# A curious subquotient of divided power algebra

Lê Minh Hà

Vietnam National University

The Second East-Asia Conference in Algebraic Topology - Singapore, 2008

# Introduction

- H, Sub-Hopf Algebra of the Steenrod Algebra and the Singer Transfer, Geometry and Topology Monograph, 2008.
- Phan H. Chon and H, Lambda algebra and the Singer transfer, preprint.

# Introduction

- The mod 2 homology of the infinite real projective space:

$$H_*(\mathbb{R}P^\infty) \cong \{a^{(0)}, a^{(1)}, \dots a^{(n)}, \dots\} = \Gamma(a)$$

- Product: $a^{(m)} \cdot a^{(n)} = \binom{m+n}{m} a^{(m+n)}$.
- Coproduct: $\Delta(a^{(m)}) = \sum_{i+j=m} a^{(i)} \otimes a^{(j)}$.
- Action of the Steenrod algebra $\mathscr{A}$: $a^{(m)} Sq^k = \binom{m-k}{k} a^{m-k}$.
-

$$H_*(B(\mathbb{Z}/2)^s) \cong \Gamma(a_1, \dots, a_s).$$

is a commutative and cocomutative Hopf algebra.

- additive basis $a^K = a_1^{(k_1)} \dots a_s^{(k_s)}$ where $K = (k_1, \dots, k_s)$ is a s-tuple of non-negative integers.

# Introduction

- The mod 2 homology of the infinite real projective space:

$$H_*(\mathbb{R}P^\infty) \cong \{a^{(0)}, a^{(1)}, \dots a^{(n)}, \dots\} = \Gamma(a)$$

- Product: $a^{(m)} \cdot a^{(n)} = \binom{m+n}{m} a^{(m+n)}$.
- Coproduct: $\Delta(a^{(m)}) = \sum_{i+j=m} a^{(i)} \otimes a^{(j)}$.
- Action of the Steenrod algebra $\mathscr{A}$: $a^{(m)} Sq^k = \binom{m-k}{k} a^{m-k}$.
-

$$H_*(B(\mathbb{Z}/2)^s) \cong \Gamma(a_1, \dots, a_s).$$

  is a commutative and cocomutative Hopf algebra.
- additive basis $a^K = a_1^{(k_1)} \dots a_s^{(k_s)}$ where $K = (k_1, \dots, k_s)$ is a s-tuple of non-negative integers.

# Product of infinite real projective spaces

- From James's theorem:

$$\tilde{H}_*(\Omega\Sigma\,\mathbb{R}P^\infty) = \oplus_s \tilde{H}_*(B(\mathbb{Z}/2)^s) = \Gamma(a_1, a_2, \ldots).$$

- Let $G(s) = GL(s;\mathbb{F}_2)$ the general linear group on the vector space generated by $a_1, \ldots, a_s$. There is natural action of G(s) on $\Gamma_s = \Gamma(a_1, \ldots, a_s)$; commutes with the $\mathscr{A}$-action.
- Let $P_{\mathscr{A}}\Gamma_s$ be the subring of elements of $\Gamma_s$ which are annihilated by all positively graded Steenrod operations.

Examples:

- $P_{\mathscr{A}}H_*(B\,\mathbb{Z}/2) = \{a^{(0)}, a^{(1)}, a^{(3)}, \ldots, a^{(2^t-1)}, \ldots\}$.
- $P_{\mathscr{A}}H_*(B(\mathbb{Z}/2)^2)$ contains the products $a_1^{(2^{t_1}-1)}a_2^{(2^{t_2}-1)}$ and others, such as

$$a_1^{(1)}a_2^{(2)} + a_1^{(2)}a_2^{(1)}.$$

# Product of infinite real projective spaces

- From James's theorem:

$$\tilde{H}_*(\Omega\Sigma\,\mathbb{R}P^\infty) = \oplus_s \tilde{H}_*(B(\mathbb{Z}/2)^s) = \Gamma(a_1, a_2, \dots).$$

- Let $G(s) = GL(s;\mathbb{F}_2)$ the general linear group on the vector space generated by $a_1, \dots, a_s$. There is natural action of G(s) on $\Gamma_s = \Gamma(a_1, \dots, a_s)$; commutes with the $\mathscr{A}$-action.
- Let $P_{\mathscr{A}}\Gamma_s$ be the subring of elements of $\Gamma_s$ which are annihilated by all positively graded Steenrod operations.

Examples:

- $P_{\mathscr{A}} H_*(B\,\mathbb{Z}/2) = \{a^{(0)}, a^{(1)}, a^{(3)}, \dots, a^{(2^t-1)}, \dots\}$.
- $P_{\mathscr{A}} H_*(B(\mathbb{Z}/2)^2)$ contains the products $a_1^{(2^{t_1}-1)} a_2^{(2^{t_2}-1)}$ and others, such as

$$a_1^{(1)} a_2^{(2)} + a_1^{(2)} a_2^{(1)}.$$

# Product of infinite real projective spaces

- From James's theorem:

$$\tilde{H}_*(\Omega\Sigma\,\mathbb{R}\,P^\infty) = \oplus_s \tilde{H}_*(B(\mathbb{Z}/2)^s) = \Gamma(a_1, a_2, \ldots).$$

- Let $G(s) = GL(s; \mathbb{F}_2)$ the general linear group on the vector space generated by $a_1, \ldots, a_s$. There is natural action of G(s) on $\Gamma_s = \Gamma(a_1, \ldots, a_s)$; commutes with the $\mathscr{A}$-action.
- Let $P_{\mathscr{A}}\Gamma_s$ be the subring of elements of $\Gamma_s$ which are annihilated by all positively graded Steenrod operations.

Examples:

- $P_{\mathscr{A}} H_*(B\,\mathbb{Z}/2) = \{a^{(0)}, a^{(1)}, a^{(3)}, \ldots, a^{(2^t-1)}, \ldots\}$.
- $P_{\mathscr{A}} H_*(B(\mathbb{Z}/2)^2)$ contains the products $a_1^{(2^{t_1}-1)} a_2^{(2^{t_2}-1)}$ and others, such as

$$a_1^{(1)} a_2^{(2)} + a_1^{(2)} a_2^{(1)}.$$

# The subring of $\mathscr{A}$-annihilated elements

- $P_{\mathscr{A}}\tilde{H}_*(\Omega\Sigma\,\mathbb{R}\,P^\infty) = \oplus_s P_{\mathscr{A}}\Gamma_s$.
- Since $G(s)$-action and Steenrod action are commutes, there is induced action of G(s) on $P_{\mathscr{A}}\Gamma_s$.
- Let $P_{\mathscr{A}}\Gamma_s$ be the subring of elements of $\Gamma_s$ which are annihilated by all positively graded Steenrod operations.

Examples:

- $P_{\mathscr{A}}H_*(B\,\mathbb{Z}/2) = \{a^{(0)}, a^{(1)}, a^{(3)}, \ldots, a^{(2^t-1)}, \ldots\}$.
- $P_{\mathscr{A}}H_*(B(\mathbb{Z}/2)^2)$ contains the products $a_1^{(2^{t_1}-1)}a_2^{(2^{t_2}-1)}$ and others, such as

$$a_1^{(1)}a_2^{(2)} + a_1^{(2)}a_2^{(1)}.$$

# The subring of $\mathscr{A}$-annihilated elements

- $P_{\mathscr{A}}\tilde{H}_*(\Omega\Sigma\mathbb{R}P^\infty) = \oplus_s P_{\mathscr{A}}\Gamma_s$.
- Since $G(s)$-action and Steenrod action are commutes, there is induced action of G(s) on $P_{\mathscr{A}}\Gamma_s$.
- Let $P_{\mathscr{A}}\Gamma_s$ be the subring of elements of $\Gamma_s$ which are annihilated by all positively graded Steenrod operations.

Examples:

- $P_{\mathscr{A}}H_*(B\mathbb{Z}/2) = \{a^{(0)}, a^{(1)}, a^{(3)}, \ldots, a^{(2^t-1)}, \ldots\}$.
- $P_{\mathscr{A}}H_*(B(\mathbb{Z}/2)^2)$ contains the products $a_1^{(2^{t_1}-1)}a_2^{(2^{t_2}-1)}$ and others, such as

$$a_1^{(1)}a_2^{(2)} + a_1^{(2)}a_2^{(1)}.$$

# The subring of $\mathscr{A}$-annihilated elements

- $P_{\mathscr{A}} \tilde{H}_*(\Omega\Sigma \mathbb{R} P^\infty) = \oplus_s P_{\mathscr{A}} \Gamma_s$.
- Since $G(s)$-action and Steenrod action are commutes, there is induced action of G(s) on $P_{\mathscr{A}} \Gamma_s$.
- Let $P_{\mathscr{A}} \Gamma_s$ be the subring of elements of $\Gamma_s$ which are annihilated by all positively graded Steenrod operations.

Examples:

- $P_{\mathscr{A}} H_*(B\mathbb{Z}/2) = \{a^{(0)}, a^{(1)}, a^{(3)}, \ldots, a^{(2^t-1)}, \ldots\}$.
- $P_{\mathscr{A}} H_*(B(\mathbb{Z}/2)^2)$ contains the products $a_1^{(2^{t_1}-1)} a_2^{(2^{t_2}-1)}$ and others, such as

$$a_1^{(1)} a_2^{(2)} + a_1^{(2)} a_2^{(1)}.$$

# G(s)-coinvariant

- For each $s$, $P_{\mathscr{A}} H_*(B(\mathbb{Z}/2)^s)$ is also a G(s)-module. So can take $(P_{\mathscr{A}}\Gamma_s)_{G(s)}$.
- There is induced product

$$(P_{\mathscr{A}}\Gamma_s)_{G(s)} \otimes (P_{\mathscr{A}}\Gamma_t)_{G(t)} \to (P_{\mathscr{A}}\Gamma_{s+t})_{G(s+t)}$$

- Let $\mathscr{G}_{\mathscr{A}} = \oplus_s (P_{\mathscr{A}}\Gamma_s)_{G(s)}$.

**Problem**

Determine the structure of the graded algebra $\mathscr{G}_{\mathscr{A}}$.

# G(s)-coinvariant

- For each $s$, $P_{\mathscr{A}} H_*(B(\mathbb{Z}/2)^s)$ is also a G(s)-module. So can take $(P_{\mathscr{A}}\Gamma_s)_{G(s)}$.
- There is induced product

$$(P_{\mathscr{A}}\Gamma_s)_{G(s)} \otimes (P_{\mathscr{A}}\Gamma_t)_{G(t)} \to (P_{\mathscr{A}}\Gamma_{s+t})_{G(s+t)}$$

- Let $\mathscr{G}_{\mathscr{A}} = \oplus_s (P_{\mathscr{A}}\Gamma_s)_{G(s)}$.

**Problem**

Determine the structure of the graded algebra $\mathscr{G}_{\mathscr{A}}$.

# G(s)-coinvariant

- For each $s$, $P_{\mathscr{A}} H_*(B(\mathbb{Z}/2)^s)$ is also a G(s)-module. So can take $(P_{\mathscr{A}}\Gamma_s)_{G(s)}$.
- There is induced product

$$(P_{\mathscr{A}}\Gamma_s)_{G(s)} \otimes (P_{\mathscr{A}}\Gamma_t)_{G(t)} \to (P_{\mathscr{A}}\Gamma_{s+t})_{G(s+t)}$$

- Let $\mathscr{G}_{\mathscr{A}} = \oplus_s (P_{\mathscr{A}}\Gamma_s)_{G(s)}$.

## Problem

Determine the structure of the graded algebra $\mathscr{G}_{\mathscr{A}}$.

# Examples

- $(P_{\mathscr{A}}\Gamma_1)_{G(1)} = \{a^{(0)}, a^{(1)}, \dots, a^{(2^k-1)}, \dots\}$.
- $(P_{\mathscr{A}}\Gamma_2)_{G(2)} = \{a_1^{(2^{k_1}-1)} a_2^{(2^{k_2}-1)} | 0 \le k_1 \le k_2\}$.
- Note that

$$a_1^{(1)}a_2^{(2)} + a_1^{(2)}a_2^{(1)} = (a_1+a_2)^{(3)} + a_1^{(3)} + a_2^{(3)} \sim a_2^{(3)}.$$

- There are relations among $a_1^{(2^{k_1}-1)} a_2^{(2^{k_2}-1)}$. For example,

$$[0,1] = a_1^{(0)}a_2^{(1)} = (a_1+a_2)^{(1)} + a_1^{(1)} \sim 0.$$

- More generally, $a_1^{(2^k-1)} a_2^{(2^{k+1}-1)} \sim 0$.

# Examples

- $(P_{\mathscr{A}}\Gamma_1)_{G(1)} = \{a^{(0)}, a^{(1)}, \ldots, a^{(2^k-1)}, \ldots\}$.
- $(P_{\mathscr{A}}\Gamma_2)_{G(2)} = \{a_1^{(2^{k_1}-1)} a_2^{(2^{k_2}-1)} | 0 \leq k_1 \leq k_2\}$.
- Note that

$$a_1^{(1)}a_2^{(2)} + a_1^{(2)}a_2^{(1)} = (a_1 + a_2)^{(3)} + a_1^{(3)} + a_2^{(3)} \sim a_2^{(3)}.$$

- There are relations among $a_1^{(2^{k_1}-1)} a_2^{(2^{k_2}-1)}$. For example,

$$[0,1] = a_1^{(0)}a_2^{(1)} = (a_1 + a_2)^{(1)} + a_1^{(1)} \sim 0.$$

- More generally, $a_1^{(2^k-1)} a_2^{(2^{k+1}-1)} \sim 0$.

# Examples

- $(P_{\mathscr{A}}\Gamma_1)_{G(1)} = \{a^{(0)}, a^{(1)}, \dots, a^{(2^k-1)}, \dots\}$.
- $(P_{\mathscr{A}}\Gamma_2)_{G(2)} = \{a_1^{(2^{k_1}-1)} a_2^{(2^{k_2}-1)} | 0 \leq k_1 \leq k_2\}$.
- Note that

$$a_1^{(1)} a_2^{(2)} + a_1^{(2)} a_2^{(1)} = (a_1 + a_2)^{(3)} + a_1^{(3)} + a_2^{(3)} \sim a_2^{(3)}.$$

- There are relations among $a_1^{(2^{k_1}-1)} a_2^{(2^{k_2}-1)}$. For example,

$$[0,1] = a_1^{(0)} a_2^{(1)} = (a_1 + a_2)^{(1)} + a_1^{(1)} \sim 0.$$

- More generally, $a_1^{(2^k-1)} a_2^{(2^{k+1}-1)} \sim 0$.

# Rank 3

- $(P_{\mathscr{A}}\Gamma_3)_{G(3)}$ contains $a_1^{(2^{k_1}-1)}a_2^{(2^{k_2}-1)}a_3^{(2^{k_3}-1)}$.
- And an "exotic" element:

$$c_0 = [1,1,6] + [1,2,5] + [1,4,3] + [2,3,3]$$

- In fact, a whole family of exotic elements, starting from $c_0$.
- There are relations among $a_1^{(2^{k_1}-1)}a_2^{(2^{k_2}-1)}a_3^{(2^{k_3}-1)}$. For example

$$a_1^{(0)}a_2^{(0)}a_3^{(3)} \sim a_1^{(1)}a_2^{(1)}a_3^{(1)}.$$

# Rank 3

- $(P_{\mathscr{A}}\Gamma_3)_{G(3)}$ contains $a_1^{(2^{k_1}-1)}a_2^{(2^{k_2}-1)}a_3^{(2^{k_3}-1)}$.
- And an "exotic" element:

$$c_0 = [1,1,6] + [1,2,5] + [1,4,3] + [2,3,3]$$

- In fact, a whole family of exotic elements, starting from $c_0$.
- There are relations among $a_1^{(2^{k_1}-1)}a_2^{(2^{k_2}-1)}a_3^{(2^{k_3}-1)}$. For example

$$a_1^{(0)}a_2^{(0)}a_3^{(3)} \sim a_1^{(1)}a_2^{(1)}a_3^{(1)}.$$

# Rank 3

- $(P_{\mathscr{A}}\Gamma_3)_{G(3)}$ contains $a_1^{(2^{k_1}-1)}a_2^{(2^{k_2}-1)}a_3^{(2^{k_3}-1)}$.
- And an "exotic" element:

$$c_0 = [1,1,6] + [1,2,5] + [1,4,3] + [2,3,3]$$

- In fact, a whole family of exotic elements, starting from $c_0$.
- There are relations among $a_1^{(2^{k_1}-1)}a_2^{(2^{k_2}-1)}a_3^{(2^{k_3}-1)}$. For example

$$a_1^{(0)}a_2^{(0)}a_3^{(3)} \sim a_1^{(1)}a_2^{(1)}a_3^{(1)}.$$

# Rank 3

- $(P_{\mathscr{A}}\Gamma_3)_{G(3)}$ contains $a_1^{(2^{k_1}-1)}a_2^{(2^{k_2}-1)}a_3^{(2^{k_3}-1)}$.
- And an "exotic" element:

$$c_0 = [1,1,6] + [1,2,5] + [1,4,3] + [2,3,3]$$

- In fact, a whole family of exotic elements, starting from $c_0$.
- There are relations among $a_1^{(2^{k_1}-1)}a_2^{(2^{k_2}-1)}a_3^{(2^{k_3}-1)}$. For example

$$a_1^{(0)}a_2^{(0)}a_3^{(3)} \sim a_1^{(1)}a_2^{(1)}a_3^{(1)}.$$

# Rank 4: $P_{\mathscr{A}}\Gamma_4$

There are exotic elements such as:

$$d_0 = x + (2,3)x + (1,3)x + (3155 + 5513 + 5135 + 5315 + 5333),$$

where $x$ is

$$(2255+2165+1256+1166+4253+4163+3263+2435+1436+2336+4433).$$

# Why do we care about $\mathscr{G}_A$?

For each $s > 0$, there exists a diagram:

$$\begin{array}{ccc} P_{\mathscr{A}}\Gamma_{s,*} & \xrightarrow{\quad t_s^{\mathscr{A}} \quad} & H^{s,s+*}(\mathscr{A}) \\ & {\scriptstyle q}\searrow & \nearrow{\scriptstyle \varphi_s^{\mathscr{A}}} \\ & (P_{\mathscr{A}}\Gamma_{s,*})_{G(s)} & \end{array}$$

- (Singer, Mitchell) $t^{\mathscr{A}} = \oplus_s t_s^{\mathscr{A}}$ and $\varphi^{\mathscr{A}}$ are algebra homomorphisms.
- Singer's conjecture: $\varphi^{\mathscr{A}}$ is a monomorphism.

# Why do we care about $\mathscr{G}_A$?

For each $s > 0$, there exists a diagram:

$$\begin{array}{ccc} P_{\mathscr{A}}\Gamma_{s,*} & \xrightarrow{\quad t_s^{\mathscr{A}} \quad} & H^{s,s+*}(\mathscr{A}) \\ & {\scriptstyle q}\searrow \qquad \nearrow{\scriptstyle \varphi_s^{\mathscr{A}}} & \\ & (P_{\mathscr{A}}\Gamma_{s,*})_{G(s)} & \end{array}$$

- (Singer, Mitchell) $t^{\mathscr{A}} = \oplus_s t_s^{\mathscr{A}}$ and $\varphi^{\mathscr{A}}$ are algebra homomorphisms.
- Singer's conjecture: $\varphi^{\mathscr{A}}$ is a monomorphism.

# Facts about $\varphi_s^{\mathscr{A}}$

- (Singer, Kameko-Boardman) $\varphi_s^{\mathscr{A}}$ is an isomorphism for $s \leq 3$.
- One case left for $s = 4$.
- $\varphi^{\mathscr{A}}$ detects the Adams subalgebra of $H^{*,*}(\mathscr{A})$ generated by $h_i \in H^{1,2^i}(\mathscr{A})$.
- Also detects a few other exotic elements such as $d_0, e_0$, but does not detect $g_0$.

# The multiplicative structure of $\mathscr{G}_{\mathscr{A}}$

## Theorem (Anick)

*$P_{\mathscr{A}}\Gamma$ is a tensor algebra.*

Unfortunately, Anick's theorem is non-constructive.
There is also explicit calculation of $P_{\mathscr{A}}\tilde{H}_*(\Omega\Sigma\mathbb{R}P^2)$ by Anick and Peterson.

# Quillen stratification for $\mathscr{G}_{\mathscr{A}}$

- Consider instead $\mathscr{G}_B = \oplus_s (P_B \Gamma_s)_{G(s)}$ where B is a sub-Hopf algebra of $\mathscr{A}$.
- There exists similar transfer maps and diagram where $\mathscr{A}$ is replaced by B.
- Consider the natural algebra homomorphism

$$\mathscr{G}_{\mathscr{A}} \to \lim \mathscr{G}_E,$$

where $E$ runs over the elementary sub-Hopf algebras of $\mathscr{A}$.

# Elementary sub-Hopf algebras of $\mathscr{A}$

- An elementary sub-Hopf algebra $E$ of $\mathscr{A}$ is a bicommutative Hopf algebra such that $e^2 = 0$ for all $e$ in the augmentation ideal $IE$.
- As an algebra, $E$ is isomorphic to the exterior algebra on the $P^s_t$ (dual to $\xi_t^{2^s}$) that it contains.
- $H^*(E) \cong \mathbb{F}_2[h_{ts} | P^s_t \in E]$, where $h_{ts}$ is represented by $[\xi_t^{2^s}]$ in the cobar complex for $E$, so $|h_{ts}| = (1, 2^s(2^t - 1))$.
- The maximal elementary sub-Hopf algebras of $\mathscr{A}$ are of the form $E(m)_* = \mathscr{A}_* / (\xi_1, \ldots, \xi_{m-1}, \xi_m^{2^m}, \xi_{m+1}^{2^m}, \xi_{m+2}^{2^m}, \ldots)$.
- Thus $E(m)$ is generated by the operations $P^s_t$ where $s < m \leq t$.

# A normal sub-Hopf algebras of $\mathscr{A}$

Let $D$ be the following sub-Hopf algebra of $\mathscr{A}$

$$D_* = \mathscr{A}_* /(\xi_1^2, \xi_2^4, \ldots, \xi_t^{2^t}, \ldots).$$

- $D$ contains all $E(m)$.
- $D$ is normal in $\mathscr{A}$. (i.e. $D \cdot \overline{\mathscr{A}} = \overline{\mathscr{A}} \cdot D$.)
- Thus we can define $\mathscr{A} // D = \mathscr{A} \otimes_D \mathbb{F}_2$.
- There is a sequence of normal sub-Hopf algebras

  $$D \subset \cdots \subset D(m) \subset D(m-1) \subset \cdots \subset D(0) = \mathscr{A}$$

  where $D(m)_* = \mathscr{A}_* /(\xi_1^2, \xi_2^4, \ldots, \xi_m^{2^m})$.

# A commutative diagram

$$\begin{array}{ccccc} (P_{\mathscr{A}}\Gamma_s)_{G(s)} & \longrightarrow & (P_D\Gamma_s)_{G(s)} & \longrightarrow & \lim_{\mathscr{E}}(P_E\Gamma_s)_{G(s)} \\ \downarrow{\scriptstyle \varphi_s^{\mathscr{A}}} & & \downarrow{\scriptstyle \varphi_s^{D}} & & \downarrow{\scriptstyle \varphi_s^{\mathscr{E}}} \\ H^{s,s+*}(\mathscr{A}) & \longrightarrow & H^{s,s+*}(D) & \longrightarrow & \lim_{\mathscr{E}} H^{s,s+*}(E) \end{array}$$

The bottom row is an $F$-isomorphism by Palmieri's theorem (after taking $\mathscr{A}$-invariant).

# An $F$-isomorphism conjecture

There is an induced action of $\mathscr{A}//D$ on $P_D\Gamma_{n,*}$ and $(P_D\Gamma_s)_{G(s)}$.

**Conjecture**

The following induced homomorphisms of algebras:

$$\mathscr{G}_{\mathscr{A}} \to (\mathscr{G}_D)^{\mathscr{A}//D},$$

and

$$\mathscr{G}_D \to \lim_{E\in\mathscr{E}} \mathscr{G}_E,$$

are $F$-isomorphism.

# An application

- We construct an invariant-theoretic chain level representation of $t_{st}$ and $\varphi_{\mathscr{A}}$ in the lambda algebra.
- Computed $\mathscr{G}_{E(2)}$.
- As an application, showed that $d_0 \in H^{4,24}(\mathscr{A})$ is in the image of the transfer (and a few others).