Beyond p-compact groups

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joint work with Natàlia Castellana and Juan A. Crespo

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Theorem (Dwyer-Wilkerson)

The mod p cohomology $H^*(BX; \mathbb{F}_p)$ is Noetherian.

Completion

All spaces with the same mod p homology are identified with one preferred p-*complete* space $X_p^\wedge$.

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1. $K(\mathbb{Z}, n)_p^\wedge = K(\mathbb{Z}_p^\wedge, n) = K(\mathbb{Z}_{p^\infty}, n-1)_p^\wedge$,
2. $K(\mathbb{Z}/p, n)$ is p-complete,
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When X is a simply connected space of finite type, then $X_p^\wedge$ is a space in which the p-torsion is kept unchanged, the other torsion is eliminated, and copies of the integers are converted into copies of the p-adics.

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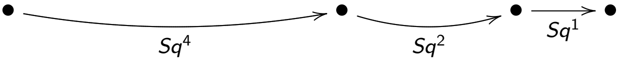
The space $BDI(4)$ was constructed by Dwyer and Wilkerson. Its cohomology realizes the Dickson invariants, whence its name.

$$H^*(BDI(4); \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, x_3, x_4]^{GL_4(2)} \cong \mathbb{F}_2[y_8, y_{12}, y_{14}, y_{15}]$$

The action of the mod 2 Steenrod algebra on the generators can be described as follows:



Cohomology of Eilenberg-Mac Lane spaces

Serre computed the cohomology of $K(\mathbb{Z}/2, n)$ and Cartan did the analogue for odd primes. For example

$$H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \cong \mathbb{F}_2[x_2, Sq^1 x_2, Sq^2 Sq^1 x_2, Sq^4 Sq^2 Sq^1 x_2, \dots]$$

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In particular this is finitely generated as an algebra over the Steenrod algebra. In fact the module of indecomposable elements is "very small":

This is the suspension of the free unstable module $F(1)$, generated by an element in degree 1.

The object of study

Definition

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Example

1. p-compact groups are p-Noetherian groups;
2. $K(\mathbb{Z},3)$ and $K(\mathbb{Z}/2,2)$ are p-Noetherian groups;
3. For any simply connected compact Lie group G, the p-completion of the 4-connected cover $BG\langle 4\rangle$ is a p-Noetherian group.

The structure of BX

Let us look at a concrete example: $(BS^3)\langle 4\rangle$. It is constructed as the homotopy fiber of $BS^3 \to K(\mathbb{Z}, 4)$. In other words we have a fibration

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For any p-Noetherian group we have a principal fibration

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Theorem

For any p-Noetherian group we have a principal fibration

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1. BY *is a p-compact group;*
2. $K \simeq K(\mathbb{Z}_p^\wedge, 3)^r \times K(P, 2)$ *and* P *is a finite abelian p-group.*

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This is not very precise. Why is it for example that only the Eilenberg-Mac Lane spaces $K(\mathbb{Z}, 3)$ and $K(\mathbb{Z}/p, 2)$ are p-Noetherian groups, but $K(\mathbb{Z}, n)$ and $K(\mathbb{Z}/p, n-1)$ are not for $n > 3$?

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A good way to measure the size of these cohomologies is to look at:

Definition

The module of indecomposable elements $QH^*(BX) = \tilde{H}^*(BX)/\tilde{H}^*(BX)\cdot\tilde{H}^*(BX)$.

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This is an unstable module over the Steenrod algebra.

The analysis of $QH^*(BX)$

There is a Krull type filtration introduced by Schwartz on the category $\mathcal{U}$ of unstable modules. It starts with $\mathcal{U}_0$ the subcategory of locally finite modules.

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Since $QH^*(K(\mathbb{Z}/2, n); \mathbb{F}_2) \cong \Sigma F(n-1)$, this explains that the cohomology of $K(\mathbb{Z}/2, n)$ is larger than that of $K(\mathbb{Z}/2, n-1)$.

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Theorem

The module $QH^*(BX; \mathbb{F}_p)$ belongs to $\mathcal{U}_1$.

Back to 4-connected covers

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There is always a map

$$QH^*((BG)\langle 4\rangle; \mathbb{F}_2) \to \Sigma F(1)$$

with finite kernel and finite cokernel.