On attractors derived from expanding maps

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Discrete dynamical system

Given closed manifold M and diffeomorphism f: M→M

Recurrence behaviour

$\Omega(f)$,the non-wandering set of f, the set of non-wandering point of f

- x non-wandering : if for any neighbourhood *U* of *x* and there are infinitely many positive integers *n*, such that the iterated map is intersecting:

$$f^n(U) \cap U \neq \emptyset$$

Axiom A

- Let M be a smooth manifold. We say that a diffeomorphism f:M→M satisfies (Smale's) Axiom A (or that f is an Axiom A diffeomorphism) if

  The nonwandering set Ω(f) has a hyperbolic structure;
  The set of periodic points of f is dense in Ω(f)

- A subset X of a M is said to be hyperbolic with respect to f, when TM_X may be split into two invariant subbundles, Df is contracting on one of them, and expanding on the another .

- A map f is called Anosov if M is hyperbolic with respect to f.

- A map f is called *expanding* if f is Anosov and Df is expanding on TM

Spectral Decomposition Theorem

For an Axiom A diffeomorphism f

$\Omega(f)$ can be decomposed into the union of so-called basic sets:

Group 0

Group A and Group DA

Group DE which are attractors derived from expanding maps.

DE attractors derived from expanding maps

Let M be a (p + q)-manifold , f : M →M be a diffeomorphism.

embedding N→M

(oriented)disk bundle D^q→N→X^p such that

f restricted to N is "equivalent" to a "fibre map" which shrinks the D^q factor by some constant 0 < t < 1

intersection over n>0 of f^n(N) is called (oriented) DE attractor of type (p, q) derived from expanding maps

Two maps f:X → X , g:Y → Y are called topologically conjugate if there exists a homeomorphism h:X→ Y such that g。h=h 。f

Smale Conjecture

If f:M→M is an Anosov diffeomorphism, then the non-wandering set Ω(*f*) =M

Negative answer for 3-manifolds by J. Franks; B. Williams

Negative answer for 2n-manifolds if Ω(*f*) consists of two attractors derived from expanding maps of type (k,k) with some transversality condition

Question

For which positive integer n, there is a diffeomorphism f: M → M on a closed n-manifold M with $\Omega(f)$ a union of finitely many attractors derived from expanding maps?

Structure of non-wandering set

Theorem(DPWY)

If there exists a diffeomorphism f: M → M on a closed n-manifold M such that Ω(*f*) consists of finitely many oriented attractors derived from expanding maps, then M is a rational homology sphere. Moreover all those attractors are of type (n-2, 2).

Structure of non-wandering set

Theorem(DPWY)

Let M be a closed n-manifold. There is no diffeomorphism f: M → M such that $\Omega(f)$ consists of finitely many oriented attractors derived from expanding maps, if

n=2k>4 and attractors are restricted to be of type (k, k).

expanding map on cohomology

Lemma(DPWY)

Let f: M → M be an expanding map on the closed manifold M, then the induced homomorphisms $f_*:H_*(X, R)\to H_*(X, R)$ and $f^*:H^*(X, R) \to H^*(X, R)$ are both expanding.

Theorem(Gromov)

An expanding self-map of any compact manifold is topolojically conjugate to an infra-nil-endomorphism

M is infra-nil-manifold which is finitely covered by a nilmanifold

Nilmanifold

Let G be a simply connected nilpotent Lie group and L be a discrete subgroup with compact quotient
M=G/L (such M is called a nilmanifold)

An automorphism A:G → G which sends L to L induces a self map of M and this map is expanding iff the induced map on the Lie algebra has all eigenvalues bigger than one in absolute value.

Aff(G)=semidirect product of Aut(G) on G.

Let L be a subgroup of Aff(G) which acts freely and discretely on G. If M=G/L is compact, it is called infra-nil-manifold
An expanding automorphism of G which respects L induces an expanding map of M. Such a map is called an infra-nil-endomorphism.

- **Nomizu Theorem**

  For each nilmanifold N = G/L, the De Rham cohomology of N is isomorphic to the cohomology of the Chevalley-Eilenberg complex $(\wedge g^*, d)$ associated with the Lie algebra g of G, that is $H^*(N;R) = H^*(\wedge g^*, d)$.

- The dual map of an expanding map is expanding.
- The exterior product of an expanding map is expanding.
- An expanding chain map induces an expanding map on the homology.

Topology of sphere bundle

Let g be an expanding map on the closed manifold X and ξ be the disk bundle $N = X \tilde{\times} D^q \rightarrow X$ such that g can be lifted to a hyperbolic bundle embedding on N. Then

(a) the Euler class of disk bundle, $e(\xi) = 0 \in H^q(X)$;

(b) $H_l(\partial N) = H_l(X) \oplus H_{l-q+1}(X)$

g^* is an expanding map

Let $\Omega(f)=N_1 \cup N_2$

$P=N_1 \cap N_2$

Then $\partial N_{\_i} \subset P$ induces isomorphism in homology

Lemma 1: Let $f : X \to X$ be an expanding map and $g : Y \to Y$ be a homotopy equivalence. If there exists a map $h : X \to Y$ such that $h \circ f = g \circ h$, then h induces trivial map in constant rational (co)homology in positive dimension.

$$\begin{array}{ccc} H_l(N_1; \mathbb{Z}) & \xrightarrow{\delta_1} & H_l(M; \mathbb{Z}) \\ {\scriptstyle (f|N_1)_*} \downarrow & & \downarrow {\scriptstyle f_*} \\ H_l(N_1; \mathbb{Z}) & \xrightarrow[\delta_1]{} & H_l(M; \mathbb{Z}) \end{array} .$$

Consider the Mayer-Vietoris sequence for the pair (N_1, N_2) :

$$H_l(P) \xrightarrow{\varphi_l=(r_1,r_2)} H_l(N_1) \oplus H_l(N_2) \xrightarrow{\psi_l=s_1-s_2} H_l(M).$$

$$n = p_1 + q_1 = p_2 + q_2$$

$$For$$

$$l > 0$$

$$H_l(X_k) \oplus H_{n-1-l}(X_k) = H_l(P) \to H_l(X_1) \oplus H_l(X_2)$$

$$\dim H_{n-1-l}(X_1) = \dim H_l(X_2)$$

$$H_l(P) = H_l(X_1) \oplus H_l(X_2)$$

$$H_l(M) = 0$$

$$except$$

$$l = 0, n$$

Thank You