

Fundamental groups of complements of
Arrangements and plane curves - Part II

Singapore, December 2008

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Known results on fundamental groups of complements of arrangements

Structure of the group

Zariski, Deligne-Fulton: If \mathcal{L} is a union of n lines in general position, then

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \mathbb{Z}^n.$$

Dethloff-Orevkov-Zaidenberg (1998): If \mathcal{L} is a union of n lines not in general position, then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ is big, i.e. $\mathbb{F}_2 \leq \pi_1(\mathbb{C}^2 - \mathcal{L})$.

Randell (1982): Let $L_i, 1 \leq i \leq n$, be lines such that $\bigcap_{i=1}^n L_i = \{\text{pt}\}$.

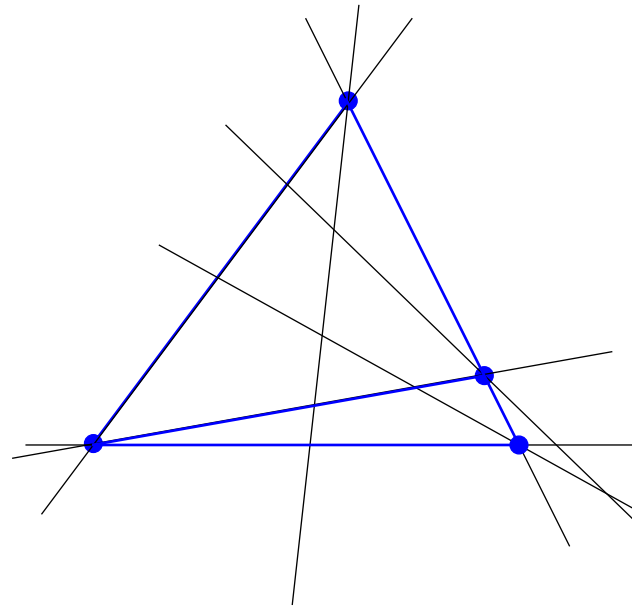
Then: $\pi_1(\mathbb{C}^2 - \bigcup_{i=1}^n L_i) = \mathbb{Z} \oplus \mathbb{F}_{n-1}$.

Graph of multiple points

$G(\mathcal{L})$:

Vertices: Multiple points

Edges: Segments on lines with more than two multiple points.



[Fan \(1997\)](#): Let \mathcal{L} be an arrangement of n lines and $S = \{a_1, \dots, a_p\}$ be the set of multiple points of \mathcal{L} (multiplicity ≥ 3). Suppose that $\beta(\mathcal{L}) = 0$ (i.e. the graph $G(\mathcal{L})$ has no cycles). Then:

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^r \oplus \mathbb{F}_{m(a_1)-1} \oplus \dots \oplus \mathbb{F}_{m(a_p)-1}$$

where $r = n + p - 1 - m(a_1) - \dots - m(a_p)$.

[G-Teicher](#): Part of this result by braid monodromy techniques.

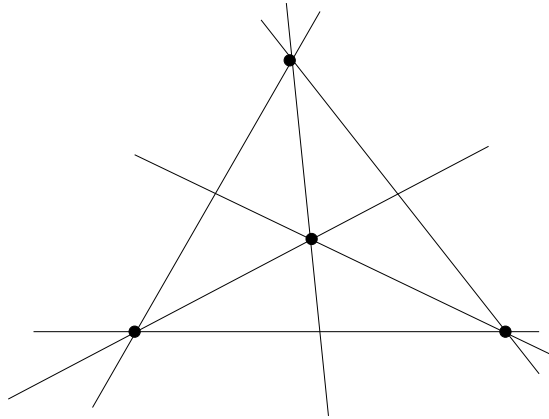
[Eliyahu-Liberman-Schaps-Teicher \(2008\)](#): If the fundamental group is a sum of free graphs, then $G(\mathcal{L})$ has no cycles.

Liberman: Let \mathcal{L} be a real line arrangement with n lines, and let $G = \pi_1(\mathbb{C}^2 - \mathcal{L})$. Let $\Gamma_1, \dots, \Gamma_n$ be the topological generators of G . $\tilde{G} \cong \langle \Gamma_2, \dots, \Gamma_n \rangle$. Then:

$$G = \tilde{G} \rtimes \mathbb{Z}$$

G (2005): Under some conditions,

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) = \mathbb{Z} \oplus \pi_1(\mathbb{CP}^2 - \mathcal{L})$$



Suciu: If \mathcal{L} is the Ceva arrangement, then:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \mathbb{F} \times \mathbb{F}_2 \times \mathbb{F}_3$$

Eliyahu-G-Teicher (2008): If the graph of \mathcal{L} is a cycle of length 3, then:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \mathbb{F} \times \mathbb{F}_2 \times (\mathbb{Z} \star \mathbb{Z}^2).$$

Question: Can one generalize it to a cycle of length n ?

Conjugation-free geometric presentation of fundamental group

A *conjugation-free geometric presentation* of a fundamental group is a presentation with the natural topological generators x_1, \dots, x_n and the cyclic relations:

$$x_{i_k} x_{i_{k-1}} \cdots x_{i_1} = x_{i_{k-1}} \cdots x_{i_1} x_{i_k} = \cdots = x_{i_1} x_{i_k} \cdots x_{i_2}$$

with no conjugations on the generators.

Importance: One can read this presentation directly from the arrangement.

Eliyahu-G-Teicher (2008): if $G(\mathcal{L})$ is a cycle of any length, then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ has a conjugation-free geometric presentation.

If \mathcal{L} is the Ceva arrangement, then its fundamental group has no conjugation-free geometric presentation.

Conjecture (Eliyahu-G-Teicher, 2008): If $G(\mathcal{L})$ is a *K_4 -free graph*, then $\pi_1(\mathbb{C}^2 - \mathcal{L})$ has a conjugation-free geometric presentation.

Torsion-free fundamental groups

Question (Falk-Randell, 2000): Are all arrangement groups torsion-free?

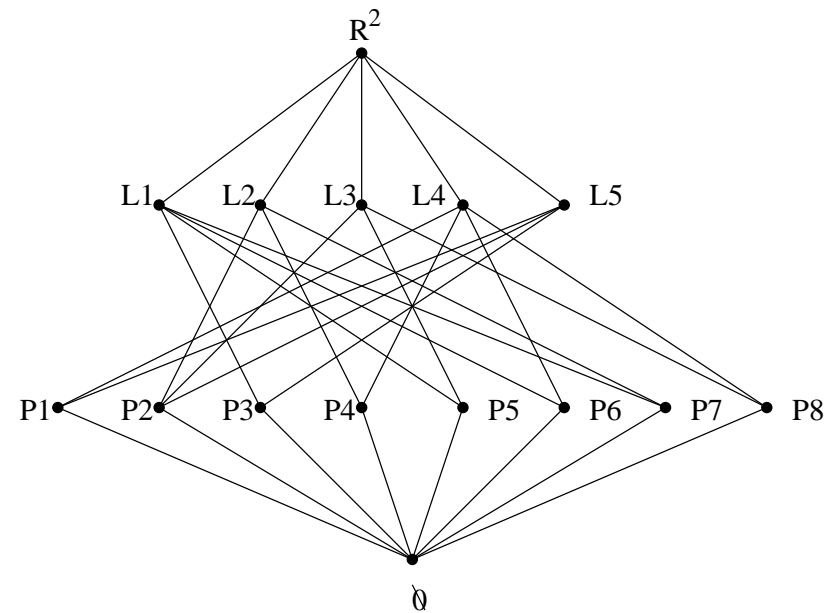
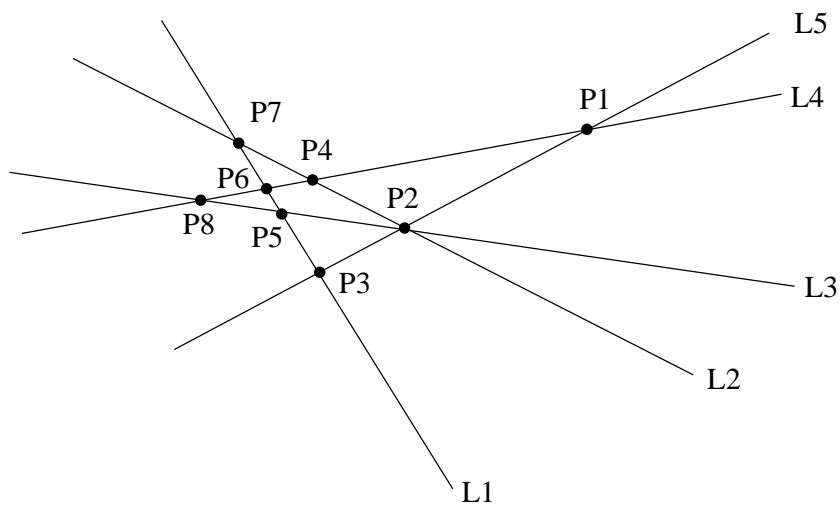
For some families (real reflection, fiber-type): Yes. (Ceva arrangement is a fiber-type arrangement.)

Falk-Randell's suggested approach: Show that all arrangement groups are *orderable*. (a group G is *orderable* if there is a linear order $<$ on G so that $g < h$ implies $cg < ch$ for all $c \in G$.) An orderable group has no torsion.

Examples: Braid groups are orderable (Dehornoy, 1994). Groups of fiber-type arrangements are orderable (Paris).

Connection between invariants of arrangements

Combinatorics vs. Topology: does the lattice determine the fundamental group?



Computation of a lattice

Rybnikov (1994): Example of complex arrangements with 13 lines, with the same lattice but different fundamental groups of the complement.

Idea: The example is constructed by two different gluings of a pair of two MacLane configurations (8 lines, 8 triple points, 4 nodes).

The two groups are non-isomorphic since their G/G_3 are different.

Question: Can Rybnikov's example be generalized?

Nice arrangements (Jiang-Yau)

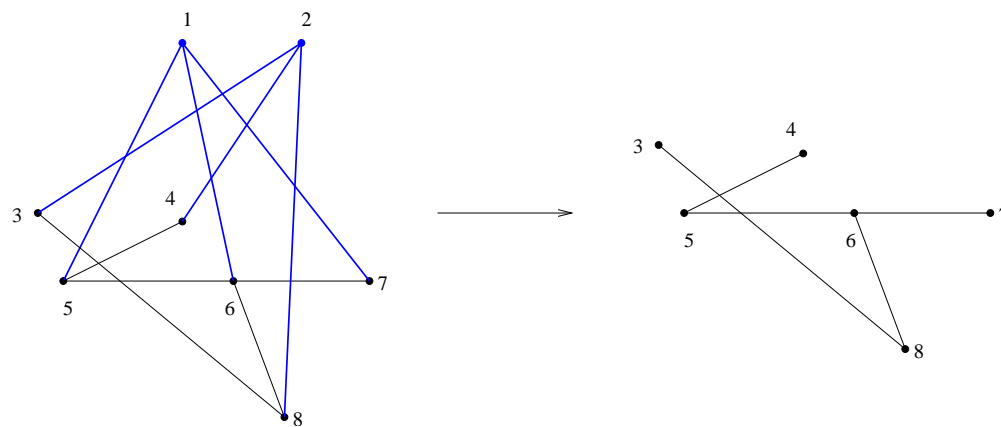
For \mathcal{L} , define a graph $G(V, E)$: The vertices are the multiple points of \mathcal{L} .

u, v are connected if there exists $\ell \in \mathcal{L}$ such that $u, v \in \ell$

For $v \in V$, define a subgraph $G_{\mathcal{L}}(v)$: The vertex set is v and all his neighbors from G .

u, v are connected if there exists $\ell \in \mathcal{L}$ such that $u, v \in \ell$

\mathcal{L} is *nice* if there is $V' \subset V$ such that $G_{\mathcal{L}}(v) \cap G_{\mathcal{L}}(u) = \emptyset$ for all $u, v \in V'$, and if we delete the vertex v and the edges of its subgraph $G_{\mathcal{L}}(v)$ from G , for all $v \in V'$, we get a forest (a graph without cycles).



Jiang-Yau (1994): Let \mathcal{L}_1 and \mathcal{L}_2 be two nice projective arrangements in $\mathbb{C}\mathbb{P}^2$. If their lattices are isomorphic, then their complements are diffeomorphic. In particular,

$$\pi_1(\mathbb{C}\mathbb{P}^2 - \mathcal{L}_1) \cong \pi_1(\mathbb{C}\mathbb{P}^2 - \mathcal{L}_2)$$

Simple arrangements (Wang-Yau)

A *net* in a graph: Let B be a circle and let C be a circle or a vertex, a *net of B and C* , $\text{Net}(B, C)$, is a subgraph of G :

Vertices: $V^{\text{Net}(B, C)} = V^B \cup V^C \cup N$,

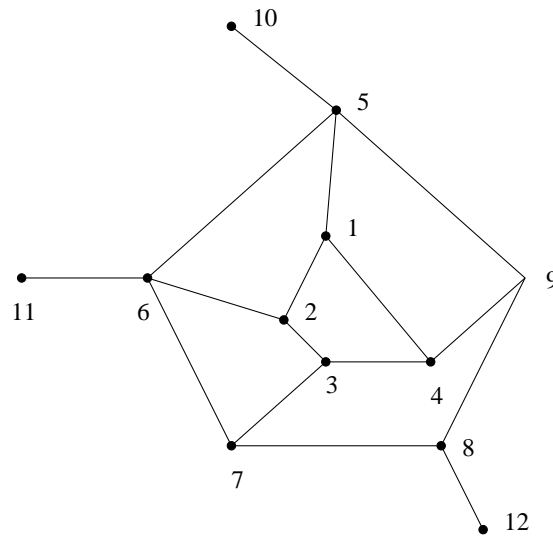
where $N = \{v \in V^G - (V^B \cup V^C) \mid \exists! u \in V^B \cup V^C, (u, v) \in E^G\}$.

Edges: $\{(v, u) \in E^G \mid v, u \in V^{\text{Net}(B, C)}\}$.

C , B and N are called *center*, *base* and *node* of the net respectively.

Simple net: $\text{Net}(B, C)$ is a simple net, if any two non-adjacent vertices of center C or two non-adjacent vertices of base B are not connected by an edge, any two vertices of center C do not connect a same vertex of the base B by two edges, moreover, any two vertices of net do not connect a same vertex in $V^G - V^{\text{Net}(B, C)}$ by two edges.

Free vertex: A vertex $v \in V^B$ is called *free* in G if there is no edge connecting v and C . A simple net with free vertex is called *free simple net*.



$$C = \{1, 2, 3, 4\}, \quad B = \{5, 6, 7, 8, 9\}, \quad 8 \text{ free}$$

Simple arrangement: An arrangement \mathcal{L} in $\mathbb{C}\mathbb{P}^2$ is called *simple* if the graph G has the following property: There are finitely many stars and free simple nets, which are pairwise disjoint in G and if we delete them, we get a forest.

Clearly, a nice graph is simple.

Wang-Yau (2005): Let \mathcal{L}_1 and \mathcal{L}_2 be two simple arrangements in C^3 and \mathcal{L}_1^* and \mathcal{L}_2^* be the corresponding projective arrangements in $\mathbb{C}\mathbb{P}^2$. If the lattices of \mathcal{L}_1 and \mathcal{L}_2 are isomorphic, then the complements of \mathcal{L}_1^* and \mathcal{L}_2^* are diffeomorphic. In fact, the moduli space of simple arrangements with fixed lattice is connected.

The real case

What is about real case?

Fan (1997): Up to 6 lines, the lattice determines the fundamental group

Cordovil (1998): Introduction of a new order *shellability order* on the lines of an arrangement.

If there is a bijection of two arrangements which induces an isomorphism of the associated lattices that preserves the shellability order, then their *affine fundamental groups* are isomorphic.

G-Teicher-Vishne (2002-3): Up to 8 lines, the lattice determines the fundamental group.

Idea of proof: classification of arrangements to equivalence classes.

[Artal-Bartolo, Carmona, Cogolludo, Marco \(2003\)](#): Existence of complexified real arrangements (with 10 lines) with the same combinatorics but different embeddings in \mathbb{P}^2 (i.e. the pairs $(\mathbb{P}^2, \mathcal{L}_1)$ and $(\mathbb{P}^2, \mathcal{L}_2)$ are not homeomorphic).

But: $\pi_1(\mathbb{P}^2 - \mathcal{L}_1)$ and $\pi_1(\mathbb{P}^2 - \mathcal{L}_2)$ have isomorphic profinite completion, isomorphism between $\pi_1(\mathbb{P}^2 - \mathcal{L}_1)$, $\pi_1(\mathbb{P}^2 - \mathcal{L}_2)$ is not known.

[Remark \(Cohen-Suciu, 1997\)](#): The braid monodromy of a curve is more closely tied to the homeomorphism type of the complement than to the fundamental group of the complement.

Conjecture: For the real case, the lattice does determine the fundamental group.

What is about the other direction: Topology \Rightarrow Combinatorics?

Falk (1993): An example of two arrangements (with 6 lines, 2 triple points and 9 double points) with non-isomorphic lattices, but with isomorphic fundamental group of the complement.

Jiang-Yau (1998): Let \mathcal{L}_1 and \mathcal{L}_2 be two projective arrangements in $\mathbb{C}\mathbb{P}^2$. If their complements are homeomorphic, then their lattices are isomorphic.

Corollary: There are two projective arrangements whose complements have the same homotopy type, but not the same topological type.

Results on conic-line arrangements

Amram-Teicher-Uludag (2003): Presentation of families:

- (1) a smooth quadric Q with n tangent lines to Q .
- (2) a quadric Q with n lines passing through a point $p \notin Q$.
- (3) a quadric Q with n lines passing through a point $p \in Q$.

Amram-Teicher (2006): Presentation of families:

- (1) a union of n quadrics, which are tangent to each other at two common points.
- (2) n quadrics which are tangent to each other at one common point.
- (3) n quadrics, $n - 1$ of them are tangent to the n th one and each one of the $n - 1$ quadrics is transversal to the other $n - 2$ ones.

[Amram-G-Teicher \(2007\)](#): Presentations (and structure) of arrangements with two tangent conics which are tangent at two points and up to two additional lines in any position

[Amram-G-Teicher \(2008\)](#): Presentations of arrangements with two tangent conics which are tangent at two points and n additional lines tangent to the conics.

Idea: Computing the braid monodromy of the arrangement by induction.

Known results on fundamental groups of complements of plane curves

Structure and computations of the group

Zariski (1929): If C is an irreducible smooth curve of degree d , then $\pi_1(\mathbb{CP}^2 - C) \cong \mathbb{Z}/d\mathbb{Z}$.

Zariski (1929): If C is a nodal curve, then $\pi_1(\mathbb{CP}^2 - C)$ is abelian. (proved finally by Deligne-Fulton, 1980)

Degtyarev: Systematic study of fundamental groups of complements of curves of degree up to 5.

Curves of degree 6: Artal, Carmona, Cogolludo, Degtyarev, Eyrat, Oka, Tokunaga and more.

Oka (1975): Let $C_{p,q}$ be the projective curve of degree pq :

$$(X^p + Y^p)^q + (Y^q + Z^q)^p = 0.$$

Then:

(1) $C_{p,q}$ has pq cusps.

(2) $\pi_1(\mathbb{CP}^2 - C_{p,q}) \cong \mathbb{Z}_p \star \mathbb{Z}_q$.

(3) The commutator group of $\pi_1(\mathbb{CP}^2 - C_{p,q})$ is a free group of rank $(p-1)(q-1)$.

Oka (1975): Let D_{2q} be the following curve $X^{2q-1}Y + (Y^q + Z^q)^2 = 0$. Then, $\pi_1(\mathbb{CP}^2 - D_{2q}) \cong \mathbb{Z}_{2q}$.

Remark: These curves are generalizations of the curves in Zariski's original pair.

Oka (1976): Let $C = C_1 \cup C_2 \cup \dots \cup C_r$ be a curve such that its irreducible components $\{C_j\}$ are in general position. Then, $\pi_1(\mathbb{CP}^2 - C)$ is abelian if and only if $\pi_1(\mathbb{CP}^2 - C_j)$ is abelian for each $1 \leq j \leq r$.

Oka-Sakamoto (1978): Let C_1 and C_2 be algebraic plane curves in \mathbb{C}^2 . Assume that the intersection $C_1 \cap C_2$ consists of distinct $d_1 \cdot d_2$ points, where d_i ($i = 1, 2$) are the respective degrees of C_1 and C_2 . Then:

$$\pi_1(\mathbb{C}^2 - (C_1 \cup C_2)) \cong \pi_1(\mathbb{C}^2 - C_1) \oplus \pi_1(\mathbb{C}^2 - C_2)$$

Combinatorics vs. Topology of curves: Zariski pairs

Zariski pair: A pair of curves with the same combinatorics but different topology of the complement.

Zariski: Two curves of degree 6 with 6 cusps. If the cusps are on a conic, then π_1 is $(\mathbb{Z}/2\mathbb{Z}) \star (\mathbb{Z}/3\mathbb{Z})$, otherwise, π_1 is abelian.

More families and examples of pairs and triples: E. Artal-Bartolo, J. Carmona-Ruber and J. I. Cogolludo Agustin; Tokunaga; Degtyarev; Eyral and Oka; Greuel, Lossen and Shustin; Ozguner; Shimada; Uludag; G.

Examples of Zariski k-ples: Artal-Bartolo & Tokunaga; Degtyarev.

Construction of new Zariski pairs from existing ones by using Cremona transformation: Uludag, G.

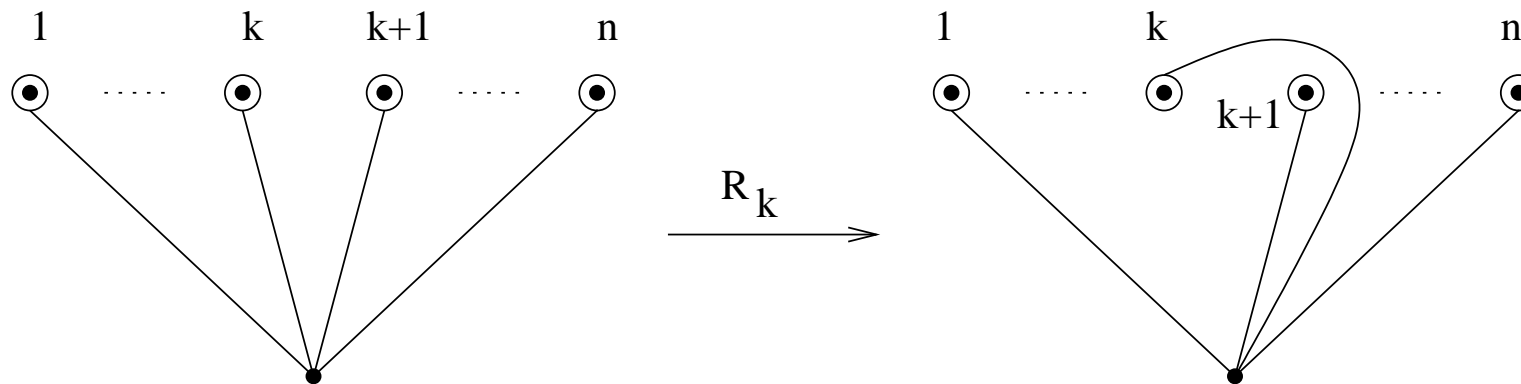
Application of the fundamental group to classification of surfaces

Artin: Let S be a curve transversal to the line in infinity, and φ is a braid monodromy of S , $\varphi : \pi \rightarrow B_m$. Let δ_i be a geometric (free) base (g-base) of π , and Δ^2 is the generator of $Z(B_m)$. Then:

$$\Delta^2 = \prod \varphi(\delta_i)$$

This product is also defined as *the braid monodromy factorization (BMF) related to a curve S* .

Definition: Let $\vec{t} = (t_1, \dots, t_m) \in G^m$. We say that $\vec{s} = (s_1, \dots, s_m) \in G^m$ is obtained from \vec{t} by *the Hurwitz move R_k* if $s_i = t_i$ for $i \neq k, k+1$, $s_k = t_{k+1}$, $s_{k+1} = t_{k+1}^{-1}t_k t_{k+1}$.



Definition: Let G be a group, $t \in G$. Let $t = t_1 \cdots t_m = s_1 \cdots s_m$ be two factorizations of t . We say that $s_1 \cdots s_m$ is obtained from $t_1 \cdots t_m$ by a Hurwitz move R_k if (s_1, \dots, s_m) is obtained from (t_1, \dots, t_m) by a Hurwitz move R_k .

Definition: Two factorizations are *Hurwitz equivalent* if they are obtained from each other by a finite sequence of Hurwitz moves.

Kulikov-Teicher (2000): If the Braid Monodromy Factorizations (BMF) of the branch loci of two surfaces X_1 and X_2 are Hurwitz-equivalent, then the surfaces are diffeomorphic. Moreover, if the factorizations are not Hurwitz-equivalent, then X_1 and X_2 are not projectively deformation equivalent.

Friedman-Teicher (2008): An example of two (polarized K3) surfaces whose fundamental groups of the complement of the branch curves are not isomorphic, but the fundamental groups of their related Galois covers are isomorphic.