Loop Spaces and Representation Theory I

Cohen Groups and an Application to Stiefel Manifolds

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Outline

The James Construction

Let X be a pointed space. The **James construction** $J(X)$ is the free monoid generated by X subject to the single relation $* = e$.

- **The James filtration** is the word length filtration: $J_n(X) = \{x_1 \cdots x_k \mid x_i \in X,\ k \leq n\}$.
- $J_n(X) = X^n / \sim$

  $$(x_1, \ldots, x_{i-1}, *, x_i, \ldots, x_{n-1}) \sim (x_1, \ldots, x_{j-1}, *, x_j, \ldots, x_{n-1})$$

  for any $1 \leq i < j \leq n$.
- $J_n(X)/J_{n-1}(X) \cong X^{\wedge n}$.

Properties of the James Construction

- **Model for loop suspensions:** $J(X) \simeq \Omega\Sigma X$ for any path-connected CW-complex X.
- **Example:** $J(S^1) \simeq \Omega S^2$, $J(S^2) \simeq \Omega S^3$.
- **James-Hopf Maps:** $H_k \colon J(X) \longrightarrow J(X^{\wedge k})$

$$H_k(x_1 x_2 \cdots x_n) = \prod_{1 \le i_1 < i_2 < \cdots < i_k \le n} (x_{i_1} \wedge \cdots \wedge x_{i_k})$$

with right lexicographic order.

- **Example:**

$$\begin{aligned} H_3(x_1x_2x_3x_4x_5) \;&=\; (x_1 \wedge x_2 \wedge x_3)(x_1 \wedge x_2 \wedge x_4)(x_1 \wedge x_3 \wedge x_4) \\ &\quad (x_2 \wedge x_3 \wedge x_4)(x_1 \wedge x_2 \wedge x_5)(x_1 \wedge x_3 \wedge x_5) \\ &\quad (x_2 \wedge x_3 \wedge x_5)(x_1 \wedge x_4 \wedge x_5)(x_2 \wedge x_4 \wedge x_5) \\ &\quad (x_3 \wedge x_4 \wedge x_5). \end{aligned}$$

Suspension Splittings

- **Natural Splitting:** $\Sigma J_n(X) \simeq \Sigma \bigvee_{j=1}^{n} X^{\wedge j}$ for $1 \leq n \leq \infty$, where $J(X) = J_\infty(X)$.
- The inclusion $\Sigma J_{n-1}(X) \to \Sigma J_n(X)$ admits a (natural) retraction map.
- The quotient map $\Sigma q_n \colon \Sigma X^n \to \Sigma J_n(X)$ admits a (natural) cross-section.

The set of homotopy classes $[J_n(X), Y]$

Let
$d^i\colon X^{n-1} \to X^n,\ (x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{i-1}, *, x_i, \ldots, x_{n-1})$.
Then

$$[J_n(X), Y] \xrightarrow{q_n^*} [X^n, Y] \quad \overset{d^{1*}}{\underset{d^{n*}}{\vdots}} \quad [X^{n-1}, Y]$$

- **Theorem.** (Wu, Memoirs AMS 2006, Theorem 1.1.5) Let X and Y be path-connected spaces. Suppose that X is a co-H-space or Y is an H-space. Then
  1. $q_n^*\colon [J_n(X), Y] \to [X^n, Y]$ is injective.
  2. $q_n^*([J_n(X), Y])$ is the **equalizer** of $d^{1*}, d^{2*}, \ldots, d^{n*}\colon [X^n, Y] \to [X^{n-1}, Y]$.
  3. $[J(X), Y] = \lim_n [J_n(X), Y]$ the inverse limit.

Constructing the Cohen Groups

Let $\theta_n \colon F_n = F(z_1, \ldots, z_n) \to [X^n, \Omega\Sigma X]$ be the group such that $\theta_n(z_i)$ is represented by the composite

$$X^n \xrightarrow{\pi_i} X \hookrightarrow \Omega\Sigma X,$$

where $\pi_i(x_1, \ldots, x_n) = x_i$. There is a commutative diagram

$$\begin{array}{ccc} F_n & \xrightarrow{\theta_n} & [X^n, \Omega\Sigma X] \\ \downarrow d_i & & \downarrow d^{i*} \\ F_{n-1} & \xrightarrow{\theta_{n-1}} & [X^{n-1}, \Omega\Sigma X], \end{array}$$

where $d_i : \begin{pmatrix} z_1 & \cdots & z_{i-1} & z_i & z_{i+1} & \cdots & z_n \\ z_1 & \cdots & z_{i-1} & 1 & z_i & \cdots & z_{n-1}. \end{pmatrix}$

Cohen Lemma

- **Cohen Lemma:** Suppose that the reduced diagonal $\bar{\Delta}\colon X \to X \wedge X$ is null homotopic. Then

$$\theta_n([[z_{i_1}, z_{i_2}], \ldots, z_{i_t}]) = 1$$

if $z_{i_p} = z_{i_q}$ for some $1 \le p < q \le t$.

Proof. The element $\theta_n([[z_{i_1}, z_{i_2}], \ldots, z_{i_t}]) \in [X^n, \Omega\Sigma X]$ is represented by the composite

$$\begin{array}{ccccccc}
X^n & \xrightarrow{\ \phi\ } & X^t & \twoheadrightarrow & X^{\wedge t} & \xrightarrow{\ S_t\ } & \Omega\Sigma X \\
 & \searrow & \uparrow & & \uparrow\ \bar{\Delta}' \simeq * & & \\
 & & X^{t-1} & \twoheadrightarrow & X^{\wedge t-1}, & &
\end{array}$$

where $\phi(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_t})$, S_t is the iterated Samelson product and $\bar{\Delta}'$ is a reduced diagonal.

The Cohen group K_n and $\mathfrak{H}_n$

- The group K_n is defined to be the quotient of F_n subject to the relations
$$[[z_{i_1}, z_{i_2}], \ldots, z_{i_t}] = 1$$
if $z_{i_p} = z_{i_q}$ for some $1 \leq p < q \leq t$.
- The group $\mathfrak{H}_n$ is defined to be the **equalizer** of the homomorphisms $d_i \colon K_n \to K_{n-1}$ given by
$$d_i : \begin{pmatrix} z_1 & \cdots & z_{i-1} & z_i & z_{i+1} & \cdots & z_n \\ z_1 & \cdots & z_{i-1} & 1 & z_i & \cdots & z_{n-1}. \end{pmatrix}$$
- The group $\mathfrak{H} := \lim_n \mathfrak{H}_n$ the inverse limit.

Some Properties of K_n and $\mathfrak{H}_n$

- For the descending central series, $\gamma_q(K_n) = 1$ for $q > n$ and, for $q \leq n$, $\gamma_q(K_n)/\gamma_{q+1}(K_n)$ has a basis

$$[[z_{i_1}, z_{i_{\sigma(2)}}], \dots, z_{i_{\sigma(q)}}]$$

  for $1 \leq i_1 < i_2 < \cdots < i_q \leq n$, $\sigma \in \Sigma_{q-1}$ acting on $\{2, \dots, q\}$.
- $\gamma_n(K_n)$ is $\mathrm{Lie}(n)$.
- The elements $\prod_{1\leq i_1<\cdots<i_q\leq n}[[z_{i_1}, z_{i_{\sigma(2)}}], \dots, z_{i_{\sigma(q)}}] \in \mathfrak{H}_n$ represented by the composite

$$J_n(X) \xrightarrow{H_q} J(X^{\wedge q}) \xrightarrow{J(\mathrm{id}_X \wedge \sigma)} J(X^{\wedge q}) \xrightarrow{JS_q} \Omega\Sigma X,$$

  where JS_q is the H-map induced by $S_q \colon X^{\wedge q} \to \Omega\Sigma X$.
- **Proposition.** (Wu, Memoirs AMS 2006, Proposition 1.1.9) Any pointed map $f \colon Y \to \Omega Z$ extends uniquely (up to homotopy) to an H-map $Jf \colon J(Y) \to \Omega Z$.

Some elements in $\mathfrak{H}_n$

Let $X = \Sigma Y$ for some Y.

- $J(X) \xrightarrow{\simeq} \Omega\Sigma X \longleftrightarrow z_1 z_2 \cdots\cdots$.
- $J_n(X) \hookrightarrow J(X)\Omega\Sigma X \longleftrightarrow z_1 z_2 \cdots z_n$.
- Let $[q]\colon X \to X$ is the map of degree q. Then

$$J([q])|_{J_n(X)} \longleftrightarrow z_1^q z_2^q \cdots z_n^q.$$

- Let $k\colon J(X) \simeq \Omega\Sigma X \to J(X) \simeq \Omega\Sigma X$ be the k th power map for $k \in \mathbb{Z}$. Then

$$k|_{J_n(X)} \longleftrightarrow (z_1 z_2 \cdots z_n)^k.$$

Example

The commutator $[a, b] = a^{-1}b^{-1}ab$. Thus $ab = ba[a, b]$.

$$\begin{aligned} z_1^2 z_2^2 &= z_1 z_1 z_2 z_2 \\ &= z_1 z_2 z_1 [z_1, z_2] z_2 \\ &= z_1 z_2 z_1 z_2 [z_1, z_2][[z_1, z_2], z_2] \\ &= (z_1 z_2)^2 [z_1, z_2] \end{aligned}$$

in K_2 with $z_1^2 z_2^2, (z_1 z_2)^2, [z_1, z_2] \in \mathfrak{H}_2$.

- **Topological meaning:** Let α be the composite

$$J(X) \xrightarrow{H_2} J(X \wedge X) \xrightarrow{JS_2} \Omega\Sigma X.$$

Then

$$[\Omega[2]|_{J_2(X)}] = [2|_{J_2(X)}] \cdot [\alpha|_{J_2(X)}]$$

in $[J_2(X), \Omega\Sigma X]$ provided that $\bar{\Delta} \simeq * \colon X \to X \wedge X$.

Degree of maps between manifolds

Let M^m and N^m be oriented m-manifolds. Let $f\colon M \to N$ be any map. Then

$$f_*\colon H_m(M) = \mathbb{Z} \longrightarrow H_m(N) = \mathbb{Z}$$

is given by the multiple of an integer called **degree** of f, denoted by $\deg(f)$.
Let

$$\mathrm{DS}(M) = \{a \in \mathbb{Z} \mid a = \deg(f) \text{ for some } f\colon M \to M\}.$$

The Stiefel Manifold $V_{2n+1,2}$

We consider the Stiefel manifold:

$$\begin{aligned} V_{2n+1,2} &= \{(x,y) \in \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \mid \|x\| = \|y\| = 1, \langle x, y\rangle = 0\} \\ &= \tau(S^{2n}). \end{aligned}$$

- **Theorem.** (Wu, Memoirs AMS 2003, Theorem 2.19) Let $n \geq 1$. Then

$$\mathrm{DS}(V_{4n+1,2}) = \{4k, 4k+1 \mid k \in \mathbb{Z}\},$$

$$\mathrm{DS}(V_{4n+3,2}) = \{8k, 8k+5 \mid k \in \mathbb{Z}\}.$$

Proof of the Theorem

- $V_{2n+1,2} \smallsetminus \mathrm{point} \simeq P^{2n}(2) = \Sigma^{2n-2}\mathbb{R}\mathrm{P}^2$, the mod 2 Moore space.
- $V_{2n+1,2} = P^{2n}(2) \cup_{\lambda_{2n}} e^{4n-1}$ for a map $\lambda_{2n}\colon S^{4n-2} \to P^{2n}(2)$.
- **Cohen-Wu, 1995; Mahowald knew earlier:** the order of $[\lambda_{4n}]$ is 4 and the order of $[\lambda_{4n+2}]$ is of 8 in the homotopy groups for $n \geq 1$.
- Let $f\colon V_{2n+1,2} \to V_{2n+1,2}$ be a map. Then

$$\begin{array}{ccccc}
S^{4n-2} & \xrightarrow{\lambda_{2n}} & P^{2n}(2) & \hookrightarrow & V_{2n+1,2} \\
\downarrow{\scriptstyle \deg(f)} & & \downarrow{\scriptstyle g = f|_{P^{2n}(2)}} & & \downarrow{\scriptstyle f} \\
S^{4n-2} & \xrightarrow{\lambda_{2n}} & P^{2n}(2) & \hookrightarrow & V_{2n+1,2}.
\end{array}$$

Proof of the Theorem

- **Well-known, for instance, Wu, Memoirs, 2003, Prop. 2.6.** $[P^{2n}(2), P^{2n}(2)] = \mathbb{Z}/4\mathbb{Z}$.
- **Trivial Cases:** $[g] = 0 \Longrightarrow \deg(f) = 0 \mod 4 \text{ or } 8$ and $[g] = 1 \Longrightarrow \deg(f) = 1 \mod 4 \text{ or } 8$.
- **Case:** $g \simeq [2]\colon P^{2n}(2) \to P^{2n}(2)$. Then g is homotopic to the composite

$$P^{2n}(2) \xrightarrow{p} S^{2n} \xrightarrow{\eta} S^{2n-1} \hookrightarrow P^{2n-1}(2).$$

  (Well-known, for instance, Wu, Memoirs, 2003, Prop. 2.5.)
- Since the map $p\colon P^{2n}(2) \to S^{2n}$ factors through $V_{2n+1,2}$, $g \circ \lambda_{2n} \simeq *$ and so $\deg(f) = 0 \mod 4 \text{ or } 8$ in this case.

Proof of Theorem, last case $g \simeq [-1]$.

Let $\lambda'_{2n} \colon S^{4n-3} \to \Omega P^{2n}(2)$ be the adjoint map of λ_{2n}.

- λ'_{2n} lifts to $J_2(P^{2n-1}(2))$ because the inclusion $J_2(P^{2n-1}(2)) \hookrightarrow J(P^{2n-1}(2)) \simeq \Omega P^{2n}(2)$ induces an isomorphism on homology **up to dimension** $3(2n-1)-1 = 6n-4 > 4n-3$. Thus there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{4n-3} & & \\ \downarrow{\scriptstyle \tilde{\lambda}_{2n}} & \searrow^{\lambda'_{2n}} & \\ J_2(P^{2n-1}(2)) & \hookrightarrow & \Omega P^{2n}(2). \end{array}$$

- $[2] \circ \lambda_{2n} \simeq * \Longrightarrow (\Omega[2]|_{J_2(P^{2n-1}(2))})_*([\tilde{\lambda}_{2n}]) = 0$ for $(\Omega[2]|_{J_2(P^{2n-1}(2))})_* \colon \pi_{4n-3}(J_2(P^{2n-1}(2))) \to \pi_{4n-3}(\Omega P^{2n}(2))$.

Proof of Theorem, last case $g \simeq [-1]$.

Let $\alpha\colon J(X) \xrightarrow{H_2} J(X \wedge X) \xrightarrow{JS_2} \Omega\Sigma X$.

$$\begin{aligned} 0 = (\Omega[2]|_{J_2(P^{2n-1}(2))})_*([\tilde{\lambda}_{2n}]) &= (2|_{J_2(X)})_*([\tilde{\lambda}_{2n}]) + (\alpha|_{J_2(X)})_*([\tilde{\lambda}_{2n}]) \\ &= 2[\lambda'_{2n}] + \alpha_*([\lambda'_{2n}]) \end{aligned}$$

- $\alpha_*([\lambda'_{2n}]) = -2[\lambda'_{2n}]$.
- $\Omega[-1] \longleftrightarrow z_1^{-1}z_2^{-1} = z_2^{-1}z_1^{-1}[z_1^{-1}, z_2^{-1}] = (z_1z_2)^{-1}[z_1, z_2]$ in K_2 because $\gamma_3 = 0$.

Thus

$$(\Omega g)_*([\lambda'_{2n}]) = (-1)_*([\lambda'_{2n}]) + \alpha_*([\lambda'_{2n}]) = -[\lambda'_{2n}] - 2[\lambda'_{2n}] = -3[\lambda'_{2n}].$$