Twisted Reidemeister torsion for twist knots

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Twist knots

J(2, n) and J(−2, n), n > 0.

Example: In D. Rolfsen's table the trefoil knot is 3_1 = J(2, 2), the figure eight knot is 4_1 = J(2, −2), 5_2 = J(2, 4), 6_1 = J(2, −4).
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\[ J(2, n) \text{ and } J(-2, n), \quad n > 0. \]
Twist knots

\[ n\text{-crossings} \]

**Figure:** Twist knots \( J(2, n) \) and \( J(-2, n), \ n > 0 \).

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Properties of twist knots

are two-bridge knots (i.e. rational knots).

the fundamental group has two generators and one relation.

\[ \pi_1(J(2,2m)) = \langle x, y | w^m x = y^m w \rangle \]

where \( w \) is the word \([y, x^{-1}] = yx^{-1}y^{-1}x \).

\[ \pi_1(J(2,2m+1)) = \langle x, y | w^m x = y^m w \rangle \]

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Reidemeister torsion

Reidemeister torsion is a classical topological invariant, studied since the 1930s.

Twisted Reidemeister torsion associated with a representation of the fundamental group to $GL(n, F)$ has been studied since the early 1990's. Here we are interested in the adjoints of representations to $SL_2(C)$, in connections with hyperbolic structures and the theory of character varieties (as well as Chern-Simons theory).


Twisted Alexander polynomial detects the unknot, decides fibrerness for some classes of knots (2006, 2007).

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Torsion of a chain complex

Let $C^\bullet = (0 \rightarrow C^m \xrightarrow{d_m} C^{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} C^0 \rightarrow 0)$ be a chain complex of finite dimensional vector spaces over $\mathbb{C}$. Choose a basis $c_i$ for $C^i$, and a basis $h_i$ for $H^i = H^i(C^\bullet)$. Let $b_i$ be a sequence of vectors in $C^i$ such that $d_i(b_i)$ is a basis of $B_{i-1} = \text{im}(d_i)$. Let $\tilde{h}_i$ be a lift of $h_i$ in $Z_i = \ker(d_i)$. The sign-determined Reidemeister torsion of $C^\bullet$ is

$$\text{Tor}(C^\bullet, c^\bullet, h^\bullet) = (-1)^{|C^\bullet|} \cdot n \prod_{i=0} \left[ d_i + 1 \right] \left( \frac{b_i}{c_i} \right) (-1)^i \in C^\bullet.$$ 

where $|C^\bullet| = \sum_{k \geq 0} \alpha_k(C^\bullet) \beta_k(C^\bullet)$, $\alpha_i(C^\bullet) = \sum_{k=0} \dim C_k$, $\beta_i(C^\bullet) = \sum_{k=0} \dim H_k$. 
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Choose a basis $c_i$ for $C_i$, and a basis $h_i$ for $H_i = H_i(C_\ast)$. Let $b_i$ be a sequence of vectors in $C_i$ such that $d_i(b_i)$ is a basis of $B_i = \text{im}(d_i)$. Let $\tilde{h}_i$ be a lift of $h_i$ in $Z_i = \ker(d_i)$. The sign-determined Reidemeister torsion of $C_\ast$ is

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Let $C_* = (0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over $\mathbb{C}$. Choose a basis $c^i$ for $C_i$, and a basis $h^i$ for $H_i = H_i(C_*)$.

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The sign-determined Reidemeister torsion of $C_*$ is

$$\text{Tor}(C_*, c^*, h^*) = (-1)^{|C_*|} \prod_{i=0}^{n} [d_{i+1}(b^{i+1}) \tilde{h}^i b^i / c^i](-1)^{i+1} \in \mathbb{C}^*.$$ 

where $|C_*| = \sum_{k \geq 0} \alpha_k(C_*) \beta_k(C_*)$, $\alpha_i(C_*) = \sum_{k=0}^{i} \dim C_k$, $\beta_i(C_*) = \sum_{k=0}^{i} \dim H_k$. 
Let $W$ be a finite CW-complex and $\rho$ be an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(W)$. Define

$$C^*(\tilde{W}; \mathbb{Z}) \otimes \mathbb{Z}[\pi_1(W)]_{\text{sl}_2(\mathbb{C})}\rho.$$ 

Here:

- $C^*(\tilde{W}; \mathbb{Z})$ is the complex of the universal cover with integer coefficients which is a $\mathbb{Z}[\pi_1(W)]$-module,
- $\text{Ad}: \text{SL}_2(\mathbb{C}) \rightarrow \text{Aut}(\text{sl}_2(\mathbb{C}))$, $A \mapsto \text{Ad}A$ is the adjoint representation,
- $\text{sl}_2(\mathbb{C})\rho$ is the $\mathbb{Z}[\pi_1(W)]$-module via the composition $\text{Ad} \circ \rho$.

Let $\tau_0 = \text{sgn}(\text{Tor}(C^*(W; \mathbb{R}), c^* B, h^*)) \in \{\pm 1\}$. Define the twisted Reidemeister torsion of $W$ to be

$$\text{TOR}(W; \text{sl}_2(\mathbb{C})\rho, h^*, o) = \tau_0 \cdot \text{Tor}(C^*(W; \text{sl}_2(\mathbb{C})\rho), c^* B, h^*) \in C^*(B).$$
Torsion of a CW-complex

Let $W$ be a finite CW-complex and $\rho$ be an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(W)$. 

Define $C^\ast(W; \text{sl}_2(\mathbb{C}) \rho) = C^\ast(\tilde{W}; \mathbb{Z}) \otimes \mathbb{Z}[\pi_1(W)] \text{sl}_2(\mathbb{C}) \rho$.

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$$\text{TOR}(W; \mathfrak{sl}_2(\mathbb{C})_\rho, \rho^*, \sigma) = \tau_0 \cdot \text{Tor}(C_*(W; \mathfrak{sl}_2(\mathbb{C})_\rho), c_B^*, h^*) \in \mathbb{C}^*.$$
Regularity for representations

Let $E_k$ be the exterior of the knot $K$. Roughly, with a notion of regularity there is a canonical way to choose bases for homologies. We say that $\rho \in \mathbb{R}_{irr}(\pi_1(K); SL_2(\mathbb{C}))$ is regular if $\dim H^1(\rho)(E_K) = 1$.

For a regular representation $\rho$, we have $\dim H^1(\rho)(E_K) = 1$, $\dim H^2(\rho)(E_K) = 1$ and $H^j(\rho)(E_K) = 0$ for all $j \neq 1, 2$.

Let $\lambda$ be the longitude of $K$. We say that an irreducible representation $\rho: \pi_1(K) \to SL_2(\mathbb{C})$ is $\lambda$-regular, if (J. Porti 1997):

1. the inclusion $\iota: \lambda \to E_K$ induces a surjective map $\iota^*: H^1(\rho)(\lambda) \to H^1(\rho)(E_K)$,
2. if $\text{trace}(\rho(\pi_1(\partial E_K))) \subset \{\pm 2\}$, then $\rho(\lambda) \neq \pm 1$. 

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2. if $\text{trace}(\rho(\pi_1(\partial E_K))) \subset \{\pm 2\}$, then $\rho(\lambda) \neq \pm 1$. 
The Reidemeister torsion for knot exteriors

Let $\rho: \pi_1(K) \to \text{SL}_2(\mathbb{C})$ be a $\lambda$-regular representation. There is a canonical way to choose a basis $o$ for the homology with real coefficients, and a basis $\{h_\rho^{(1)}(\lambda), h_\rho^{(2)}(\lambda)\}$ for the twisted homology. The Reidemeister torsion $T_K^\lambda(\rho)$ is defined to be $\text{TOR}(E_K; \text{sl}_2(C)\rho, \{h_\rho^{(1)}(\lambda), h_\rho^{(2)}(\lambda)\}, o) \in \mathbb{C}^*$. This torsion (defined by Porti, Dubois) is a numerical invariant, associated with not necessary acyclic (i.e. exact) chain complexes, generally not easy to compute. It has role in the asymptotic expansions of the colored Jones polynomial (Dubois-Kashaev, 2007).
Let $\rho: \pi_1(K) \to \text{SL}_2(\mathbb{C})$ be a $\lambda$-regular representation. There is a canonical way to choose a basis $\sigma$ for the homology with real coefficients, and a basis $\{h_{(1)}^\rho(\lambda), h_{(2)}^\rho\}$ for the twisted homology. The Reidemeister torsion $T_{K,\lambda}^{\rho}$ at $\rho$ is defined to be
\[ T_{K,\lambda}^{\rho}(\rho) = \text{TOR}(E_K; \text{sl}_2(\mathbb{C})^\rho, \{h_{(1)}^\rho(\lambda), h_{(2)}^\rho\}, \sigma) \in \mathbb{C}^\ast. \]
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Let $\rho : \pi_1(K) \to \text{SL}_2(\mathbb{C})$ be a $\lambda$-regular representation. There is a canonical way to choose a basis $o$ for the homology with real coefficients, and a basis $\{ h_\rho^0(1)(\lambda), h_\rho^0(2) \}$ for the twisted homology. The *Reidemeister torsion* $\mathbb{T}_\lambda^K$ at $\rho$ is defined to be

$$\mathbb{T}_\lambda^K(\rho) = \text{TOR} \left( E_K; \mathfrak{sl}_2(\mathbb{C})_\rho, \{ h_\rho^0(1)(\lambda), h_\rho^0(2) \}, o \right) \in \mathbb{C}^*.$$
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Twisted Reidemeister torsion polynomial

We study another torsion which is a function of one variable, associated to acyclic chain complex, which is easier to compute.

Let the CW–complex $W$ be $E_K$ and the homomorphism $\alpha: \pi_1(K) \to \mathbb{Z} = \langle t \rangle$ be the abelianization.

Define the $\tilde{sl}_2(C)$-$\rho$-twisted chain complex of $W$ to be $C^*(W; \tilde{sl}_2(C) \otimes \text{Ad} \circ \rho \otimes \alpha(\text{sl}_2(C) \otimes C(t)))$.

The sign–defined Reidemeister torsion of $W$ with respect to this $\tilde{sl}_2(C)$-twisted chain complex is defined to be $\text{TOR}(W; \tilde{sl}_2(C) \rho, h^\ast, o) = \tau_0 \cdot \text{Tor}(C^*(W; \tilde{sl}_2(C) \rho, c^\ast B, h^\ast)) \in C(t)^\ast$.

If $\rho$ is $\lambda$-regular, then all homology groups $H^\ast(E_K; \tilde{sl}_2(C) \rho)$ vanishes, the chain $C^*(W; \tilde{sl}_2(C) \rho)$ is acyclic (Y. Yamaguchi, 2005), and we define the twisted Reidemeister torsion polynomial at $\rho$ to be $T_K^{\lambda}(\rho) = \text{TOR}(W; \tilde{sl}_2(C) \rho, \emptyset, o) \in C(t)^\ast$.

The torsion is also determined up to a factor $t^m$ where $m \in \mathbb{Z}$. 
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Define the $\tilde{\mathfrak{sl}}_2(\mathbb{C})$-twisted chain complex of $W$ to be

$$C_*(W; \tilde{\mathfrak{sl}}_2(\mathbb{C}) \mathcal{R}) = C_*(\tilde{W}; \mathbb{Z}) \otimes_{Ad \circ \mathcal{R} \otimes \alpha} (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t)).$$

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We study another torsion which is a function of one variable, associated to *acyclic* chain complex, which is easier to compute. Let the CW–complex $W$ be $E_K$ and the homomorphism $\alpha : \pi_1(K) \to \mathbb{Z} = \langle t \rangle$ be the abelianization. Define the $\tilde{\mathfrak{sl}}_2(\mathbb{C})$-twisted chain complex of $W$ to be

$$C_*(W; \tilde{\mathfrak{sl}}_2(\mathbb{C})_\rho) = C_*(\tilde{W}; \mathbb{Z}) \otimes \text{Ad} \circ \rho \otimes \alpha (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t)).$$

The sign–defined Reidemeister torsion of $W$ with respect to this $\tilde{\mathfrak{sl}}_2(\mathbb{C})$-twisted chain complex is defined to be

$$\text{TOR}(W; \tilde{\mathfrak{sl}}_2(\mathbb{C})_\rho, h^*, \emptyset) = \tau_0 \cdot \text{Tor}(C_*(W; \tilde{\mathfrak{sl}}_2(\mathbb{C})_\rho, c^*_B, h^*)) \in \mathbb{C}(t)^*.$$

If $\rho$ is $\lambda$-regular, then all homology groups $H_*(E_K; \tilde{\mathfrak{sl}}_2(\mathbb{C})_\rho)$ vanishes, the chain $C_*(W; \tilde{\mathfrak{sl}}_2(\mathbb{C})_\rho)$ is acyclic (Y. Yamaguchi, 2005), and we define the twisted Reidemeister torsion polynomial at $\rho$ to be

$$T^K_\lambda(\rho) = \text{TOR}(W; \tilde{\mathfrak{sl}}_2(\mathbb{C})_\rho, \emptyset, \emptyset) \in \mathbb{C}(t)^*.$$

The torsion is also determined up to a factor $t^m$ where $m \in \mathbb{Z}$. 
Theorem (Yamaguchi (2005))

The derivative with respect to $t$ of $T^K_{\lambda}(\rho)$ at $t = 1$ is equal to $-T^K_{\lambda}(\rho)$. 
How to compute $\mathcal{T}_\chi^K(\rho)$ from Fox free differential calculus

Choose and fix a Wirtinger presentation $\pi_1(K) = \langle x_1, \ldots, x_k | r_1, \ldots, r_{k-1} \rangle$.

Let $W_K$ be the 2-dimensional CW–complex constructed from the presentation.

F. Waldhausen proved that the Whitehead group of a knot group is trivial. As a result, $W_K$ has the same simple homotopy type as $E_K$.

So, the CW–complex $W_K$ can be used to compute the twisted Reidemeister torsion polynomial.
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F. Waldhausen proved that the Whitehead group of a knot group is trivial. As a result, $W_K$ has the same simple homotopy type as $E_K$. So, the CW–complex $W_K$ can be used to compute the twisted Reidemeister torsion polynomial.
The twisted complex $C_\ast(W_K; \tilde{\mathfrak{sl}}_2(C)_\rho)$ becomes:

$$0 \to (\mathfrak{sl}_2(C) \otimes \mathbb{C}(t))^{k-1} \xrightarrow{\partial_2} (\mathfrak{sl}_2(C) \otimes \mathbb{C}(t))^k \xrightarrow{\partial_1} \mathfrak{sl}_2(C) \otimes \mathbb{C}(t) \to 0.$$
The twisted complex $C_\ast(W_K; \tilde{sl}_2(\mathbb{C})_\rho)$ becomes:

$$0 \rightarrow (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t))^{k-1} \xrightarrow{\partial_2} (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t))^k \xrightarrow{\partial_1} \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t) \rightarrow 0.$$ 

Where, writing $\Phi$ for $(\text{Ad} \circ \rho) \otimes \alpha$:

$$\partial_1 = (\Phi(x_1 - 1), \Phi(x_2 - 1), \ldots, \Phi(x_k - 1)).$$

and $\partial_2$ is expressed using the Fox’s free differential calculus

$$\partial_2 = \begin{pmatrix}
\Phi\left(\frac{\partial r_1}{\partial x_1}\right) & \ldots & \Phi\left(\frac{\partial r_{k-1}}{\partial x_1}\right) \\
\vdots & \ddots & \vdots \\
\Phi\left(\frac{\partial r_1}{\partial x_k}\right) & \ldots & \Phi\left(\frac{\partial r_{k-1}}{\partial x_k}\right)
\end{pmatrix}$$
The twisted complex $C_*(W_K; \tilde{sl}_2(\mathbb{C})_{\rho})$ becomes:

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Where, writing $\Phi$ for $(Ad \circ \rho) \otimes \alpha$:

$$\partial_1 = (\Phi(x_1 - 1), \Phi(x_2 - 1), \ldots, \Phi(x_k - 1)).$$

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\vdots & \ddots & \vdots \\
\Phi\left(\frac{\partial r_1}{\partial x_k}\right) & \ldots & \Phi\left(\frac{\partial r_{k-1}}{\partial x_k}\right)
\end{pmatrix}$$

Let $A^1_{K, Ad \circ \rho}$ denote the $3(k-1) \times 3(k-1)$–matrix obtained from the matrix of $\partial_2$ by deleting its first row.
The torsion polynomial $T_{K^\lambda}(\rho)$ can be described, up to a factor $t^m$ ($m \in \mathbb{Z}$), as:

$$T_{K^\lambda}(\rho) = \tau_0 \cdot \det A_{1K}, \text{Ad}_\circ \rho \cdot \det(\Phi(x_1 - 1)).$$

This rational function has the first order zero at $t = 1$. The twisted Reidemeister torsion $T_{K^\lambda}(\rho)$ is expressed as

$$T_{K^\lambda}(\rho) = -\lim_{t \to 1} T_{K^\lambda}(\rho)(t - 1) = -\lim_{t \to 1} (\tau_0 \cdot \det A_{1K}, \text{Ad}_\circ \rho)(t - 1) \det(\Phi(x_1 - 1)).$$
The torsion polynomial $\mathcal{T}_\lambda^K(\rho)$ can be described, up to a factor $t^m$ ($m \in \mathbb{Z}$) as:

$$\mathcal{T}_\lambda^K(\rho) = \tau_0 \cdot \frac{\det A_{K, Ad \circ \rho}}{\det(\Phi(x_1 - 1))}.$$
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$$\mathcal{T}^K_\lambda(\rho) = \tau_0 \cdot \frac{\det A^1_{K, Ad^\circ \rho}}{\det(\Phi(x_1 - 1))}.$$ 

This rational function has the first order zero at $t = 1$. The twisted Reidemeister torsion $\mathbb{T}^K_\lambda(\rho)$ is expressed as

$$\mathbb{T}^K_\lambda(\rho) = - \lim_{t \to 1} \frac{\mathcal{T}^K_\lambda(\rho)}{(t - 1)} = - \lim_{t \to 1} \left( \tau_0 \cdot \frac{\det A^1_{K, Ad^\circ \rho}}{(t - 1) \det(\Phi(x_1 - 1))} \right).$$
Using Riley's method we can parametrize a non–abelian SL$_2(\mathbb{C})$-representation $\rho$ by two parameters $u$ and $s$ as follows:

\[
\rho(x) = \left( \frac{\sqrt{s} \bar{1}}{\sqrt{s} \bar{0}} \right), \quad \rho(y) = \left( \frac{\sqrt{s} \bar{-1}}{\sqrt{s} \bar{-0}} \right).
\]

Let $W = \rho(w)$. Then $s$ and $u$ satisfy Riley's equation

\[
\phi J(2, 2^m)(s, u) = W_{1, 1} + (1 - s) W_{1, 2} = 0.
\]

Let $\xi \pm$ are the eigenvalues of $W$, given by explicit expressions in terms of $s$ and $u$. 

Formulas for the torsion of twist knots
Using Riley’s method we can parametrize a non–abelian \( SL_2(\mathbb{C}) \)-representation \( \rho \) by two parameters \( u \) and \( s \) as follows:

\[
\rho(x) = \begin{pmatrix} \sqrt{s} & 1/\sqrt{s} \\ 0 & 1/\sqrt{s} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \sqrt{s} & 0 \\ -\sqrt{su} & 1/\sqrt{s} \end{pmatrix}.
\]

Let \( W = \rho(w) \). Then \( s \) and \( u \) satisfy Riley’s equation

\[
\phi_{J(2,2m)}(s, u) = W_{1,1} + (1 - s)W_{1,2} = 0.
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\phi_{J(2,2m)}(s, u) = W_{1,1} + (1 - s)W_{1,2} = 0.
\]

Let $\xi_{\pm}$ are the eigenvalues of $W$, given by explicit expressions in terms of $s$ and $u$. 

Theorem
Let $m$ be a positive integer.

1. The Reidemeister torsion $T_J^{(2,2m)}(\rho)$ is:

\[
\frac{\tau_0}{s + s^{-1} - 2} \left[ C_1(m)s^{m-1}t_m + C_2(m)s^{m-1}t_m + C_3(m) \right].
\]

2. Similarly, $T_J^{(2,-2m)}(\rho)$ is

\[
\frac{\tau_0}{s + s^{-1} - 2} \left[ -C_1(-m)s^{-m-1}t_m - C_2(-m)s^{-m-1}t_m + C_3(-m) \right].
\]

Where $C_1(m), C_2(m), C_3(m), t_m, \xi_+, \xi_-$ are explicit expressions in terms of $s, u, m$ (the formulas are available in our paper).
Torsion at the holonomy representation

Formulas of the twisted Reidemeister torsion associated to twist knots are complicated. But formulas for the twisted Reidemeister torsion at holonomy representations are simpler (could be efficiently computed using computer).

Every twist knots except the trefoil knot are hyperbolic. The exterior of a hyperbolic knot admits a hyperbolic structure which determines a unique discrete faithful representation of the knot group in $\text{PSL}_2(\mathbb{C})$, called the holonomy representation. Such a representation lifts to $\text{SL}_2(\mathbb{C})$ and determines two representations in $\text{SL}_2(\mathbb{C})$. Such lifts are $\lambda$-regular representations.
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Lemma

Let $K$ be a hyperbolic two–bridge knot and suppose that its knot group admits a presentation $\pi_1(K) = \langle x, y \mid wx = yw \rangle$.

If $\rho_0$ denotes a lift in $\text{SL}_2(\mathbb{C})$ of the holonomy representation, then $\rho_0$ is given by, up to conjugation,

\[ x \mapsto \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \pm \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}, \]

where $u$ is a root of Riley’s equation $\phi_K(1, u) = 0$ of $K$. 


Theorem

Let $m > 0$, then

1. \( \mathbb{T}_{\lambda}^{J(2,2m)}(\rho_u) = \frac{-\tau_0}{u^2 + 4} \left[ (4 + m(u^2 - 4u + 8)) \, t_m(\xi^m_+ + \xi^m_-) 
+ (t_m(\xi^{m-1}_+ + \xi^{m-1}_-) - 1) \, (u^2 - 4)m 
+ (-5u^2 - 8u + 4)t^2_m \right] , \)

2. \( \mathbb{T}_{\lambda}^{J(2,-2m)}(\rho_u) = \frac{-\tau_0}{u^2 + 4} \left[ (-4 + m(u^2 - 4u + 8)) \, t_m(\xi^m_+ + \xi^m_-) 
+ (t_m(\xi^{m+1}_+ + \xi^{m+1}_-) + 1) \, (u^2 - 4)m 
+ (-5u^2 - 8u + 4)t^2_m \right] . \)
Asymptotic behavior of torsion at the holonomy
Asymptotic behavior of torsion at the holonomy

Figure: Graph of $|\mathbb{T}_\lambda^{J(2,-2m)}(\rho_0)|$ and $f(m) = C(\# J(2,-2m))^3$, where $\#K$ is the number of crossings of $K$. 
Observation
The sequence \( \left( |\mathbb{T}^{J(2, -2m)}(\rho_0)| \right)_{m \geq 1} \) has the same behavior as the sequence \( \left( C(\#J(2, -2m))^3 \right)_{m \geq 1} \), for some constant \( C \).
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Twist knots can be obtained by surgery on the Whitehead link. The above observation can be justified by using the Product Formula for Reidemeister torsion.

We do not know yet the precise value of the constant \( C \).
Observation
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Thank you!