega$, wave function $\psi_{LLL}$ automatically includes all the kinetic energy

• since LLL wave functions play a crucial role in the quantum Hall effect (two-dimensional electrons in a strong magnetic field), this LLL regime has been called “mean-field quantum-Hall” limit [30]

• note that we are still in the regime governed by GP equation, so there is still a BEC

• corresponding many-body ground state is simply a Hartree product with each particle in same one-body solution $\psi_{LLL}(r)$, namely

$$\Psi_{GP}(r_1, r_2, \cdots, r_N) \propto \prod_{n=1}^{N} \psi_{LLL}(r_n)$$

• this is coherent (superfluid) state, since a single GP state $\psi_{LLL}$ has macroscopic occupation
Take this LLL trial function seriously

- for any LLL state $\psi_{LLL}$ that is linear combination of $\psi_{0m}$, can show that (use oscillator units with $\omega_\perp$ and $d_\perp$ for energy and length) [28, 30, 31]

$$\int d^2r r^2 |\psi_{LLL}|^2 = 1 + \int d^2r \psi^*_{LLL} L_z \psi_{LLL}$$

namely $M \omega_\perp \langle r^2 \rangle = \hbar + \langle L_z \rangle$ in dimensional units

- dimensionless energy functional becomes

$$E'[\psi_{LLL}] = \Omega + \int d^2r \left[ (1 - \Omega) r^2 |\psi_{LLL}|^2 + \frac{1}{2}g_{2D} |\psi_{LLL}|^4 \right]$$

- unrestricted variation with respect to $|\psi|^2$ would lead to inverted parabola

$$|\psi|^2 = n(r) = \frac{2}{\pi R_0^2} \left( 1 - \frac{r^2}{R_0^2} \right)$$

where $\pi R_0^4 = 2g_{2D}/(1 - \Omega)$ fixes condensate radius

- looks like earlier TF profile, but here include all kinetic energy explicitly

- these results ignore vortices and violate form of $\psi_{LLL}$
• to include effect of vortices, study logarithm of the particle density for any LLL state

• use $\psi_{LLL}$ to find

$$\ln n_{LLL}(\mathbf{r}) = -\frac{r^2}{d_\perp^2} + 2 \sum_j \ln |\mathbf{r} - \mathbf{r}_j|$$

where $\mathbf{r}_j$ is the position of the $j$th vortex

• apply two-dimensional Laplacian: use standard result $\nabla^2 \ln |\mathbf{r} - \mathbf{r}_j| = 2\pi \delta^{(2)} (\mathbf{r} - \mathbf{r}_j)$ to obtain

$$\nabla^2 \ln n_{LLL}(\mathbf{r}) = -\frac{4}{d_\perp^2} + 4\pi \sum_j \delta^{(2)} (\mathbf{r} - \mathbf{r}_j)$$

• here, sum over delta functions is precisely the vortex density $n_v(\mathbf{r})$

• this result relates particle density $n_{LLL}(\mathbf{r})$ in LLL approximation to vortex density $n_v(\mathbf{r})$ [28, 30, 31]

$$n_v(\mathbf{r}) = \frac{M \omega_\perp}{\pi \hbar} + \frac{1}{4\pi} \nabla^2 \ln n_{LLL}(\mathbf{r})$$
• if vortex lattice is exactly uniform (so $n_v$ is constant), then density profile is strictly Gaussian, with $n_{LLL}(r) \propto \exp(-r^2/\sigma^2)$ and $\sigma^{-2} = M\omega_\perp/\hbar - \pi n_v = M(\omega_\perp - \Omega)/\hbar$

• note that $\sigma^2 \gg d_\perp^2$, so that this resulting Gaussian is much bigger than original ground state

• to better minimize the energy, mean density profile $\bar{n}_{LLL}$ should approximate inverted parabolic shape $\bar{n}_{LLL}(r) \propto 1 - r^2/R_\perp^2$

• then find nonuniform vortex density with

$$n_v(r) \approx \frac{1}{\pi d_\perp^2} - \frac{1}{\pi R_\perp^2} \frac{1}{(1 - r^2/R_\perp^2)^2}$$

similar to result at lower $\Omega$ [20] (in both cases, small correction term is of order $\sim N_v^{-1}$)

• independently, numerical work by Cooper et al. [32] shows that allowing the vortices in the LLL to deviate from the triangular array near the outer edge lowers the energy
5 Behavior for $\Omega \rightarrow \omega_\perp$

What happens beyond the “mean-field quantum Hall” regime is still subject to vigorous debate.

Predict quantum phase transition from coherent BEC states to various correlated many-body states

- define the ratio $\nu \equiv N/N_v = \text{the number of atoms per vortex}$

- because of similarities to a two-dimensional electron gas in a strong magnetic field, $\nu$ is called the “filling fraction” [33, 34]

- current experiments [18] have $N \sim 10^5$ and $N_v \sim$ several hundred, so $\nu \sim$ a few hundred

- numerical studies [34] for small number of vortices ($N_v \lesssim 8$) and variable $N$ indicate that the coherent GP state is favored for $\nu \gtrsim 6 - 8$

- softening of Tkachenko spectrum for rapid rotation can induce melting of vortex lattice at similar filling fraction $\nu$ [35, 36]
• in either scenario, BEC and the associated coherent state disappears at a quantum phase transition for $\nu \sim 5-10$

• replaced by qualitatively different correlated states that are effectively incompressible vortex liquids

• for smaller $\nu$, there is a sequence of highy correlated states similar to some known from the quantum Hall effect

• in particular, theorists have proposed boson version of the Laughlin state [34] (here $z_n = x_n + iy_n$ refers to $n$th particle)

$$
\Psi_{\text{Lau}}(r_1, r_2, \cdots, r_N) \propto \prod_{n<n'} (z_n - z_{n'})^2 \exp \left( - \sum_{n=1}^{N} \frac{|z_n|^2}{2d_{\perp}} \right)
$$
Note: these correlated many-body states are *qualitatively different* from the coherent GP state

- the GP state $\Psi_{GP}(r_1, r_2, \cdots, r_N) \propto \prod_n \psi(r_n)$ is the Hartree product of $N$ factors of *same* one-body function $\psi(r)$

- in the Laughlin state $\Psi_{Lau}(r_1, r_2, \cdots, r_N)$, the 2-body product $\prod_{n<n'}(z_n - z_{n'})^2$ involves $N(N-1)/2$ factors for all possible *pairs* of particles

- note that this correlated wave function vanishes when any two particles approach each other

- thus it minimizes total interaction energy for short-range repulsive potentials, which is the source of the correlations

- for large $N$, the correlated Laughlin state is much more difficult to use
What is physics of this phase transition? Why does it occur for relatively large \( \nu \sim 10 \)?

- for small \( N \) and large \( L \) (namely small \( \nu \)), exact ground states have correlated form [33]
- different symmetry of ground states for small and large \( \nu \) requires a quantum phase transition at some intermediate critical \( \nu_c \)
- For \( N \) particles in two dimensions, there are \( 2N \) degrees of freedom
- each vortex has one collective degree of freedom
- hence \( \mathcal{N}_v \) vortices have \( \mathcal{N}_v \) collective degrees of freedom
- usually treated as additive, but in fact only \( 2N - \mathcal{N}_v \) particle degrees of freedom remain
- for \( N \gg \mathcal{N}_v \) (namely large \( \nu \)), depletion of particle degrees is unimportant
- eventually, when \( \mathcal{N}_v \) is a finite fraction of \( N \) (namely \( \nu \sim 10 \)), original description fails and get transition to new correlated ground state
How to reach the highly correlated regime?

• need to reduce the ratio $\nu = N/N_v$ (number of atoms per vortex)

• very challenging since need small $N \lesssim 100$ and rapid rotation $\Omega \gtrsim 0.999 \omega_\perp$

• one possibility is to use one-dimensional array of small pancake condensates trapped in optical lattice

• need to rotate each condensate to a relatively high angular velocity

• several experimental groups working on this option
References


