Some nontrivial elements in the stable homotopy groups of sphere

For connected finite type spectra $X, Y$, there exists Adams spectral sequence (ASS) $\{E^{r,s}_r, d_r\}$ such that:

1. $d_r : E^{s,t}_r \to E^{s+r,t+r-1}_r$ is the differential,
2. $E^{s,t}_2 \cong \text{Ext}^s_A(H^*X, H^*Y)$ and
3. converges to $[\Sigma^{t-s}Y, X]_p$

i.e. $E^{s,t}_2 \cong \text{Ext}^s_A(H^*X, H^*Y) \Longrightarrow [\Sigma^{t-s}Y, X]_p$. When $Y$ is sphere spectrum $S$, it is $E^{s,t}_2 \cong \text{Ext}^s_A(H^*X, Z_p) \Longrightarrow \pi_{t-s}(X)_p$. When $X$ is sphere spectrum $S$, Moore spectrum $M$, Toda-Smith spectrum $V(1), V(2)$ respectively, $\pi_{t-s}(X)_p$ is respectively the stable homotopy group of $S, M, V(1), V(2)$. Today, we detected some new nonzero elements of the stable homotopy groups of sphere and Toda-Smith spectrum $V(1)$ by using of the ASS. If a family of homotopy generators $x_i$ in $E^{s,*}_2$ converges non-trivially in the ASS, then we get a family of homotopy elements $f_i$ in $\pi_\ast S$ and we say that $f_i$ is represented by $x_i \in E^{s,*}_2$ and has filtration $s$ in the ASS. so far, not so many families of homotopy elements in $\pi_\ast S$ have been detected. For example, a family $\xi_0 \in \pi_p^{n,q+q-3}S(n \geq 2)$, which has filtration 3 in the ASS and is represented by $h_0b_{n-1} \in \text{Ext}^3_A, p^{n+q+q}(Z_p, Z_p)$.

This thesis contains four chapters. In the first chapter, we find the convergence of the products $\widetilde{\gamma}_l b_1g_0 \in \text{Ext}^{t+5,(t+1)p^2q+(t+2)pq+q+t-3}_A(Z_p, Z_p)(3 \leq t < p-2)$ in Adams spectral sequence, where $A$ is mod $p$ Steenrod algebra, $\gamma_l \in \text{Ext}^{t+2p^2q+(t-1)pq+(t-2)q+t-3}_A(Z_p, Z_p)$ converges to $\gamma_l = j_0j_1j_2\gamma^l j_2i_1i_0 \in \pi_\ast S$.

In the second chapter, by the algebraic method, we prove the existence of a new nontrivial family $\widetilde{\gamma}_s h_n h_m(m > n + 2 > 5, s < p-3)$ which filtration is $s + 5$ in the stable homotopy groups of spheres $\pi_q(p = p + (s + 3)p^{2+(s+2)p+(s+1)} - 5S$.

In the third chapter, we use the estimation about $\text{Ext}^{s,t}_p(Z_p, Z_p)$, which is the subalgebra of mod $p$ Steenrod algebra $A$ which is generated by all $P^i(i \geq 0)$, we find $\text{Ext}_A^{4+s+p^2q+q+s}(H^*V(1), Z_p) = Z_p\{b_{0,0}^0\} (s \geq 1). At the same time, Massey product and Toda brackets are very important to determine some new families of homotopy elements in $\pi_\ast S$. In this chapter, a nontrivial product $i_1i_0(\xi_1) \cdot i_1j_1\beta i_1i_0 \neq 0$ was detected by Moss. In the end, the relation $i_1i_0(\xi_1) \cdot i_1j_1\beta i_1i_0 = (\beta')^p\alpha''\beta i_1i_0 \neq 0$ (where $p \geq 7$) in $\pi_\ast V(1)$ was obtained.

In the forth chapter, we discuss the property of the ring spectrum $V_r(2)$. At
Abstract

last, we get the convergence of $\gamma_{tp^n/r}(p \geq 7, t \geq 1, 1 \leq r \leq 2^n < \frac{p-3}{2})$ in the Adams-Novikov spectral sequence (ANSS).

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