Fundamental groups of complements of Arrangements and plane curves - Part I

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- Computations of braid monodromy & fundamental groups of curves

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- Known results on conic-line arrangements
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Importance

- Used by Chisini, Kulikov and Kulikov-Teicher in order to distinguish between connected components of the moduli space of surfaces of general type.

- The Zariski-Lefschetz hyperplane section theorem:

$$\pi_1(\mathbb{CP}^N \setminus S) \cong \pi_1(H \setminus H \cap S),$$

  where S is an hypersurface and H is a generic 2-plane. This invariant can be used also for computing the fundamental group of complements of hypersurfaces in $\mathbb{CP}^N$.

- Getting more examples of Zariski pairs: A pair of plane curves is called a *Zariski pair* if they have the same combinatorics, but their complements are not homeomorphic.

- Exploring new finite non-abelian groups which are serving as fundamental groups of complements of plane curves in general.

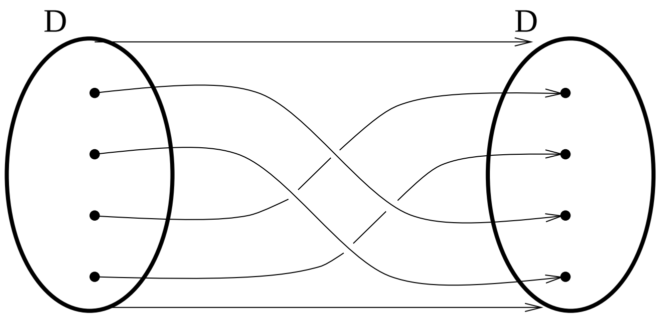
- Computing the fundamental group of the Galois cover of a surface: By the fundamental group of a complement of a branch curve of a surface, we can find the fundamental group of the Galois cover of the surface, with respect to a generic projection of the surface onto $\mathbb{CP}^2$.

Braid group

Topological definition of the braid group $B_n[D, K]$:

D disk, $K = \{a_1, \ldots, a_n\} \subset D$

$$\mathcal{B} = \left\{ \beta : \begin{array}{c} \beta : D \to D \text{ diffeomorphism,} \\ \beta(K) = K, \beta|_{\partial D} = \text{Id}|_{\partial D} \end{array} \right\} / \sim$$

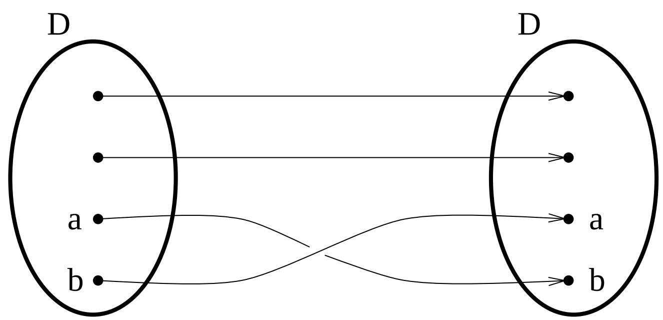


Half-twist: $\sigma \subset (D - \partial D - K) \cup \{a, b\}$ a simple path such that σ connects a with b. Choose $f : \mathbb{R}^2 \to \mathbb{C}$ such that:

$$f(\sigma) = [-1, 1], \; f(U) = \{z \in \mathbb{C} \; : \; |z| < 2\}, \; \alpha(x) = \begin{cases} 1 & x \in [0, \frac{3}{2}] \\ 0 & x \geq 2 \end{cases}$$

Define a diffeomorphism $h : \mathbb{C} \to \mathbb{C}$ as follows: $h(z) = re^{i(\varphi + \alpha(r)\pi)}$.

$$H(\sigma) = (f \cdot h \cdot f^{-1})|_D$$

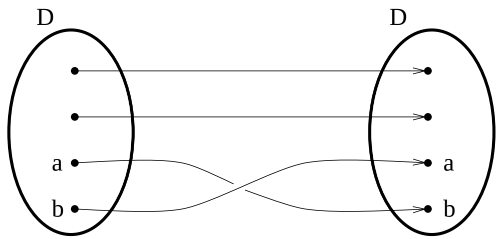


Frame of $B_n[D,K]$

$K = \{a_1, a_2, \cdots, a_n\}$. $\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$ - system of simple paths in $D - \partial D$ such that each σ_i connects a_i with a_{i+1}.
Let $H_i = H(\sigma_i)$. $(H_1, H_2, \cdots, H_{n-1})$ is called **a frame of $B_n[D,K]$ defined by $\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$**.

$$B_n \cong \left\langle H_1, \ldots, H_{n-1} \;\middle|\; \begin{array}{cc} H_iH_j = H_jH_i & \text{for } |i-j| \geq 2 \\ H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1} & \end{array} \right\rangle$$

which is the *Artin algebraic presentation* of the braid group.



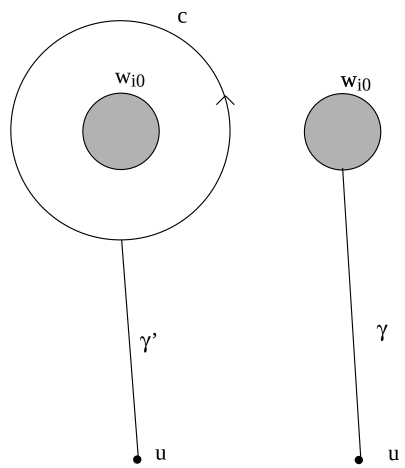
Skeleton in (D, K, K'')

$K'' \subset K, K'' = \{b_1, \cdots, b_m\}$. A *skeleton* in (D, K, K'') is simple paths $(p_1, \cdots, p_{m-1})$ in $D - \partial D$ such that each p_i connects b_i to b_{i+1}. $(p_1, \cdots, p_{m-1}) \sim (\tilde{p}_1, \cdots, \tilde{p}_{m-1})$, if $H(p_i) = H(\tilde{p}_i)$, $\forall i$.

Braid Monodromy

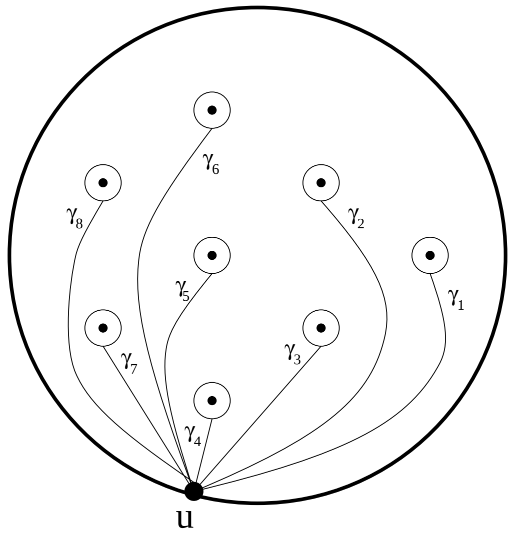
Monodromy in general: From π_1 to a group.

$\ell(\gamma)$, **g-base**



$K \subset D$, $\# K < \infty$, $u \in D - K$. $\{\gamma_i\}$ is a **bush** in (D, K, u), if $\forall i, j$, $\gamma_i \cap \gamma_j = u$; $\forall i$, $\gamma_i \cap K = *_i$, and γ_i are ordered counterclockwise around u.

$\Gamma_i = l(\gamma_i) \in \pi_1(D - K, u)$. $\{\Gamma_i\}$ is called a **g-base** of $\pi_1(D - K, *)$.



Braid monodromy of C with respect to $E \times D, \pi_1, M$:
All in $\mathbb{C}^2$. $E \subset x$-axis, $D \subset y$-axis, C algebraic curve in $E \times D$.

$\pi_1 : E \times D \to E$, $\pi_2 : E \times D \to D$ canonical projections.

$\pi = \pi_1|_C : C \to E$, $\deg \pi = n$. $N = \{x \in E \mid \#\pi^{-1}(x) < n\}$.

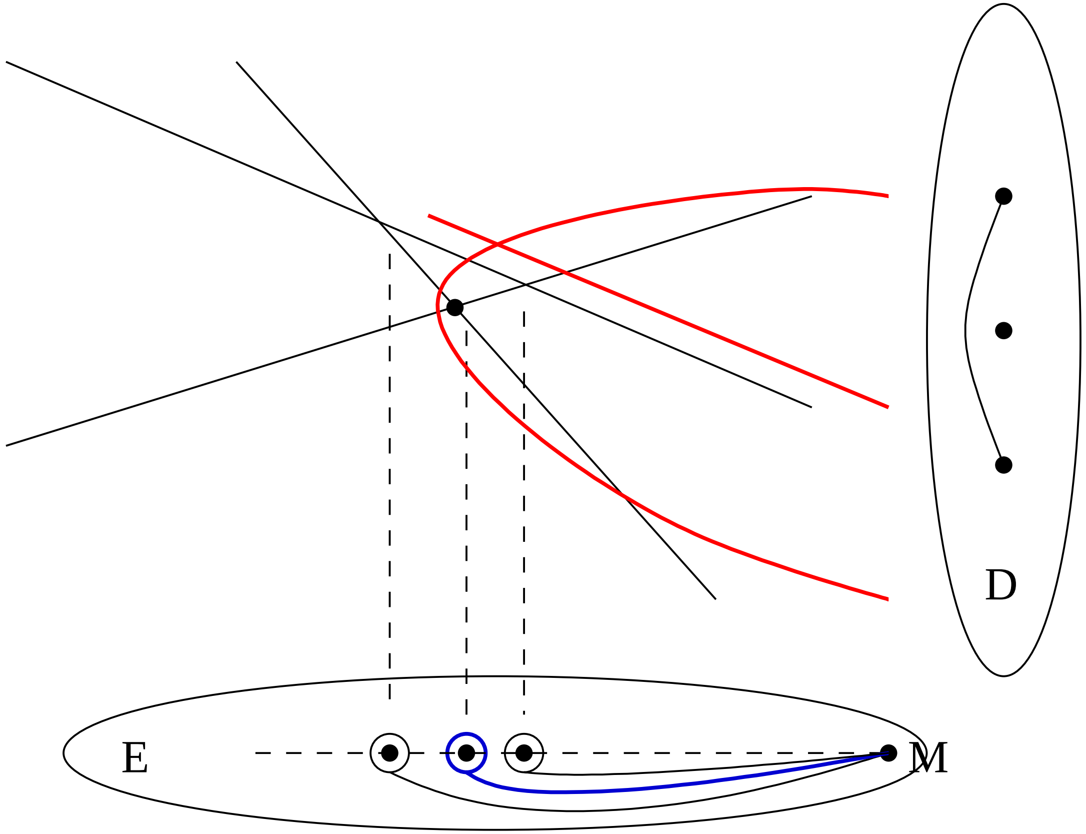
Choose $M \in \partial E$, $K = K(M) = \pi^{-1}(M)$. $K = \{a_1, a_2, \cdots, a_n\}$.

Each loop in $E - N$ defines a braid in $B_n[M \times D, K]$.

We get a group homomorphism

$$\varphi : \pi_1(E - N, M) \to B_n[M \times D, K]$$

which is called **the braid monodromy of C with respect to $E \times D, \pi_1, M$**.



An illustration for the braid monodromy

Moishezon-Teicher's algorithm for braid monodromy of a real line arrangement

More ways for computing $\pi_1(\mathbb{C}^2 - \mathcal{L})$: Randell (1982), Arvola (1993), Dung-Vui (1995), Cohen-Suciu (1997).

Line arrangement in $\mathbb{CP}^2$: An algebraic curve in $\mathbb{CP}^2$ which is a union of projective lines. An arrangement is called *real* if its defining equations can be written with real coefficients.

Theorem (Moishezon-Teicher): Let γ be a simple path in $E - N$ connecting x_j with M ($\in \partial E$), $[x_j, x_j'] \subset \gamma$. Let γ' be the part of γ from x_j' to M.

$$\varphi : \pi_1(E - N, M) \to B_n[M \times D, K_M]$$

be the braid monodromy of C w.r.t. $E \times D, \pi_1$, M. $\Gamma = \ell(\gamma)$. Then,

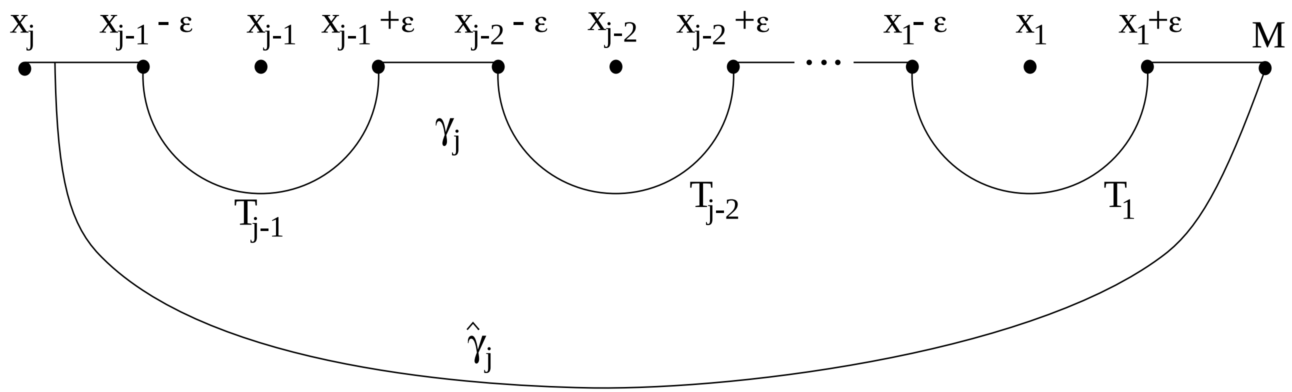
$$\varphi(\Gamma) = \Delta^2 < \mathcal{L}.\mathrm{V.C.}(\gamma', H(< \xi_x >)) >$$

Moishezon-Teicher's algorithm

$N = \{x_1, x_2, \cdots, x_q\}$ with $x_q < x_{q-1} < \cdots < x_2 < x_1$, $M \in \partial E \cap$ (real axis), with $M > x_1$, and $\epsilon > 0$ very small number. T_j $(1 \leq j \leq q)$ - the path from $x_j - \epsilon$ to $x_j + \epsilon$ along the semicircle below real axis centered at x_j.

$$\gamma_j = [x_j, x_{j-1} - \epsilon] \cdot T_{j-1} \cdot [x_{j-1} + \epsilon, x_{j-2} - \epsilon]$$

$$\cdot T_{j-2} \cdots T_1 \cdot [x_1, M]$$



A_j - the singular point over x_j.
(k_j, l_j) - the Lefschetz pair of A_j.
$< k_j, l_j >$ - the skeleton in $(\tilde{D}, \tilde{K}, (k_j, k_j + 1, \cdots, l_j - 1, l_j))$ representing local $\mathcal{L}$.V.C. of A_j.
γ'_j be the part of γ_j from $x'_j = x_j + \epsilon$ to M.

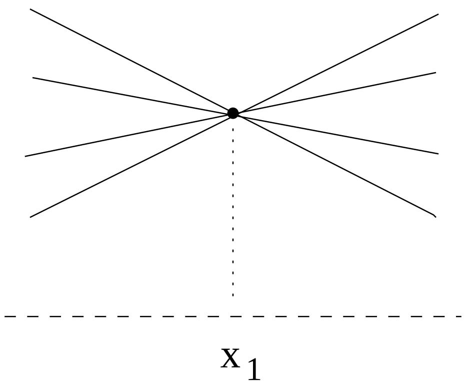
Then,

$$\mathcal{L}.\text{V.C.}(\hat{\gamma}'_j) = \mathcal{L}.\text{V.C.}(\gamma'_j) = \beta_M^{-1}(< k_j, l_j > \cdot \prod_{m=j-1}^{1} \Delta < k_m, l_m >)$$

and

$$\mathcal{L}.\text{V.C.}(\hat{\gamma}'_1) = \mathcal{L}.\text{V.C.}(\gamma'_1) = \beta_M^{-1}(< k_1, l_1 >)$$

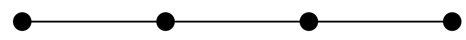
Example 1:



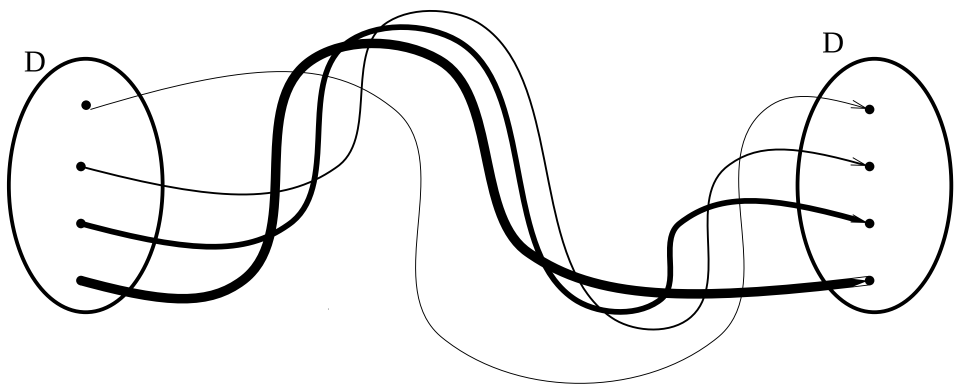
The Lefschetz pair:

j	λ_{x_j}
1	$(1,4)$

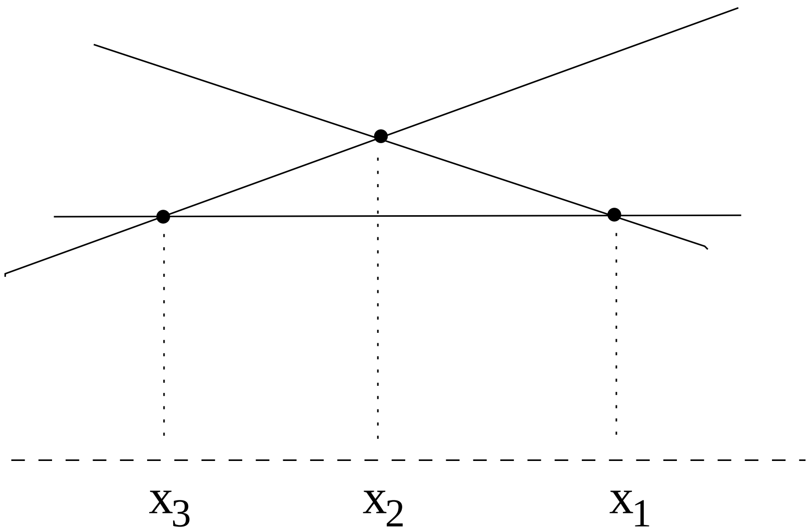
Therefore, the skeleton of the braid is:



Therefore, the braid is:



Example 2:



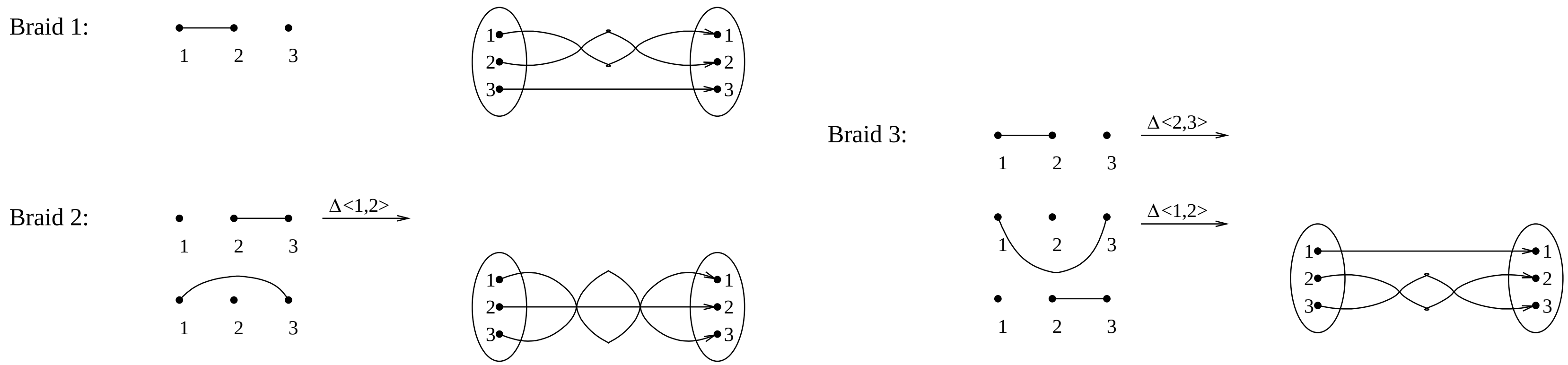
The Lefschetz pairs:

j	λ_{x_j}
1	$(1,2)$
2	$(2,3)$
3	$(1,2)$

The Lefschetz pairs:

j	λ_{x_j}
1	$(1,2)$
2	$(2,3)$
3	$(1,2)$

Therefore, the braids are:



van Kampen's Theorem

$S \in \mathbb{C}^2$ ($p = \text{deg}(S)$), $\pi = \pi_1 : \mathbb{C}^2 \to \mathbb{C}$ canonical projection.
$\mathbb{C}_x = \pi^{-1}(x)$, $K_x = \mathbb{C}_x \cap S$, $N = \{x \mid \#K_x < p\} = \{c_i\}$.
Choose $u \in \mathbb{C}$, u real, such that $x \ll u$, $\forall x \in N$.
Define: $B_p = B_p[\mathbb{C}_u, \mathbb{C}_u \cap S]$.
$\varphi_u : \pi_1(\mathbb{C} - N, u) \to B_p$ be the braid monodromy of S w.r.t π, u.
Choose $u_0 \in \mathbb{C}_u$, $u_0 \notin S$.
There exists an epimorphism $\pi_1(\mathbb{C}_u - S, u_0) \to \pi_1(\mathbb{C}^2 - S, u_0)$.

van Kampen (classic): Let $\{\delta_i\}$ be a g-base of $\pi_1(\mathbb{C} - N, u)$.
Let $\{\Gamma_j \mid 1 \leq j \leq p\}$ ($p = \text{deg}(S)$) be a g-base for $\pi_1(\mathbb{C}_u - S, u_0)$.
Then, $\pi_1(\mathbb{C}^2 - S, u_0)$ is generated by the images of Γ_j in $\pi_1(\mathbb{C}^2 - S, u_0)$ and the set of relations is:

$$(\varphi_u(\delta_i))(\Gamma_j) = \Gamma_j; \forall i \forall j$$

van Kampen's theorem for cuspidal curves

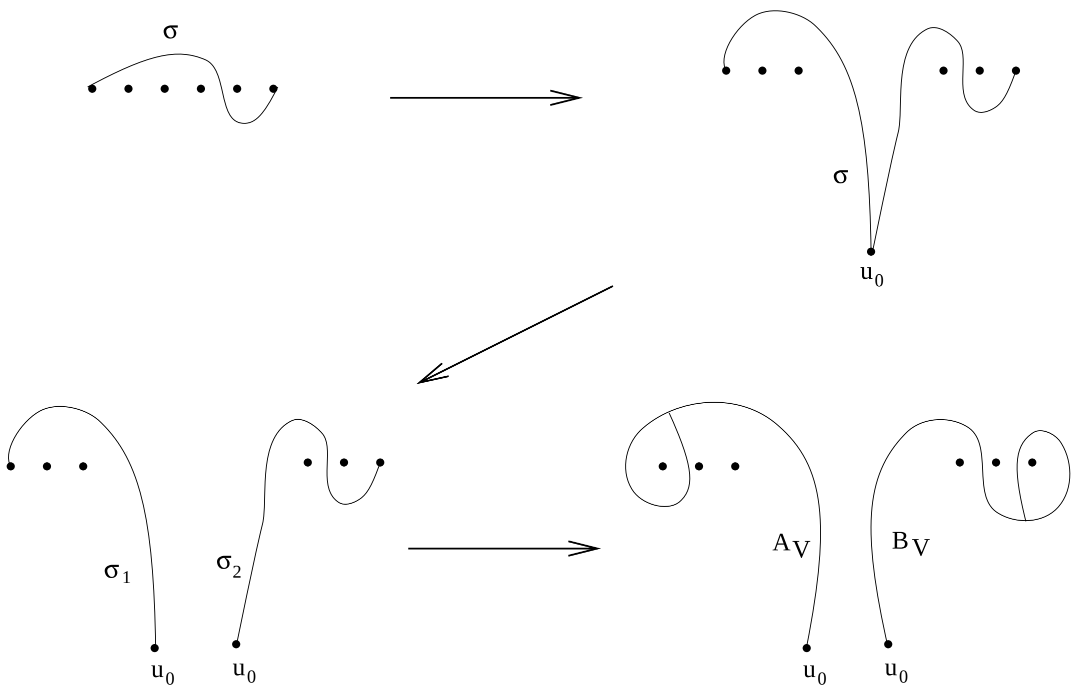
S is an cuspidal curve, $u, u_0, \varphi_u, A_{V_i}, B_{V_i}$.
Let $\{\delta_i\}$ be a g-base of $\pi_1(\mathbb{C} - N, u)$.
Let $\varphi_u(\delta_i) = V_i^{\nu_i}$, V_i is a half-twist, $\nu_i = 1, 2, 3$.
Let $\{\Gamma_j \mid 1 \leq j \leq p\}$ $(p = degS)$ be a g-base for $\pi_1(\mathbb{C}_u - S, u_0)$.

Then, $\pi_1(\mathbb{C}^2 - S, u_0)$ is generated by the images of Γ_j in $\pi_1(\mathbb{C}^2 - S, u_0)$ and the induced relations are

(a) $A_{V_i} = B_{V_i}$, when $\nu_i = 1$.

(b) $[A_{V_i}, B_{V_i}] = 1$, when $\nu_i = 2$.

(c) $< A_{V_i}, B_{V_i} >= 1$, when $\nu_i = 3$.


σ
σ
u0
σ1
σ2
u0
u0
AV
BV
u0
u0

van Kampen's theorem for a single multiple point:

$$\bigcap_{i=1}^{k} \ell_i = \{p\}$$

$\{\delta\}$ - a loop in $\pi_1(E - N, u_0)$ around $x(p)$.

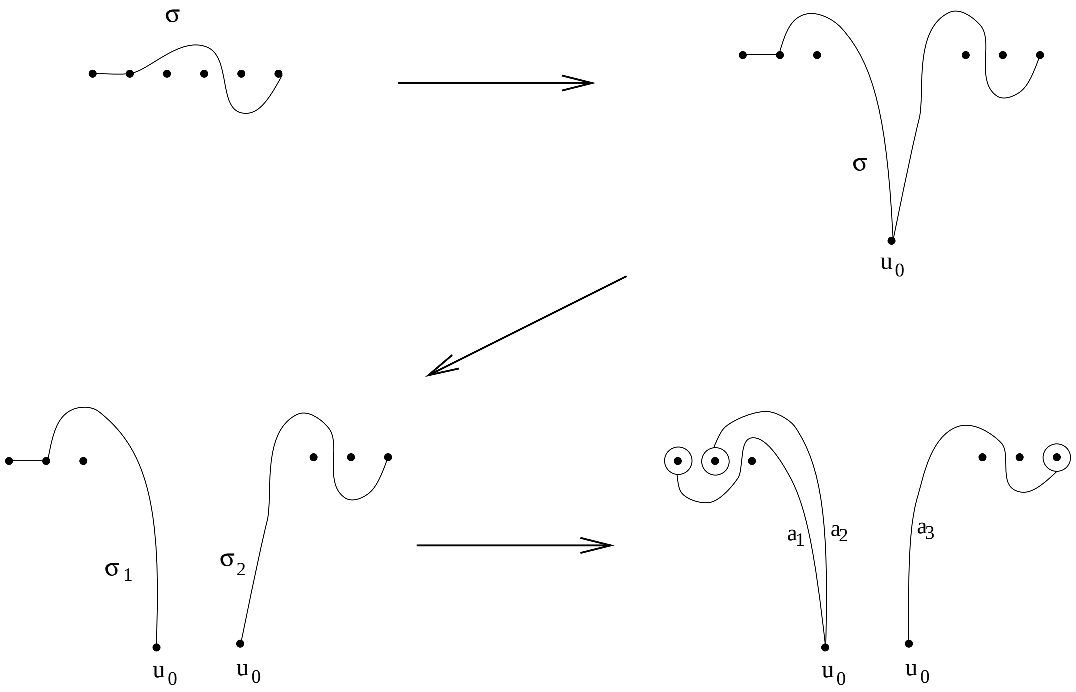
Let $\{\Gamma_1, \cdots, \Gamma_k\}$ be a g-base of $\pi_1\left(\mathbb{C}_{u_0} - \bigcup_{i=1}^{k} \ell_i\right)$.

Then, the relations which are induced from this intersection point are:

$$\Gamma_k \Gamma_{k-1} \cdots \Gamma_1 = \Gamma_1 \Gamma_k \cdots \Gamma_3 \Gamma_2 = \cdots = \Gamma_{k-1} \Gamma_{k-2} \cdots \Gamma_1 \Gamma_k$$

Claim: This set is equivalent to:

$$[\Gamma_k \Gamma_{k-1} \cdots \Gamma_1, \Gamma_i] = 1, 1 \leq i \leq k$$


σ
σ
u0
σ1
σ2
u0
u0
a1
a2
a3
u0
u0

Summary of the method

1. Calculation of the braid monodromy of $\mathcal{L}$:

   - Lefschetz pairs.

   - Calculate the Lefschetz vanishing cycle.

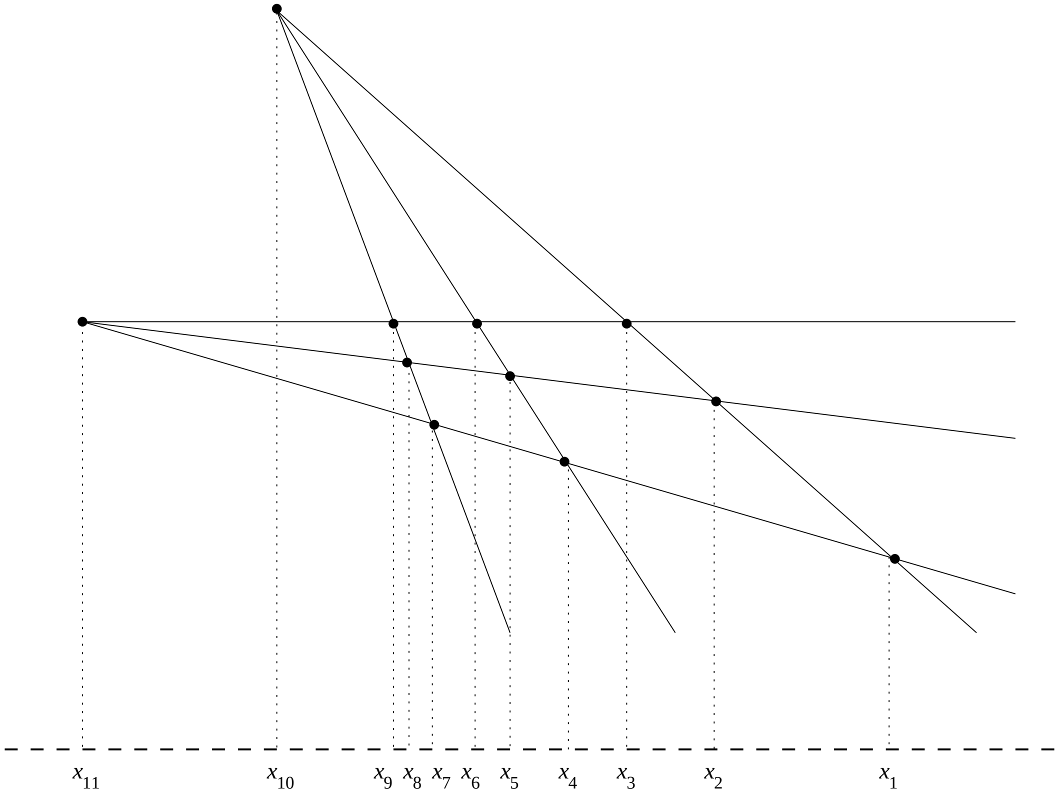
2. Calculation of the relations induced on $\pi_1(\mathbb{C}^2 - \mathcal{L})$

   - Calculate the A_{V_i}, B_{V_i} for every singular point.

   - Find the induced relations.

3. Computing a simplified presentation of $\pi_1(\mathbb{C}^2 - \mathcal{L})$.

A complete example

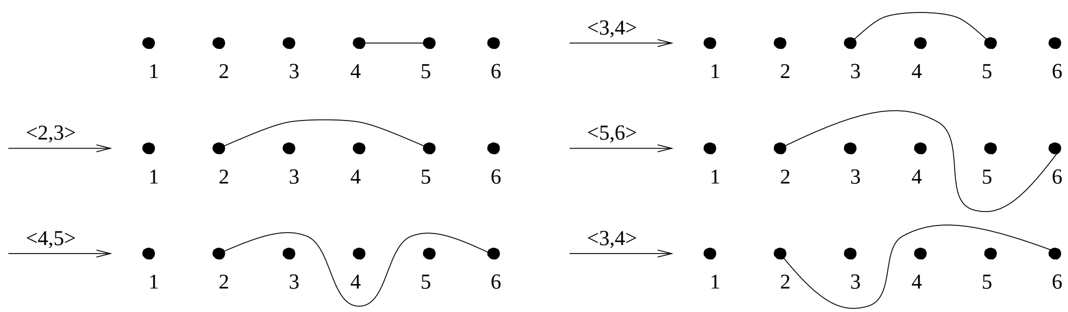


First step: Computing the Lefschetz pairs.

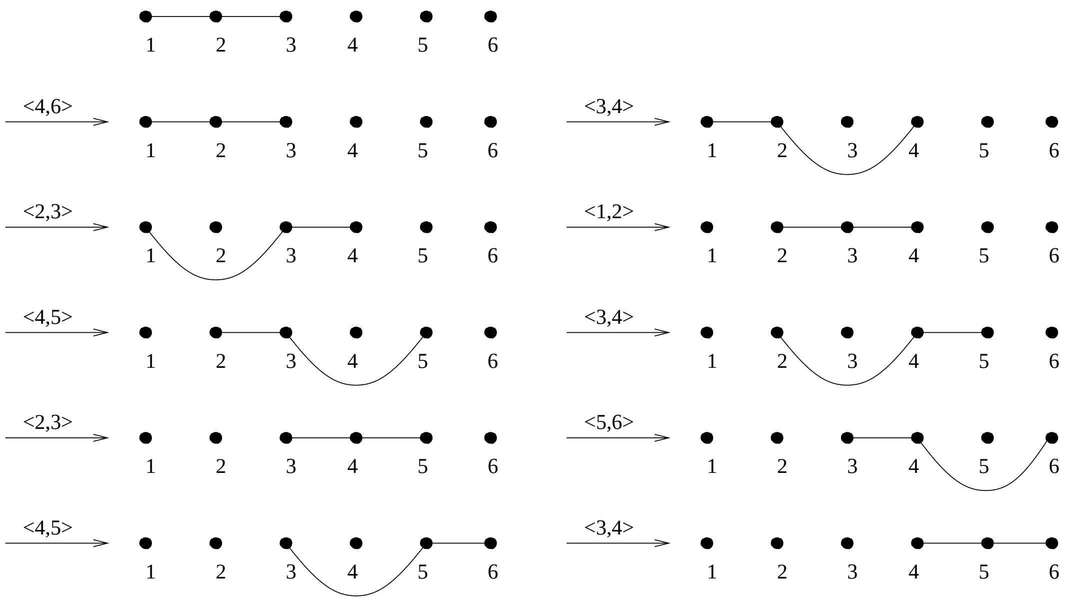
j	λ_{x_j}
1	$(3,4)$
2	$(4,5)$
3	$(5,6)$
4	$(2,3)$
5	$(3,4)$
6	$(4,5)$
7	$(1,2)$
8	$(2,3)$
9	$(3,4)$
10	$(4,6)$
11	$(1,3)$

Second step: Calculations of its braid monodromy.

$\varphi(\delta_6)$**'s skeleton**:

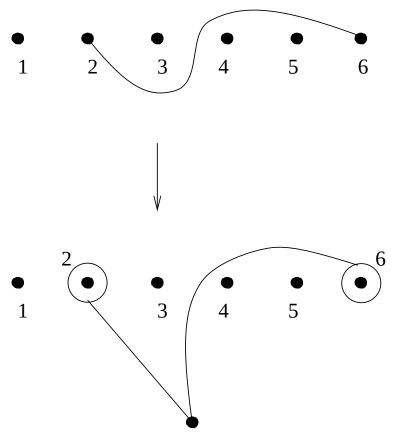


$\varphi(\delta_{11})$**'s skeleton**:



Third step: The induced van Kampen's relations.

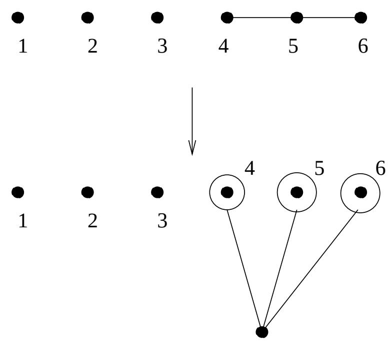
The relation which is induced from $\varphi(\delta_6)$:



Therefore, the relation is:

$$\Gamma_2 \cdot \Gamma_4^{-1}\Gamma_5^{-1}\Gamma_6\Gamma_5\Gamma_4 = \Gamma_4^{-1}\Gamma_5^{-1}\Gamma_6\Gamma_5\Gamma_4 \cdot \Gamma_2$$

The relations which are induced from $\varphi(\delta_{11})$:



Therefore, the relations are:

$$\Gamma_6 \cdot \Gamma_5 \cdot \Gamma_4 = \Gamma_5 \cdot \Gamma_4 \cdot \Gamma_6 = \Gamma_4 \cdot \Gamma_6 \cdot \Gamma_5$$

Summary of the relations:

(1) $\Gamma_3\Gamma_4 = \Gamma_4\Gamma_3$

(2) $\Gamma_3\Gamma_4^{-1}\Gamma_5\Gamma_4 = \Gamma_4^{-1}\Gamma_5\Gamma_4\Gamma_3$

(3) $\Gamma_3\Gamma_4^{-1}\Gamma_5^{-1}\Gamma_6\Gamma_5\Gamma_4 = \Gamma_4^{-1}\Gamma_5^{-1}\Gamma_6\Gamma_5\Gamma_4\Gamma_3$

(4) $\Gamma_2\Gamma_4 = \Gamma_4\Gamma_2$

(5) $\Gamma_2\Gamma_4^{-1}\Gamma_5\Gamma_4 = \Gamma_4^{-1}\Gamma_5\Gamma_4\Gamma_2$

(6) $\Gamma_2\Gamma_4^{-1}\Gamma_5^{-1}\Gamma_6\Gamma_5\Gamma_4 = \Gamma_4^{-1}\Gamma_5^{-1}\Gamma_6\Gamma_5\Gamma_4\Gamma_2$

(7) $\Gamma_1\Gamma_4 = \Gamma_4\Gamma_1$

(8) $\Gamma_1\Gamma_4^{-1}\Gamma_5\Gamma_4 = \Gamma_4^{-1}\Gamma_5\Gamma_4\Gamma_1$

(9) $\Gamma_1\Gamma_4^{-1}\Gamma_5^{-1}\Gamma_6\Gamma_5\Gamma_4 = \Gamma_4^{-1}\Gamma_5^{-1}\Gamma_6\Gamma_5\Gamma_4\Gamma_1$

(10) $\Gamma_3\Gamma_2\Gamma_1 = \Gamma_2\Gamma_1\Gamma_3 = \Gamma_1\Gamma_3\Gamma_2$

(11) $\Gamma_6\Gamma_5\Gamma_4 = \Gamma_5\Gamma_4\Gamma_6 = \Gamma_4\Gamma_6\Gamma_5$

Fourth step: Simplification of the relations. We get:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \mathbb{F}^2 \oplus \mathbb{F}^2 \oplus \mathbb{Z}^2$$

Moishezon-Teicher's algorithm for braid monodromy of a plane curve

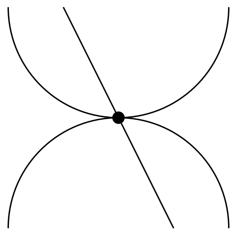
Changes from arrangements:

- More types of singular points - need to compute first the corresponding local braid monodromy.

- Complex points: a different model for computation.

First example

Local equation: $(2x + y)(y + x^2)(y - x^2) = 0$.

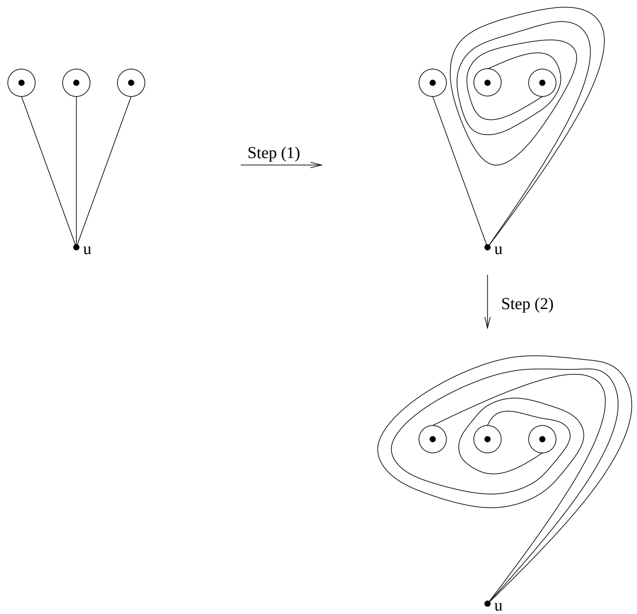


Take a loop $x = e^{2\pi i t}$ around $x = 0$ and look at the fibers:

- $t = 0$: means $x = 1$ and the points over $x = 1$ are $y = 1, -1, -2$.
- $t = \frac{1}{2}$: means $x = -1$ and the points over $x = -1$ are $y = -1, 1, 2$.

Hence, from $t = 0$ to $t = \frac{1}{2}$, the points $y = 1, -1$ do a counter-clockwise full-twist and the point $y = -2$ does a counterclockwise half-twist around the points $y = 1, -1$.

By van-Kampen Theorem:



The induced relations of this point are:

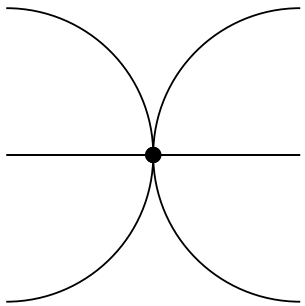
$$x_1x_3x_2 = x_3x_2x_1$$

$$x_3x_2x_1x_3x_2 = x_2x_3x_2x_1x_3$$

where $\{x_1, x_2, x_3\}$ are the generators of the standard g-base.

Second example

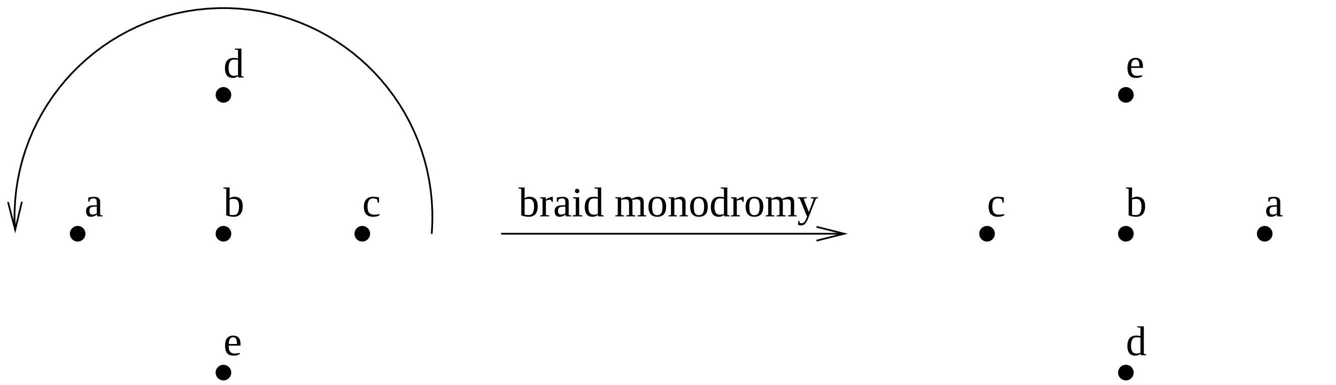
Local equation: $y(y^2+x)(y^2-x)=0$



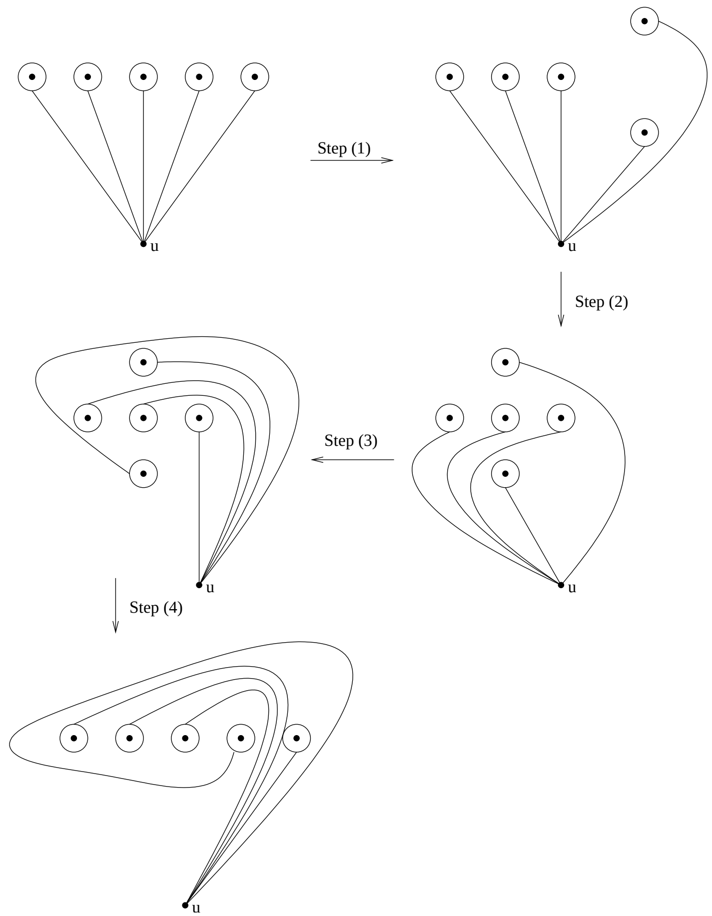
Take a loop $x = e^{2\pi it}$ around $x = 0$ and look at the fibers:

- $t = 0$: means $x = 1$, the points over $x = 1$ are $y = -1, 0, 1, i, -i$.
- $t = \frac{1}{2}$, $x = -1$, the points over $x = -1$ are again $y = -1, 0, 1, i, -i$.

By checking, the action of local braid monodromy is a 180° rotation counterclockwise of the four points around the central point:



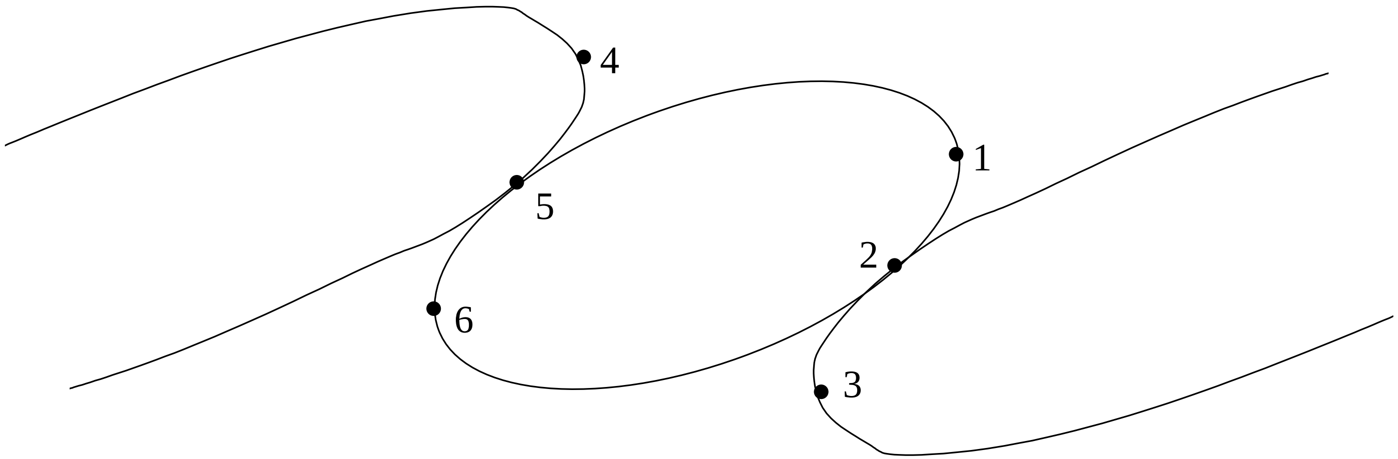
By van-Kampen Theorem:



The induced relations:

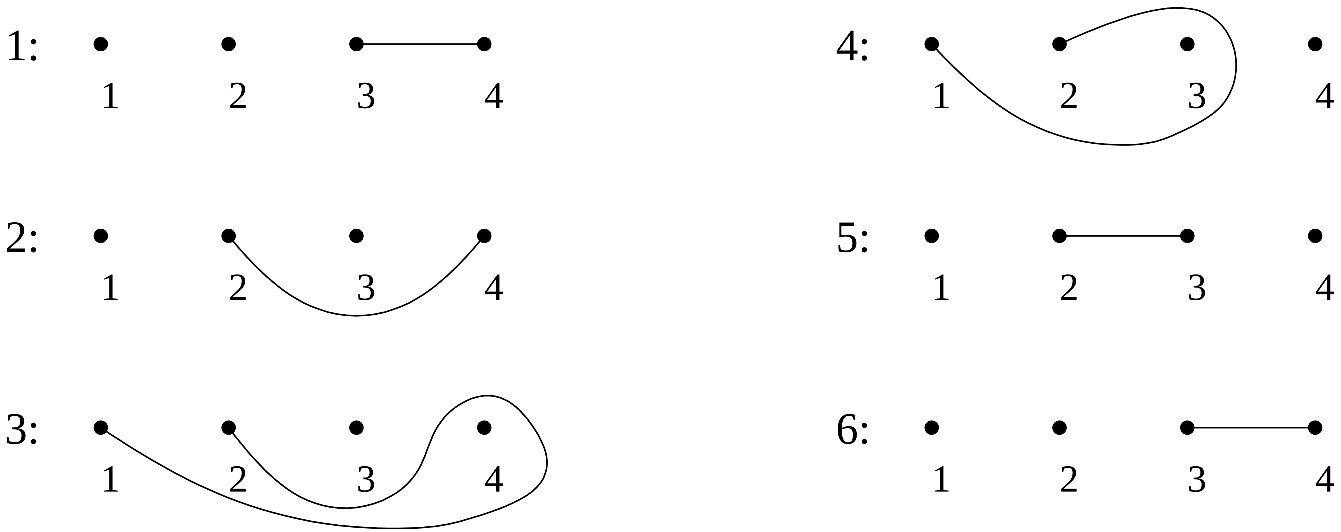
1. $x_4 x_3 x_2 = x_2 x_4 x_3$
2. $x_3 x_2 x_4 x_3 x_4 = x_4 x_3 x_2 x_4 x_3$
3. $x_1 = x_4 x_3 x_4^{-1}$
4. $x_4 = x_5$

Example for computing a whole curve



j	λ_{x_j}	ϵ_{x_j}	δ_{x_j}
1	P_3	1	$\Delta^{\frac{1}{2}}_{\mathbb{R}I_2} < 3 >$
2	$< 2,3 >$	4	$\Delta^2 < 2,3 >$
3	$< 1,2 >$	1	$\Delta^{\frac{1}{2}}_{I_2\mathbb{R}} < 1 >$
4	P_3	1	$\Delta^{\frac{1}{2}}_{\mathbb{R}I_2} < 3 >$
5	$< 2,3 >$	4	$\Delta^2 < 2,3 >$
6	$< 1,2 >$	1	$\Delta^{\frac{1}{2}}_{I_2\mathbb{R}} < 1 >$

By the Moishezon-Teicher algorithm:



Presentation of the group by van-Kampen Theorem

Generators: $\{x_1, x_2, x_3, x_4\}$.

Relations:

1. $x_4 x_3 x_2 x_1 = e$ (projective relation)
2. $x_3 = x_4$
3. $(x_2 x_4)^2 = (x_4 x_2)^2$
4. $x_1 = x_4 x_2 x_4^{-1}$
5. $x_1 = x_3 x_2 x_3^{-1}$
6. $(x_2 x_3)^2 = (x_3 x_2)^2$
7. $x_3 = x_4$

By simplifications, we get:

$$\pi_1(\mathbb{CP}^2 - C) \cong \langle x_1, x_2 \mid (x_1 x_2)^2 = (x_2 x_1)^2 = e \rangle$$