Goodwillie Calculus and Lie(n)

Lê Minh Hà

Vietnam National University

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Goal

It is possible to write endlessly about Elliptic Curves. (This is not a threat.)

(S. Lang)

Plan

- Introduction to Lie(n)
- The combinatorial approach: Lie(n) and the partition lattice
- Goodwillie calculus: Lie(n) and homotopy theory
- Some applications: Lie(n) and robot arms

Definition of Lie(n)

- Let R be a commutative ring with unit $(\mathbb{C}, \mathbb{Z}, \mathbb{Z}/p)$, n positive integer.
- $Lie_R(x_1, \ldots, x_n)$ is the free Lie algebra over R generated by x_i.
- $Lie_R(n)$ is the submodule spanned by all bracket monomials containing x_i exactly once.

Example:

- Lie(2) is spanned by a single monomial $[x_1, x_2]$.
- Lie(3) is spanned by two monomials $[x_1, [x_2, x_3]]$ and $[x_2, [x_1, x_3]]$.

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Main properties

1. More generally, Lie(n) has an additive basis consisting of monomials

$$[x_{\sigma(1)}[x_{\sigma(2)}[\dots[x_{\sigma(n-1)},x_{\sigma(n)}]\dots] \quad \sigma\in\Sigma_n, \sigma(n)=n$$

2. Lie(n) has a natural action of Σ_n.
3. The restriction of Lie(n) to Σ_{n-1} is the regular representation.
4. Over the complex numbers, $Lie(n)=\mathtt{Ind}_{C_n}^{\Sigma_n}\rho_n$ where ρ_n is a faithful linear character of the subgroup generated by an n-cycle in Σ_n.

Poset topology

1. Work of Stanley, Hanlon, Joyal, Klyachko, Björner relates Lie(n) and the top homology group of the partition lattice Π_n.
2. Π_n is the set of all partitions of $\{1, 2, \ldots, n\}$, ordered by refinement.
3. Π_n has a natural action of the symmetric group Σ_n.
4. There exists smallest element $\hat{0} = \{1\}, \ldots, \{n\}$ and maximum element $\hat{1} = \{1, 2, \ldots n\}$.
5. Consider Π_n as a category whose objects are partitions and morphisms are induced from refinements.

The nerve of a category

Given a small category $\mathscr{C}$, can associate a simplicial set $N\mathscr{C}$ called the *nerve* of $\mathscr{C}$.

1. $N\mathscr{C}_0$ is the set of elements of $\mathscr{C}$.
2. $N\mathscr{C}_k$ is the set of diagrams

$$C_1 \to C_2 \to \ldots \to C_k$$

   of objects and morphisms in $\mathscr{C}$.
3. Face map d_i is composition of the i-th node in the diagram.
4. Degeneracy maps s_j are given by inserting identity morphism in the j-th node.

The geometric realization of $N\mathscr{C}$ gives a topological space associated to the category $\mathscr{C}$.

Homology of the partition lattice

1. Since Π_n contains the minimum and maximum elements, its realization is contractible (for two reasons!).
2. Consider $\bar{\Pi}_n = \Pi_n - \{\hat{0}, \hat{1}\}$. Let $\tilde{K}_n$ be the realization of $\bar{\Pi}_n$.
3. Let K_n be the unreduced suspension of $\tilde{K}_n$.

Facts:

1. (Folkman, Björner) $\tilde{H}_*(K_n)$ concentrates in a single degree.
2. In fact, $K_n \simeq \bigvee_{(n-1)!} S^{n-2}$.
3. (Stanley, Hanlon, Joyal, Klyachko, Lehrer-Solomon) The homology above concentrate in degree $n-2$ and there is an isomorphism of right Σ_n-modules,

$$\tilde{H}_{n-2}(K_n) \cong \epsilon Lie^*(n).$$

Where $Lie^*(n)$ denote the dual of $Lie(n)$, ϵ is the sign representation.

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Fresse's theorem

There is a Koszul duality between the operad of commutative algebras and the cooperad dual of the Lie operad corresponding to Lie algebras.
The isomorphism of Σ_n modules above is a direct consequence of this duality.

Goodwillie calculus - Motivation

1. Derived functors of non-additive functors.
2. Relate a homotopy functor to "nicer" ones in a systematic way.

Homotopy functor

Functors

$$F\colon \mathscr{C} \to \mathscr{D},$$

where $\mathscr{C}$ and $\mathscr{D}$ are either Top_* (based spaces) or Spec (spectra). Can also take spaces (spectra) over and under a given space (spectra).

- Homotopy functor means F preserves weak equivalences. That is, if $X \simeq Y$ then $F(X) \simeq F(Y)$.
- For simplicity, assume $\mathscr{C} = \mathscr{D} = Top_*$ and F preserves filtered homotopy colimits.

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Goodwillie tower

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_2F(X) \arrow[d, "p_2"] \\
 & P_1F(X) \arrow[d, "p_1"] \\
F(X) \arrow[r] \arrow[ur] \arrow[uur] & P_0F(X)
\end{tikzcd}$$

- P_kF is a polynomial functor of degree at most k (or k-excisive), and
- $F \to P_kF$ is “universal” among natural transformations to a k-excisive functor.

Example

If F is the identity functor. Then

- $P_1(X) = QX$.
- $P_2(X)$ is the homotopy fiber of the natural map

$$QX \to Q(X^{\wedge 2} \wedge_{\Sigma_2} E\Sigma_{2+}).$$

The layer

Let $D_n F$ denote the homotopy fiber of the natural transformation $P_n F \rightarrow P_{n-1} F$.

Theorem

$D_n F(X) = \Omega^\infty(\partial_n F \wedge (\Sigma^\infty X)^{\wedge n})_{h\Sigma_n}$ for some spectrum $\partial_n F$ with Σ_n-action.

$\partial_n F$ is called the n-th derivative of F at a point, or the "coefficient" of $D_n F$.

The n-th layer and Lie(n)

Consider the $F = Id$ - the identity functor from based spaces to itself.

Theorem

1. *(B. Johnson) There exists a certain space Δ_n such that*

$$D_n(Id)(X) \simeq \Omega^\infty(Map_*(\Delta_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n})$$

2. *(Arone-Mahowald) There exists an Σ_n-equivariant equivalent*

$$\Delta_n \simeq SK_n.$$

The derivative $\partial_* Id$ of the identity functor is the Spanier-Whitehead dual of the partition lattice.

Construction of the tower

construction

Cubical diagrams

- Let S be a finite set. The power set of S is $\mathscr{P}(S)$ is a poset via inclusion. Let $\mathscr{P}_0(S) = \mathscr{P}(S) - \emptyset$, $\mathscr{P}_1(S) = \mathscr{P}(S) - S$.
- A d-cube in $\mathscr{C}$ is a functor $\chi\colon \mathscr{P}(S) \to \mathscr{C}$ with $|S| = d$.

Examples:

- A 1-cube is a diagram $\chi(\emptyset) \to \chi(1)$.
- A 2-cube is a square:

$$\begin{tikzcd}
\chi(\emptyset) \arrow[r] \arrow[d] & \chi(\{2\}) \arrow[d] \\
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Cartesian and coCartesian cubes

- A d-cube χ is Cartesian if the natural map

  $$\chi(\emptyset) \rightarrow \mathtt{holim}\ _{T \in \mathscr{P}_0(S)}\, \chi(T),$$

  is a weak equivalence.
- A d-cube χ is coCartesian if the natural map

  $$\mathtt{hocolim}\ _{T \in \mathscr{P}_1(S)} \rightarrow \chi(S),$$

  is a weak equivalence.

Example:

- A 2-cube is Cartesian if it is a homotopy pullback square.
- A 2-cube is coCartesian if it is a homotopy pushout square.

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Strongly coCartesian cube

- A d-cube χ is strongly coCartesian if $\chi|_{\mathscr{P}(T)}\colon \mathscr{P}(T) to \mathscr{C}$ is coCartesian for all subset $T \subseteq S$ with $|T| \geq 2$.
- Important example:
  For $T \subseteq \{1, 2, \ldots, d\}$, let $X \star T$ be the homotopy cofiber of the folding map $\vee_T X \to X$. The assignment

  $$\chi\colon T \mapsto X \star T$$

  is a strongly coCartesian d-cube.

Strongly coCartesian cube

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Polynomial functors

- A functor F is called d-excisive (or polynomial of degree at most d) if f takes strongly coCartesian (d+1)-cubes to Catesian cubes.
- 0-excisive means $F(X) \to F(*)$ is an equivalence for all X. So F is homotopy constant.
- 1-excisive means F takes pushout squares to pullback squares.

Construction of the Goodwillie tower

The construction of P_dF is done in two steps.

1. We construct a functor $T_dF \colon \mathscr{C} \to \mathscr{D}$ together with a natural transformation $t_d(F) \colon F \to T_dF$ so that this is an equivalent if F is d-excisive.
2. P_dF is obtained from T_dF via a "telescope" construction.

$$P_dF(X) = \mathtt{hocolim}\,\{F(X) \xrightarrow{t_d(F)} T_dF(X) \xrightarrow{t_d(T_dF)} T_dT_dF(X) \to \ldots\}$$

Definition

$$T_dF(X) = \mathtt{holim}\;_{T\in\mathscr{P}_0(d+1)}\,F(X\star T).$$

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The case d=1

1. The strongly coCartesian 2-cube $\chi(T) = X \star T$ has the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd}$$

2. $T_1 F$ is the homotopy pullback of

$$\begin{tikzcd} & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

3. So $T_1 F(X) \simeq \Omega F(\Sigma X)$.
4. $P_1 F(X) \simeq \Omega^\infty F(\Sigma^\infty X)$.

Details about the layer D_nF

- The layer D_nF is a n-homogeneous functor. That is, D_nF is n-excisive and $P_{n-1}(D_nF) \simeq \star$.

Theorem (Goodwillie)

If F is homogeneous of degree n and X is a space, then

$$F(X) \simeq \Omega^\infty (C \wedge X^{\wedge n})_{h\Sigma_n}$$

where C is a spectrum with Σ_n-action and $h\Sigma_n$ denotes the homotopy orbit spectrum.

The layer of the identity functor and the partition lattice

Theorem (B. Johnson)

There exists a space Δ_n *such that*

(i) *The group* Σ_n *acts on* Δ_n*.*

(ii) *Non-equivariantly,* $\Delta_n \simeq \bigvee_{(n-1)!} S^{n-1}$*,*

(iii) *The n-th derivative of the identity is* $Map_*(\Delta_n, \Sigma^\infty S^0)$*, the Spanier-Whitehead dual of* Δ_n*, considered as a spectrum with an action of* Σ_n*.*

$$D_n(X) \simeq \Omega^\infty(Map_*(\Delta_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}.$$

The space Δ_n

- Consider the n^2-dimensional cube:

$$I^{n^2} = \{t = (t_{11}, t_{12}, \dots, t_{n1}, t_{n2} \dots, t_{nn}) | 0 \leq t_{ij} \leq 1\}.$$

- Let $I^{n(n-1)}$ be the subspace of I^{n^2} consisting of those t where $t_{ii} = 0$ for all i.
- For $1 \leq i < j \leq n$, let $W_{ij} = \{t \in I^{n^2} | t_{ik} = t_{jk}$ for all $k\}$.
- Let $Z = \{t \in I^{n^2} | t_{ij} = 1$ for some $i, j\}$.

Theorem (B. Johnson)

$$\Delta_n = I^{n(n-1)} / \{Z \cup \bigcup_{i,j} W_{ij}\}.$$

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Δ_n and the partition lattice

Theorem (Arone-Mahowald)

There is a Σ_n-equivariant equivalence

$$\Delta_n \simeq SK_n.$$

Proof

- Since $I^{n(n-1)}$ is Σ_n-equivariantly contractible, Δ_n is equivariantly equivalent to the suspension of $Z \cup \bigcup_{i,j} W_{ij}$. Therefore, one needs to show that $K_n \simeq Z \cup \bigcup_{i,j} W_{ij}$.
- Since K_n is unreduced suspension of $\tilde{K}_n$ and both Z and $\bigcup_{i,j} W_{ij}$ are equivariantly contractible, $Z \cup \bigcup_{i,j} W_{ij}$ is equivariantly equivalent to the unreduced suspension of $Z \cap \bigcup_{i,j} W_{ij}$.
- Thus the problem is reduced to showing that there is an equivariant map

  $$\tilde{K}_n \to Z \cap \bigcup_{i,j} W_{ij}$$

  which is a non-equivariant equivalent.
- The spaces $U_{ij} = Z \cap W_{ij}$ and all possible intersections of U_{ij} are contractible.
- The poset associated to this covering is isomorphic to the opposite category of the partition lattice.

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Proof

- For each set A of pairs (i,j), let $U_A = \cap_{(i,j)\in A} U_{ij}$.
- A determine a partition of $\{1,2,\ldots,n\}$. U_A depends only on this partition.
- $U_A = \emptyset$ if A is $\hat{1}$, contractible otherwise.
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Homology of Σ_n with coefficient in $Lie(n)$

- When X is an odd-dimensional sphere localized at p and n is not a power of p, then $D_nX \simeq \Omega^\infty Map_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \simeq *$.
- The homology of $Map_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$ decomposes into various pieces, depending on the p-adic representation of n.
- When $n = p^k$, there is an isomorphism as module over the Steenrod algebra between the above homology and the k-fold desuspension of the free graded $\mathbb{Z}/p$-module on the "completely inadmissible" Kudo-Araki-Dyer-Lashof words of length k.
- If W is a based space (spectrum) with an action of Σ_n and $\tilde{H}_*(W)$ is concentrate in degree k, then

$$\tilde{H}_*(W_{h\Sigma_n}) \cong H_{*+k}(\Sigma_n; \tilde{H}_k(W)).$$

Thus the above calculation gives $H_*(\Sigma_n; \epsilon Lie^*(n))$.

Application - The Schwartz genus

$$q_n \colon F(\mathbb{C}, n) \to F(\mathbb{C}, n)_{\Sigma_n}$$

The Schwartz genus of q_n gives a lower bound for the number of functions needed to write the solutions of a polynomial equation of degree n in terms of the coefficients. Facts:

- Since $F(\mathbb{C}, n)_{\Sigma_n}$ is equivalent to a CW-complex of dim $(n-1)$, the genus $g_n \leq n$.
- (De Concini, Procesi, Salvetti) The genus of q_n is less than n iff the induced homomorphism on homology

  $$\rho_* \colon H_{n-1}(Br_n; H^{n-1}(F(\mathbb{C}, n)) \to H_{n-1}(\Sigma_n; H^{n-1}(F(\mathbb{C}, n))$$

  is zero.

Application - The Schwartz genus

- (Fred Cohen) $H^{n-1}(F(\mathbb{C},n)) \cong \epsilon Lie^*(n)$.
- (Arone-Mahowald) Calculation of $H_*(\Sigma_n; \epsilon Lie^*(n))$ shows that if n is not a power of a prime, or twice the power of a prime, then $H_*(\Sigma_n; H^{n-1}(F(\mathbb{C},n))$ vanishes. Thus, the genus g_n is strictly less than n.
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