

Loop Spaces and Representation Theory II

Natural Decompositions of Looped Co- H -spaces and Representation Theory

Jie Wu

Department of Mathematics
National University of Singapore

December 12, 2008

Outline

Tensor Algebras

$\text{coalg}(T, T)$

Decompositions of Looped $\text{co-}H$ -spaces

The functor A^{\min} and the Cohen Conjecture

The Questions

- **Duan's Talk:** Compute $\text{End}^{\text{alg}}(H^*(X))$.
- We consider

Natural Transformations $^{\text{alg}}(H^*(\Omega X))$

for X running through co- H -spaces.

- The applications to topology is to give **functorial decompositions of ΩX** for co- H -spaces X .

Hopf algebras

Consider $H_*(\Omega X)$ with coefficients in a field.

- $\mu: \Omega X \times \Omega X \rightarrow \Omega X \implies \mu_*: H_*(\Omega X) \otimes H_*(\Omega X) \rightarrow H_*(\Omega X)$.
Thus $H_*(\Omega X)$ is a (graded) algebra.
- The diagonal map $\Delta: \Omega X \rightarrow \Omega X \times \Omega X$ induces a comultiplication $\Delta_*: H_*(\Omega X) \rightarrow H_*(\Omega X) \otimes H_*(\Omega X)$. Thus $H_*(\Omega X)$ is a (graded) coalgebra (the dual version of algebra).
- Since the diagonal map preserves the multiplication ($\Delta(xy) = \Delta(x)\Delta(y)$), $\Delta_*: H_*(\Omega X) \rightarrow H_*(\Omega X) \otimes H_*(\Omega X)$ an algebra map.
- Δ_* is cocommutative (dual of the cup product).

A **Hopf algebra** A means: 1) A is an (associative) algebra; 2) A is a (coassociative) coalgebra and 3) the comultiplication $\psi: A \rightarrow A \otimes A$ is an algebra map.

Tensor Algebras

Let V be a projective R -module. Let $T_n(V) = V^{\otimes n}$ be the n -fold self tensor product of V with $T_0(V) = R$. The tensor algebra

$$T(V) = \bigoplus_{n=0}^{\infty} T_n(V)$$

with multiplication given by the formal product

$$(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_m) = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m.$$

- **Universal Property:** Let A be any (associative) algebra. Any R -linear map $V \rightarrow A$ extends uniquely to an algebra map $T(V) \rightarrow A$.

Hopf Structure on Tensor Algebras

Let $\psi: V \rightarrow T(V) \otimes T(V)$ be given by $\psi(x) = x \otimes 1 + 1 \otimes x$.

Then ψ extends to an **algebra map** $\psi: T(V) \rightarrow T(V) \otimes T(V)$.

- $T(V)$ is a (cocommutative) Hopf algebra with comultiplications defined in this way.
- If V is a free R -module of finite rank, the graded dual $T^*(V) = \bigoplus_{n=0}^{\infty} (T_n(V))^*$ is a commutative Hopf algebra. (Its algebra structure is called **shuffle algebra**.)

Topological Meaning:

- Let X be a simply connected co- H -space and let ground ring R be a field. Then $H_*(\Omega X) \cong T(s^{-1}\bar{H}_*(X))$ is the tensor algebra generated by the desuspension of $\bar{H}_*(X)$.
- **Note.** The comultiplication in $H_*(\Omega X)$ could be different from what we defined algebraically above. If we take the augmentation ideal filtration on $H_*(\Omega X)$, then its associated Hopf algebra is $T(s^{-1}\bar{H}_*(X))$ as a Hopf algebra.

Algebraic Question

Now we consider V an arbitrary ungraded projective R -module.

- $T: V \mapsto T(V)$ a functor from R -projective modules to Hopf algebras.
- By **forgetting algebra structure**, $T: V \mapsto T(V)$ is a functor from R -projective modules to coalgebras.
- **Question.** Determine

$$\text{coalg}(T, T) = \text{alg}(T^*, T^*).$$

Namely determine the set (class) of all coalgebra natural endomorphisms of $T(V)$.

Some Properties

- $\text{coalg}(T, T)$ is a **group** under the convolution product:

$$T(V) \xrightarrow{\psi} T(V) \otimes T(V) \xrightarrow{f_V \otimes g_V} T(V) \otimes T(V) \xrightarrow{\text{mult}} T(V)$$

for $f, g \in \text{coalg}(T, T)$.

- $\text{coalg}(T, T)$ is a **semi-ring** under the convolution product and the composition operation: $(fg) \circ h = (f \circ h)(g \circ h)$ but $h \circ (fg) \neq (h \circ f)(h \circ g)$ in general.
- **Orthogonal Property:** $\text{Hom}(T_n, T_m) = 0$ if $n \neq m$, and $\text{Hom}(T_n, T_n) \cong R(S_n)$.

Thus

$$\text{coalg}(T, T) \subseteq \prod_{n=1}^{\infty} \text{Hom}(T_n, T_n) = \prod_{n=1}^{\infty} R(S_n).$$

The group coalg(T, T)

The **algebraic James construction**: $J_n(V) = \bigoplus_{j=0}^n T_j(V)$ with comultiplication given by

$$\psi|_{J_n(V)}: J_n(V) \longrightarrow J_n(V) \otimes J_n(V).$$

Then there is a **tower of groups**

$$\text{coalg}(T, T) \longrightarrow \cdots \longrightarrow \text{coalg}(J_n, T) \longrightarrow \text{coalg}(J_{n-1}, T) \longrightarrow \cdots .$$

Grbic-Wu, AGT, 2006:

- $\text{coalg}(J_n, T) \rightarrow \text{coalg}(J_{n-1}, T)$ is an epimorphism with kernel given by $\text{Lie}^R(n) = \text{Lie}(n) \otimes R$, the module $\text{Lie}(n)$ over R .
- $\text{coalg}(T, T) = \lim_n \text{coalg}(J_n, T)$ the inverse limit.

Geometric Realization

The ground ring is \mathbb{Z}/p and the homology is mod p homology:
For decomposing $\Omega\Sigma X$,

$$\begin{array}{ccc}
 [\Omega\Sigma, \Omega\Sigma] & \xrightarrow{\text{Gr. } H_*(-)} & \text{coalg}^{\text{graded}}(T, T) \\
 \uparrow & & \uparrow \\
 \mathfrak{H}^{\text{free}} & \longrightarrow \mathfrak{H} \longrightarrow & \text{coalg}(T, T).
 \end{array}$$

- For decomposing ΩX for a co- H -space X , consider

$$\Omega X \xrightarrow{\Omega s} \Omega\Sigma\Omega X \dashrightarrow \Omega\Sigma\Omega X \xrightarrow{\Omega \text{ev}} \Omega X.$$

Decompositions of Looped Co- H -spaces

Theorem, Selick-Theriault-Wu, several papers: For any functorial coalgebra decomposition $T(V) \cong A(V) \times B(V)$, given any simply connected co- H -space X , there exist functors \bar{A} and \bar{B} such that

- $\Omega X \simeq \bar{A}(X) \times \bar{B}(X)$,
- $\text{Gr}.H_*(\bar{A}(X)) = A(s^{-1}\bar{H}_*(X))$ and
- $\text{Gr}.H_*(\bar{B}(X)) = B(s^{-1}\bar{H}_*(X))$ and

Cohen Conjecture

The algebraic functor A^{\min} is defined to be the **smallest** coalgebra retract of the functor T such that $V \subseteq A^{\min}(V)$.

- **Selick-Wu, Memoirs, 2000:** The functor A^{\min} exists and is unique up to natural equivalence.
- **Note.** If A is any coalgebra retract of T with $V \subseteq A(V)$, then A^{\min} is a coalgebra retract of A .

The geometric functor $\bar{A}^{\min}(X)$ is a **geometric realization of** A^{\min} .

- **Cohen Conjecture:** The primitives $PA^{\min}(V) \cap T_n(V) = 0$ if n is not a power of p .

Determine A^{\min} using representation theory

The symmetric group S_n acts on $\text{Lie}(n)$ by permuting letters.

- $\text{Lie}^{\max}(n)$ is the **largest $\mathbf{k}(S_n)$ -projective summand** of $\text{Lie}(n)$.
- Given any (graded or ungraded) module V , let $L_n^{\max}(V) = \text{Lie}^{\max}(n) \otimes_{\mathbf{k}(S_n)} V^{\otimes n}$. Note $L_n^{\max}(V) \subseteq L_n(V)$, submodule of the Lie power.
- Let $B^{\max}(V)$ be the subalgebra of $T(V)$ generated by $L_n^{\max}(V)$ for $n \geq 2$.

Theorem (Selick-Wu, Memoirs, 2000):

$A^{\min}(V) = \mathbf{k} \otimes_{B^{\max}(V)} T(V)$ is the quotient of $T(V)$ by the subalgebra $B^{\max}(V)$.

Solution to the Cohen Conjecture (1)

Changing the Ground Fields

- **From representation theory**, the irreducible (and indecomposable) modules over $\mathbf{k}(S_n)$ only depend on the characteristic of \mathbf{k} .
- This means that, if \mathbf{K} is an extension field of \mathbf{k} , then $\text{Lie}^{\max}(n)$ with the ground field \mathbf{K} is given by $\text{Lie}^{\max} \otimes_{\mathbf{k}} \mathbf{K}$.
- It gives the property that the algebraic functor A^{\min} **only depends** on the characteristic of the ground field.
- Thus we can assume that the ground field is **algebraically closed**.

Solution to the Cohen Conjecture (2)

Constructing a Decomposition

- Given n not a power of p , choose $\zeta \in \mathbf{k}$ with $\zeta \neq 1$ and $\zeta^n = 1$. (\mathbf{k} is algebraically closed!) Define $T\zeta: T(V) \rightarrow T(V)$ by $T\zeta(a_1 \otimes \cdots \otimes a_n) = \zeta^n a_1 \otimes \cdots \otimes a_n$. $T\zeta$ is a Hopf map. (Adams-type map!)
- Let $\chi: T(V) \rightarrow T(V)$, $\chi(a_1 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes \cdots \otimes a_1$.
- Let $\phi = (T\zeta) * \chi: T(V) \rightarrow T(V)$ be the **convolution product**.

Let A_ϕ be the colimit of $T(V) \xrightarrow{\phi} T(V) \xrightarrow{\phi} T(V) \longrightarrow \cdots$.

Then

- A_ϕ is a coalgebra retract of T
- $V \subseteq A_\phi(V)$ because $\phi(a) = (\zeta - 1)a: V \rightarrow V$ is an isomorphism.
- $PA_\phi(V) \cap T_n(V) = 0$ because, for $\alpha \in PT_n(V)$, $\phi(\alpha) = \zeta^n \alpha - \alpha = 0$.

Thus $PA^{\min} \cap T_n(V) = 0$ for any given n not a power of p .

Historic Remarks

- Fred Cohen was the first topologist who uses Specht-Wever's identity: $\beta_k \circ \beta_k = k\beta_k$ (Specht, 1947; Wever, 1949) for giving the splitting

$$\Omega\Sigma X \simeq \Omega\Sigma L_k(X) \times \text{another factor}$$

for any suspension X if $k \not\equiv 0 \pmod{p}$. (**Cohen-Wu, 1995**)

- I was pushing further for this decomposition. (**Topology, 1998**)
- We solved Cohen's conjecture (**Selick-Wu, Memoirs 2000**).
- Subsequent work also involved **Jelena Grbic and Stephen Theriault**. We have several papers on this topic.
- It has attracted representation theorists such as **K. Erdmann, R. Bryant** and etc.
- $PA^{\min}(V) \cap T_{pr}(V)$ seems very tricky.