On L_∞-algebra of toric manifolds

Cheol-Hyun Cho

Seoul National University

December 16, 2008

A_∞-algebra : (V, m)

- Generalization of a differential graded algebra.

A_∞-algebra : (V, m)

- Generalization of a differential graded algebra.
- Opertations $m_1, m_2, m_3, \cdots$
- Differential m_1
- Product m_2

A_∞-algebra : (V, m)

- Generalization of a differential graded algebra.
- Opertations $m_1, m_2, m_3, \cdots$
- Differential m_1
- Product m_2

Definition

A sequence of homomorphisms $m_k : V[1]^{\otimes k} \to V[1]$ for $k = 1, 2, \cdots$, satisfying

$$\sum_{k_1+k_2=k+1} (-1)^{Koszul} m_{k_1}(x_1, \cdots, m_{k_2}(x_i, \cdots, x_{i+k_2-1}), \cdots, x_k) = 0. \tag{1}$$

The first few equations(up to sign)

$$m_1 \circ m_1 = 0$$

The first few equations(up to sign)

$$m_1 \circ m_1 = 0$$

$$(m_1 \text{ differential})$$

The first few equations(up to sign)

$$m_1 \circ m_1 = 0$$

$(m_1$ differential$)$

$$m_1(m_2(x,y)) = m_2(m_1(x),y) + m_2(x,m_1(y))$$

(Leibniz rule for the product m_2)

The first few equations(up to sign)

$$m_1 \circ m_1 = 0$$

$$(m_1 \text{ differential})$$

$$m_1(m_2(x, y)) = m_2(m_1(x), y) + m_2(x, m_1(y))$$

$$(\text{Leibniz rule for the product } \; m_2)$$

$$m_2(m_2(x, y), z) - m_2(x, m_2(y, z)) = m_1(m_3(x, y, z)) + m_3(m_1(x), y, z)$$
$$+ m_3(x, m_1(y), z) + m_3(x, y, m_1(z))$$

The first few equations(up to sign)

$$m_1 \circ m_1 = 0$$

$(m_1$ differential)

$$m_1(m_2(x,y)) = m_2(m_1(x),y) + m_2(x,m_1(y))$$

(Leibniz rule for the product m_2)

$$m_2(m_2(x,y),z) - m_2(x,m_2(y,z)) = m_1(m_3(x,y,z)) + m_3(m_1(x),y,z)$$
$$+ m_3(x,m_1(y),z) + m_3(x,y,m_1(z))$$

(Homotopy associativity)

The first few equations(up to sign)

$$m_1 \circ m_1 = 0$$

(m_1 differential)

$$m_1(m_2(x, y)) = m_2(m_1(x), y) + m_2(x, m_1(y))$$

(Leibniz rule for the product m_2)

$$m_2(m_2(x, y), z) - m_2(x, m_2(y, z)) = m_1(m_3(x, y, z)) + m_3(m_1(x), y, z)$$
$$+ m_3(x, m_1(y), z) + m_3(x, y, m_1(z))$$

(Homotopy associativity)

$$m_2(m_3(...),) + m_1(m_4(..)) + m_3(m_2(..), ..) = 0$$

(Homotopies of homotopies)

L_∞-algebra

- Associative algebra $\mapsto$ Lie algebra

$$[a, b] = ab - ba.$$

L_∞-algebra

- Associative algebra $\mapsto$ Lie algebra

$$[a, b] = ab - ba.$$

- Given any A_∞-algebra, one obtains L_∞-algebra by setting

$$l_k(x_1, \cdots, x_k) = \sum_{\sigma \in S_k} \pm m_k(x_{\sigma(1)}, \cdots, x_{\sigma(k)}).$$

L_∞-algebra

- Associative algebra $\mapsto$ Lie algebra

$$[a, b] = ab - ba.$$

- Given any A_∞-algebra, one obtains L_∞-algebra by setting

$$l_k(x_1, \cdots, x_k) = \sum_{\sigma \in S_k} \pm m_k(x_{\sigma(1)}, \cdots, x_{\sigma(k)}).$$

- l_2 satisfies homotopy Jacobi Identity.

$$l_2(x, y) = m_2(x, y) + (-1)^{(|x|+1)(|y|+1)} m_2(y, x).$$

Lagrangian Floer homology via Iterated integrals(FOOO)

- $L \subset M$ be a Lagrangian submanifold. (For example, equator of the sphere $\mathbb{C}P^1$.)

Lagrangian Floer homology via Iterated integrals(FOOO)

- $L \subset M$ be a Lagrangian submanifold. (For example, equator of the sphere $\mathbb{C}P^1$.)
- $\mathcal{M}_1(M, L, \beta, J)$ be the certain moduli space of J-holomorphic disc $u : (D^2, \partial D^2) \to (M, L)$ of homotopy class β.

Lagrangian Floer homology via Iterated integrals(FOOO)

- $L \subset M$ be a Lagrangian submanifold. (For example, equator of the sphere $\mathbb{C}P^1$.)
- $\mathcal{M}_1(M, L, \beta, J)$ be the certain moduli space of J-holomorphic disc $u : (D^2, \partial D^2) \to (M, L)$ of homotopy class β.
- $S^1 \subset \mathbb{C}P^1$ case, there are upper disc,lower disc, and higher order ones, which are holomorphic discs

Lagrangian Floer homology via Iterated integrals(FOOO)

- $L \subset M$ be a Lagrangian submanifold. (For example, equator of the sphere $\mathbb{C}P^1$.)
- $\mathcal{M}_1(M, L, \beta, J)$ be the certain moduli space of J-holomorphic disc $u : (D^2, \partial D^2) \to (M, L)$ of homotopy class β.
- $S^1 \subset \mathbb{C}P^1$ case, there are upper disc,lower disc, and higher order ones, which are holomorphic discs
- For Lagrangian torus fibers L of toric Fano manifolds, holomorphic discs with boundary on L are completely classified by Cho, Cho-Oh.

- $(\Omega^\bullet(L), d)$: the de Rham complex of L.

- $(\Omega^\bullet(L), d)$: the de Rham complex of L.
- Δ_k : the k-simplex

$$\{(t_1, t_2, \cdots, t_k) \in \mathbb{R}^k | 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

- $(\Omega^\bullet(L), d)$: the de Rham complex of L.
- Δ_k : the k-simplex

$$\{(t_1, t_2, \cdots, t_k) \in \mathbb{R}^k | 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

- Consider the maps

$$\begin{array}{ccc} \Delta_k \times \mathcal{M}_1(\beta) & \xrightarrow{\Phi_k} & L \times \cdots \times L \\ \downarrow{\scriptstyle ev_0} & & \\ L & & \end{array}$$

where $ev_0 : u \mapsto u(0)$ the projection

$$\Phi_k(t_1, \cdots, t_k, u) = (u(t_1), \cdots, u(t_k)).$$

Define

$$P_{k,\beta}(\omega_1, \cdots, \omega_k) = (ev_0)_! \Phi^*_{k,\beta}(\pi_1^* \omega_1 \wedge \cdots \wedge \pi_k^* \omega_k).$$

Define

$$P_{k,\beta}(\omega_1, \cdots, \omega_k) = (ev_0)_! \Phi^*_{k,\beta}(\pi_1^* \omega_1 \wedge \cdots \wedge \pi_k^* \omega_k).$$

$$P_k = \sum_\beta P_{k,\beta} T^{\omega(\beta)}.$$

Define

$$P_{k,\beta}(\omega_1, \cdots, \omega_k) = (ev_0)_! \Phi^*_{k,\beta}(\pi_1^*\omega_1 \wedge \cdots \wedge \pi_k^*\omega_k).$$

$$P_k = \sum_\beta P_{k,\beta} T^{\omega(\beta)}.$$

Define with formal parameter t,

$$\begin{aligned} m_1(\omega_1) &= d\omega_1 + t\ P_1(\omega_1) \\ m_2(\omega_1, \omega_2) &= \omega_1 \wedge \omega_2 + t\ P_2(\omega_1, \omega_2) \\ m_k(\omega_1, \cdots, \omega_k) &= t\ P_k(\omega_1, \cdots, \omega_k) \end{aligned}$$

$\{m_k\}$ defines an A_∞-algebra. (Fukaya, Oh, Ohta, Ono),

Define

$$P_{k,\beta}(\omega_1, \cdots, \omega_k) = (ev_0)_! \Phi^*_{k,\beta}(\pi_1^* \omega_1 \wedge \cdots \wedge \pi_k^* \omega_k).$$

$$P_k = \sum_\beta P_{k,\beta} T^{\omega(\beta)}.$$

Define with formal parameter t,

$$\begin{aligned} m_1(\omega_1) &= d\omega_1 + t\, P_1(\omega_1) \\ m_2(\omega_1, \omega_2) &= \omega_1 \wedge \omega_2 + t\, P_2(\omega_1, \omega_2) \\ m_k(\omega_1, \cdots, \omega_k) &= t\, P_k(\omega_1, \cdots, \omega_k) \end{aligned}$$

$\{m_k\}$ defines an A_∞-algebra. (Fukaya, Oh, Ohta, Ono), (This type of integral is originally due to K.T. Chen, and used to identify Hochschild cohomology of differntial forms of a manifold M with the cohomology of the **loop space of M**.)

- Homology of $m_1 \implies$ Lagrangian Floer homology

- Homology of $m_1 \Longrightarrow$ Lagrangian Floer homology
- Non-vanishing of m_1-homology $\Longrightarrow$ a Lagrangian submanifold L has an intersection with any image of L under Hamiltonian isotopy.

- Homology of $m_1 \implies$ Lagrangian Floer homology
- Non-vanishing of m_1-homology $\implies$ a Lagrangian submanifold L has an intersection with any image of L under Hamiltonian isotopy.
- Homology of m_1, product structure m_2 has been computed in the case of toric Fano manifolds (due to (C, C-O)) and some other examples.

L_∞-divisor equation

- (C-), there is a divisor equation.

$$l_{k,\beta}(x_1, \cdots, x_k) = (\int_{\partial\beta} x_i)\ l_{k-1,\beta}(x_1, \cdots, \widehat{x_i}, \cdots, x_k),$$

  if x_i is degree one differential form.
- (idea of proof) Combine $n!$ simplices to obtain a cube $[0,1]^n$ and take the $\int x_i$ alone.

L_∞-divisor equation

- (C-), there is a divisor equation.

$$l_{k,\beta}(x_1, \cdots, x_k) = (\int_{\partial\beta} x_i)\, l_{k-1,\beta}(x_1, \cdots, \widehat{x_i}, \cdots, x_k),$$

  if x_i is degree one differential form.
- (idea of proof) Combine $n!$ simplices to obtain a cube $[0,1]^n$ and take the $\int x_i$ alone.
- Corollary) $l_{k,\beta}(x_1, \cdots, df, \cdots, x_k)$ vanishes for any funtion f on L.

Cyclic inner products of A_∞-algebra

- (due to Kontsevich) If there exist an $<,>$ which satisfies

$$< m_k(x_1, \cdots, x_k), x_{k+1} >= (-1)^{Koszul} < m_k(x_2, \cdots, x_{k+1}), x_1 > . \quad (2)$$

Cyclic inner products of A_∞-algebra

- (due to Kontsevich) If there exist an $<,>$ which satisfies

$$< m_k(x_1, \cdots, x_k), x_{k+1} >= (-1)^{Koszul} < m_k(x_2, \cdots, x_{k+1}), x_1 > . \tag{2}$$

- Needed to define open Gromov-Witten potential.

$$\Phi = \sum_k < m_k(\mathbf{x}, \cdots, \mathbf{x}), \mathbf{x} >$$

where $\mathbf{x} = \sum x_i e_i$ and x_i formal parameters and e_i basis V.

Cyclic inner products of A_∞-algebra

- (due to Kontsevich) If there exist an $<,>$ which satisfies

$$< m_k(x_1, \cdots, x_k), x_{k+1} >= (-1)^{Koszul} < m_k(x_2, \cdots, x_{k+1}), x_1 > . \tag{2}$$

- Needed to define open Gromov-Witten potential.

$$\Phi = \sum_k < m_k(\mathbf{x}, \cdots, \mathbf{x}), \mathbf{x} >$$

where $\mathbf{x} = \sum x_i e_i$ and x_i formal parameters and e_i basis V.

- These are number of holomorphic discs with certain intersection property, which in general is not an invariant.
- For cyclic A_∞-homomorphism, Potential is preserved up to change of parameter.

Some open GW invariants

Theorem

(C-) For Lagrangian submanifolds with non-vanishing Floer homology, Three-point disc Gromov Witten invariants are well-defined.
Furthermore it vanishes always in such cases.

Some open GW invariants

Theorem

(C-) For Lagrangian submanifolds with non-vanishing Floer homology, Three-point disc Gromov Witten invariants are well-defined.
Furthermore it vanishes always in such cases.

Theorem

(C-H) For $dim(L) = 2$ *with non-vanishing Floer homology, five-point disc Gromov-Witten invariants are well-defined and vanishes.*

Some open GW invariants

Theorem

(C-) For Lagrangian submanifolds with non-vanishing Floer homology, Three-point disc Gromov Witten invariants are well-defined.
Furthermore it vanishes always in such cases.

Theorem

(C-H) For $dim(L) = 2$ with non-vanishing Floer homology, five-point disc Gromov-Witten invariants are well-defined and vanishes.

(idea) to use L_∞-equation, and explicit calculation of iterated integrals for Maslov index two discs. cyclic symmetry, and sign cancellations.

- Theorem

  (C-) There exist well-defined functions

$$HF^{cyc}(L) \to \Lambda_{nov}, \quad HF^{sym}(L) \to \Lambda_{nov}.$$

- Theorem

  (C-) There exist well-defined functions

$$HF^{cyc}(L) \to \Lambda_{nov}, \quad HF^{sym}(L) \to \Lambda_{nov}.$$

  - Here $HF^{cyc}(L)$ or $HF^{sym}(L)$ are cohomology of the cyclic or symmetrized bar complex of the A_∞-algebra.

- Theorem

  (C-) There exist well-defined functions

$$HF^{cyc}(L) \to \Lambda_{nov}, \quad HF^{sym}(L) \to \Lambda_{nov}.$$

  - Here $HF^{cyc}(L)$ or $HF^{sym}(L)$ are cohomology of the cyclic or symmetrized bar complex of the A_∞-algebra.
  - These may be considered as generalized counting invariants. (idea) Find a condition where the bubbling off disc cancels as a whole.

Theorem

(C-H) The canonical model of L_∞-structure of $T^2 \subset CP^2$ (or in fact any two dimensional Fano toric case) can be completely determined.

Vanishing and invariants

- If cyclic L_∞-algebra has property that l_j vanishes for all $j < k$, then l_k defines a homological invariant.

Vanishing and invariants

- If cyclic L_∞-algebra has property that l_j vanishes for all $j < k$, then l_k defines a homological invariant.
- The expression $< l_k(x_1, \cdots, x_k), x_{k+1} >$ is independent of the choice x_i in l_1-homology class.

Vanishing and invariants

- If cyclic L_∞-algebra has property that l_j vanishes for all $j < k$, then l_k defines a homological invariant.
- The expression $< l_k(x_1, \cdots, x_k), x_{k+1} >$ is independent of the choice x_i in l_1-homology class.
- To prove it, consider $x_i = d\alpha$,

$$< l_k(x_1, \cdots, d\alpha, \cdots, x_k), x_{k+1} >= \pm < l_k(x_{i+1}, \cdots, x_{i-1}), d\alpha >$$

$$= \pm < dl_k(x_{i+1}, \cdots, x_{i-1}), \alpha >$$

$$= \pm < l_{k_1}(l_{k_2}(\cdots), \cdots), \alpha >= 0$$