Densely ordered braid subgroups

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A group $G$ is **left-ordered** if there exists a strict total ordering $<$ of its elements such that $g < h \Rightarrow fg < fh$.

Given such an ordering the positive cone

$$P = \{g \in G | 1 < g\}$$

satisfies

- $P \cdot P \subset P$ and
- For every $g \in G$, exactly one of $g = 1$, $g \in P$, or $g \in P^{-1}$ holds.

Conversely, if a group has a subset $P$ satisfying the above, then a left-ordering may be defined by $g < h \iff g^{-1}h \in P$. 
Left-orderings of a group can be either

**Discrete:** every element has an immediate successor and predecessor. Equivalently, the positive cone has a least element.

or

**Dense:** whenever $f < h$, there exists $g$ with $f < g < h$. 
A discretely left-ordered group \((G, <)\) can contain a subgroup \(H\) such that \((H, <)\) is densely ordered by the SAME ordering!

**Example.** Consider \(\mathbb{Q} \times \mathbb{Z}\), with the lexicographic order:

\[(p, m) < (q, n) \iff p < q \text{ or else } p = q \text{ and } m < n.\]

Then the ordering is discrete, with least positive element \((0, 1)\), but the subgroup \(\mathbb{Q} \times \{0\}\) is densely ordered by the restriction of the lexicographic ordering.
We will see that a similar phenomenon occurs rather naturally in the braid groups $B_n$.

Recall that $B_n$ has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \text{ and }$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i - j| = 1.$$ 

For $1 \leq i \leq n$ there are natural inclusions $B_i \hookrightarrow B_n$.

Note that if the generators are allowed to commute, they all become equal. That is: $\frac{B_n}{[B_n, B_n]} \cong \mathbb{Z}$ and the commutator subgroup $[B_n, B_n]$ is precisely the set of words in the generators with total exponent zero.
**Theorem** (Dehornoy). $B_n$ is left-orderable.

The Dehornoy ordering has a positive cone $P \subset B_n$ as follows: A word in the generators $\sigma_1, \ldots, \sigma_i$ is said to be $i$-positive if the generator $\sigma_i$ occurs at least once, and with only positive exponents. A braid $\beta \in B_n$ is said to be $i$-positive if $\beta \in \langle \sigma_1, \ldots, \sigma_i \rangle$ and it admits at least one $i$-positive representative braid word.

The positive cone in $B_n$ is

$$P = \{ \beta \in B_n : \beta \text{ is } i\text{-positive for some } i \}.$$
A key fact established by Dehornoy is that for any braid $\beta \in \langle \sigma_1, \cdots, \sigma_i \rangle \subset B_n$ we have exactly one of the following:

- $\beta$ is $i$-positive

- $\beta$ is $i$-negative, i.e. $\beta^{-1}$ is $i$-positive

- $\beta$ is $i$-neutral, meaning $\beta \in \langle \sigma_1, \cdots, \sigma_{i-1} \rangle$. 
Proposition. The Dehornoy ordering of $B_n$ is discrete, with least positive element $\sigma_1$.

Proof. Clearly $\sigma_1 > 1$. If $1 < \beta < \sigma_1$, then $\beta$ is $i$-positive for some $i$.

- If $i > 1$, then $\sigma_1^{-1}\beta$ is still $i$-positive, so
  
  $$1 < \sigma_1^{-1}\beta \Rightarrow \sigma_1 < \beta.$$  

- If $i = 1$, then $\beta = \sigma_1^k$, yet $\beta < \sigma_1$. Hence $k < 1$. 

□
Let $C(r) = \text{the centralizer of } B_{r-1} \text{ in } B_r$.

**Lemma.** Suppose $N$ is a non-trivial normal subgroup of $B_n$, with $N \cap B_{n-1} = \{1\}$ and $n \geq 3$. Then if $N$ is discretely ordered by the Dehornoy ordering, the least positive element lies in $C(n)$.

**Proof:** Choose $\beta \in N \setminus C(n)$, and choose $\gamma \in B_{n-1}$ not commuting with $\beta$. We’ll find $\alpha \in N$ with $1 < \alpha < \beta$, by considering 3 cases:

Case 1: $\beta \gamma \beta^{-1}$ is $(n-1)$-neutral: Then $\beta \gamma \beta^{-1} \gamma^{-1} \in N \cap B_{n-1} = \{1\}$, contradiction – this case doesn’t occur.

Case 2: $\beta \gamma \beta^{-1}$ is $(n-1)$-positive. Then choose $\alpha := \beta \gamma \beta^{-1} \gamma^{-1}$.

Case 3: $\beta \gamma \beta^{-1}$ is $(n-1)$-negative, choose $\alpha := \beta \gamma^{-1} \beta^{-1} \gamma$. 


Proof of case 2: ( $\beta \gamma \beta^{-1}$ is $(n-1)$-positive)

We’ve chosen $\alpha := \beta \gamma \beta^{-1} \gamma^{-1}$.

Since $\beta \gamma \beta^{-1}$ is $(n-1)$-positive, the braid $\beta \gamma \beta^{-1} \gamma^{-1}$ is also $(n-1)$-positive (recall $\gamma \in B_{n-1}$ so it does not contain $\sigma_{n-1}$).

Since $\beta$ is $(n-1)$-positive, $\gamma \beta^{-1} \gamma^{-1}$ is $(n-1)$-negative, so

$$\gamma \beta^{-1} \gamma^{-1} < 1 \Rightarrow \beta \gamma \beta^{-1} \gamma^{-1} < \beta.$$ 

Together,

$$1 < \alpha < \beta.$$
The Garside Half-twist is the braid

\[ \Delta_k := (\sigma_1 \sigma_2 \cdots \sigma_{k-1}) (\sigma_2 \cdots \sigma_{k-1}) \cdots (\sigma_{k-2} \sigma_{k-1}) (\sigma_{k-1}). \]

The centre \( ZB_n, n \geq 3 \), is infinite cyclic, generated by \( \Delta^2_n \). The ordering of \( B_n \), restricted to \( ZB_n \) is necessarily discrete, and its least element is \( \Delta^2_n \), which is of course in \( C(n) \). (\( ZB_2 = B_2 \), generated by \( \Delta_1 = \sigma_1 \))
Theorem. Suppose $N$ is a discretely ordered nontrivial normal subgroup of $B_n$. Then the least positive element of $N$ is either a power of $\sigma_1$, or lies in $C(r)$, where $r$ is the largest integer such that $3 \leq r \leq n - 2$ and $N \cap B_{r-1}$ is trivial.

Proof. (Sketch) If $\sigma_1^m \in N$ for some $m > 0$, then some positive power of $\sigma_1$ is the smallest positive element.

Otherwise, $N \cap B_{r-1}$ is trivial for some $r$, while $N \cap B_r$ is not, and apply the lemma with $r$ replacing $n$ to get the smallest positive element of $N \cap B_r$. Finally, argue this is in fact the smallest positive element of $N$. \qed
Proposition (Fenn, Rolfsen, Zhu 1996). For \( r \geq 3 \), the centralizer \( C(r) \) of \( B_{r-1} \) in \( B_r \) consists of the elements

\[
\Delta_3^{2u} \sigma_1^v \quad \text{if } r = 3, \quad \Delta_r^{2u} \Delta_{r-1}^{2v} \quad \text{if } r > 3,
\]

where \( u, v \in \mathbb{Z} \).

Note that in each case, the two parts of the word commute, so \( C(r) \cong \mathbb{Z} \times \mathbb{Z} \).
Proposition. Let $N$ be a discretely ordered nontrivial normal subgroup of $B_n$, and $r$ the greatest integer such that $N \cap B_{r-1}$ is trivial. Then the least positive element $\beta \in N$ is of the form:

$$\beta = \Delta_r^{2u} \text{ or } \beta = \sigma_1^u,$$

where $u \in \mathbb{Z}$ is positive.

Proof. Set $\beta = \Delta_r^{2u} \Delta_{r-1}^{2v}$. Then if $v > 0$, we show that

$$1 < \beta \sigma_{r-1} \beta^{-1} \sigma_{r-1}^{-1} < \beta,$$

and if $v < 0$, then

$$1 < \beta^{-1} \sigma_{r-1} \beta \sigma_{r-1}^{-1} < \beta.$$

Therefore, unless $v = 0$ the braid $\beta$ is not minimal positive.  \qed
Recall $[B_n, B_n] \subset B_n$ contains exactly those braids having a representative word with zero exponent sum. Neither $\Delta_r^{2^u}$ nor $\sigma_1^u$, $u \geq 1$, have zero exponent sum, so we conclude:

- If $n \geq 3$, $[B_n, B_n]$ is densely ordered.

- Any nontrivial $N \subset [B_n, B_n]$ that is normal in $B_n$ ($n \geq 3$) is densely ordered under the Dehornoy ordering.
Further examples:

- \([P_n, P_n] \subset [B_n, B_n]\) is normal in \(B_n\), and so densely ordered.

- The Burau representation

\[
\rho_n : B_n \rightarrow GL_n(\mathbb{Z}[t, t^{-1}]),
\]

defined on generators by

\[
\sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1 - t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1},
\]

satisfies \(\ker(\rho_n) \subset [B_n, B_n]\), so the kernel is densely ordered for values of \(n\) for which the representation is unfaithful.
• The kernel of \( h : B_4 \to B_3 \) defined by

\[
\begin{align*}
  h(\sigma_1) & = \sigma_1, \\
  h(\sigma_2) & = \sigma_2, \\
  h(\sigma_3) & = \sigma_1
\end{align*}
\]

is the normal closure of \( \sigma_1\sigma_3^{-1} \) in \( B_4 \) and therefore is densely ordered.
A braid $\beta \in B_n$ is said to be Brunnian if, for each strand, deleting that strand from $\beta$ results in the trivial $(n - 1)$-braid.

- For $n \geq 3$ the subgroup of Brunnian braids in $B_n$ is densely ordered.

- For $n \geq 3$ and $1 \leq k < n - 1$ the subgroup of $k$-Brunnian braids in $B_n$ is densely ordered.

A $k$-Brunnian braid is one which becomes trivial upon removal of an arbitrary set of $k$ strands. The set of $k$-Brunnian braids is a normal subgroup of $B_n$ and is nontrivial provided $1 \leq k < n - 1$.

Clearly none of the candidates for least positive element is $k$-Brunnian.
A braid $\beta \in B_n$ is **homotopically trivial** if there is a homotopy from $\beta$ to the trivial braid, as a disjoint mapping of strings in $\mathbb{R}^2 \times \mathbb{R}$ with endpoints fixed. Thus the strings may deform to non-braids and cross themselves, but **cannot cross each other** during the homotopy. Goldsmith showed that the set of homotopically trivial braids in $B_n$, $n \geq 3$, is a **nontrivial** normal subgroup of the pure braid subgroup, thus answering a question of Artin.

- For $n \geq 3$, the subgroup of homotopically trivial braids in $B_n$ is **densely ordered**.

This is clear, for similar reasons to the above.
Thanks for listening.....