Strong torsion generators, braid groups, mapping class groups

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Key Definitions:

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2. Group normally gen'd by stg gp – *e.g.* $E(R)$ (stable gp gen'd by eltry matrices)

Why? Realization problems

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Where? Further "classical" stg groups

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$G \in \mathrm{S{\scriptstyle TG}} \Rightarrow H_1(G) = 0$

Using algebraic K-theory: $\textsc{Stg} \ni G \overset{\checkmark}{\longmapsto} \mathcal{Z}(G)$ AJB '91

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Using combinatorial group theory:
$\mathrm{S}\textsc{tg} \ni G \stackrel{\checkmark}{\longmapsto} H_1(G) = 0, H_2(G), \ldots$ AJB-Miller '92

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Theorem

Let A, $A_2, \ldots, A_5$ be any five abelian groups. Then, there exists $G \in \textsc{Stg}$ such that:

(i) $\mathcal{Z}(G) \cong A$;

(ii) $H_1(G) = 0 \qquad (\because stg)$

(iii) *For* $i = 2, \ldots, 5, H_i(G) \cong A_i$

(iv) *For infinitely many n*, $H_n(G) \cong \mathbb{Z} \oplus \diamondsuit$.

Where? Further "classical" stg groups

AJB-Matthey Comm Math Helv, to appear

$$
\begin{array}{ccccccc}
 & & \Gamma_{g,1} & \xrightarrow{\mathrm{id}} & \Gamma_{g,1} & & \\
 & \nearrow \psi_g & & & & \searrow & \\
\mathrm{Br}_g & \twoheadrightarrow & \Sigma_g & \rightarrowtail & O_g(\mathbb{Z}) & \xrightarrow{H} & \mathrm{Sp}_{2g}(\mathbb{Z}) \\
 & {\scriptstyle \mathrm{Artin}} \searrow & & & \downarrow & & \downarrow \\
 & & \mathrm{Aut}(\mathrm{Fr}_g) & \twoheadrightarrow & \mathrm{GL}_g(\mathbb{Z}) & \xrightarrow{H} & \mathrm{GL}_{2g}(\mathbb{Z})
\end{array}
$$

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Theorem

$$\mathrm{EU}(R,\Lambda) = [\mathrm{U}(R,\Lambda),\, \mathrm{U}(R,\Lambda)] \in \textsc{Stg},$$

e.g. $\mathrm{Sp}(\mathbb{Z})$, $[\mathrm{O}(\mathbb{Z}),\, \mathrm{O}(\mathbb{Z})] \in \textsc{Stg}$.

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$$\mathrm{SAut}(F_\infty),\ \mathrm{SOut}(F_\infty) \in \textsc{Stg}.$$

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Do $B_n(S^2), B_n(P^2)$ stabilize?

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Corollary

If $k \equiv \pm 1 \pmod 6$, and also $m \equiv \pm 2$ or $\pm 3 \pmod k$, then any homomorphism $B_{k+1}(S^2) \longrightarrow B_{k+1+m}(S^2)$ has image in $\mathcal{Z}(B_{k+1+m}(S^2)) \cong C_2$.

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Proposition

For $n \geq 3$, the only nontrivial homomorphism from $B_n(P^2)$ to $B_m(P^2)$ is

$$B_n(P^2) \twoheadrightarrow \Sigma_n \twoheadrightarrow C_2 \xrightarrow{\cong} \mathcal{Z}(B_m(P^2)) \hookrightarrow B_m(P^2)\,,$$

when $\gcd(n', m) = \gcd(n', m-1) = 1$ $(n' := max\ odd \leq n)$.

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Corollary

The commutator subgroup of $B_n(P^2)$ is the intersection of:
the lower central series ($n \geq 3$);
the derived series ($n \geq 5$).

Mapping class groups.

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For $m > 0$?

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(a) If both $g \geq 3$ *and* d *divides* g *or* $g-1$ *or* $4g+2$*, then* Γ_g *is strongly* d*-torsion generated.*

Theorem

(a) If both $g \geq 3$ and d divides g or $g-1$ or $4g+2$, then Γ_g is strongly d-torsion generated.
(b) Conversely, if Γ_g is strongly d-torsion generated, then both $g \geq 3$ and $d \leq 4g+2$, and

$$\text{prime } p|d \implies -\frac{2}{k} \leq p - \frac{2g}{k} \leq 1$$

for some integer k.

Theorem

For $g \geq 3$, $m \geq 1$, and:

(i) $g - 1 = pu$ *with prime* $p > 2(m + u + 2)$*;*

(ii) $g = pv$ *with prime* $p > 2(m + v + 1)$*, and* $m \geq 2$*; or*

(iii) $2g + 1 = pw$ *with prime* $p > m + w + 1$*,*

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Rigidity Conjecture

For all sufficiently large g, *there is only the trivial homomorphism* $\Gamma_g \to \Gamma_{g+m}$.

Consider P(g, m): $\exists$ prime $p|g(g-1)(2g+1)$ with

$$p - \sqrt{2g} > 2m + 2$$

Corollary

Then $P(g, m) \Rightarrow$ rigidity,
i.e. if $P(g, m)$ holds (and $m \geq 2$ if $p|g$),
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For each $m > 0$, $P(g, m)$ holds for $>$ 97% of g.

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For each $m>0$, $P(g,m)$ holds for $>$ 97% of g.

Proposition

$\exists$ *infinitely many g for which $P(g,m)$ does not hold.*

Proposition

For any $k \in \mathbb{R}$, the set

$$\{g \in \mathbb{N} \ : \ p|g(g-1)(2g+1) \implies p - \sqrt{2g} < k\}$$

is infinite.

Pf. $L := \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$, $M := 5I + 2L$, $\begin{bmatrix} u_r \\ v_r \end{bmatrix} := M^r \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $r \geq 0$.

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$$\begin{bmatrix} u_r & 2v_r \\ v_r & 3u_r \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}^r \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

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$\therefore$ prime $p | v_r^2(v_r^2 - 1)(2v_r^2 + 1) = v_r^2(v_r - 1)(v_r + 1)3u_r^2$
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$\implies p - \sqrt{2v_r^2} \leq 1 + (1 - \sqrt{2})v_r \to -\infty \quad$ as $r \to \infty$.