Parametrized Borsuk-Ulam problem for projective space bundles

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Introduction

- The n-dimensional sphere S^n is equipped with the antipodal involution. The well known Borsuk-Ulam theorem states that, if $n \geq k$ then for every continuous map $f : S^n \to \mathbb{R}^k$ there exist a point $x \in S^n$ such that $f(x) = f(-x)$.
- There are several generalizations of the theorem in many directions.
- The article [4] by H. Steinlein lists 457 publications concerned with the Borsuk-Ulam theorem.

- One natural generalization is to the setting of fiber bundles, by considering fiber preserving maps $f : SE \to E^{'}$, where SE denotes the total space of the sphere bundle $SE \to B$ associated to a vector bundle $E \to B$ and $E^{'} \to B$ is other vector bundle. This was first done by Jaworowski [2], Dold [1] and Nakaoka [3].
- This can be viewed as the parametrization of the Borsuk-Ulam theorem (parametrized by the base space).

- General formulation of the parametrized Borsuk-Ulam theorem:
  Let G be a compact Lie group. Consider a fiber bundle $\pi : E \to B$ and a vector bundle $\pi^{'} : E^{'} \to B$ such that G acts fiber preserving and freely on E and $E^{'} - 0$, where 0 stands for the zero section of the bundle $\pi^{'} : E^{'} \to B$. For a fiber preserving G-equivariant map $f : E \to E^{'}$, the parametrized version of the Borsuk-Ulam theorem deals in estimating the cohomological dimension of the set $Z_f = \{x \in E \mid f(x) = 0\}$.

Our aim

- Our Aim: To prove parametrized Borsuk-Ulam theorems for bundles whose fibers are mod-2 cohomology real or complex projective spaces with any free involution.
- Main Tool: Characteristic polynomials associated to bundles (introduced by Dold [1] and Nakaoka [3]).

Definitions and notations

- A $finitistic\ space$ (introduced by R.G. Swan) is a paracompact Hausdorff space whose every open covering has a finite dimensional open refinement, where the dimension of a covering is one less than the maximum number of members of the covering which intersect non-trivially. It is a large class of spaces including all compact Hausdorff spaces and all paracompact spaces of finite covering dimension.
- By $X \simeq_2 \mathbb{R}P^n$, we mean that X is a finitistic space having the mod-2 cohomology algebra of $\mathbb{R}P^n$.
- By $X \simeq_2 \mathbb{C}P^n$ we mean that X is a finitistic space having the mod-2 cohomology algebra of $\mathbb{C}P^n$.

- For a paracompact space X, the *cohomological dimension* of X with respect to an abelian group G is the largest positive integer n such that $H^n(X, A; G) \neq 0$ for some closed subspace A of X. Here the cohomology used is the Čech cohomology.
- We denote by $cohom.dim(X)$, the cohomological dimension of X with respect to $\mathbb{Z}_2$.

Free involutions on projective spaces

- The odd dimensional real projective spaces $\mathbb{R}P^{2m+1}$, where $m \geq 0$, admit free involutions. If we denote an element of $\mathbb{R}P^{2m+1}$ by $[x_1, x_2, ..., x_{2m+1}, x_{2m+2}]$, then the map $\mathbb{R}P^{2m+1} \to \mathbb{R}P^{2m+1}$ given by

$$[x_1, x_2, ..., x_{2m+1}, x_{2m+2}] \mapsto [-x_2, x_1, ..., -x_{2m+2}, x_{2m+1}]$$

  is a free involution.

- Similarly, the complex projective space $\mathbb{C}P^m$ admit free involutions when $m \geq 1$ is odd. If we denote an element of $\mathbb{C}P^m$ by $[z_1, z_2, ..., z_m, z_{m+1}]$, then the map

$$[z_1, z_2, ..., z_m, z_{m+1}] \mapsto [-\overline{z}_2, \overline{z}_1, ..., -\overline{z}_{m+1}, \overline{z}_m]$$

  is a free involution.

- When n is even, there is no free involution on a space $X \simeq_2 \mathbb{R}P^n$ or $\mathbb{C}P^n$, for the Floyd's formula

$$\chi(X) + \chi(X^{\mathbb{Z}_2}) = 2\chi(X/\mathbb{Z}_2)$$

gives a contradiction.
- For each $n \geq 2$, the Quaternionic projective space QP^n admit no free involution, which follows from the stronger fact that these spaces have the fixed point property.

Cohomology algebra of orbit spaces

For the purpose of our work, we want to know the cohomology algebra of orbit spaces of free involutions on mod-2 cohomology projective spaces. Before that, recall that for a group G, we have the universal principal G-bundle $G \hookrightarrow E_G \longrightarrow B_G$. For a G-space X, we consider the diagonal action on $X \times E_G$. The projection $X \times E_G \to E_G$ is G-equivariant and on passing to orbit spaces gives a fibration $X \hookrightarrow X_G \longrightarrow B_G$ (called the Borel fibration), where $X_G = (X \times E_G)/G$.

By a theorem of Leray, associated to the fibration, there is a spectral sequence of algebras $\{{E_r}^{*,*}, d_r\}$, converging to $H^*(X_G; \mathbb{Z}_2)$ as an algebra and with

$$ {E_2}^{k,l} = H^k(B_G; \mathcal{H}^l(X; \mathbb{Z}_2)). $$

Using the Leray spectral sequence associated to the fibration $X \hookrightarrow X_G \xrightarrow{\rho} B_G$, we prove:

Theorem A. *If $G = \mathbb{Z}_2$ acts freely on a finitistic space $X \simeq_2 \mathbb{R}P^n$, where n is odd, then*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[u, v]/\langle u^2, v^{\frac{n+1}{2}} \rangle,$$

where deg(u)=1 and deg(v)=2.

Theorem B. *If $G = \mathbb{Z}_2$ acts freely on a finitistic space $X \simeq_2 \mathbb{C}P^n$, where n is odd, then*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[u, v]/\langle u^3, v^{\frac{n+1}{2}} \rangle,$$

where deg(u)=1 and deg(v)=4.

Proof of Theorem A. Let $a \in H^1(X)$ be the generator of $H^*(X)$ and $t \in H^1(B_G)$ be the generator of $H^*(B_G)$. As there are no fixed points, the spectral sequence do not degenerate at the E_2 term, that is, $d_2(1 \otimes a) = t^2 \otimes 1$. Note that

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}$$

is zero for l even and an isomorphism for l odd. Also note that $d_r = 0$ for all $r \geq 3$. Hence $E_\infty^{*,*} = E_3^{*,*}$. This gives

$$E_\infty^{k,l} = \begin{cases} \mathbb{Z}_2 & \text{for } k = 0,\, 1 \text{ and } l = 0,\, 2, \ldots,\, 2m-2 \\ 0 & \text{otherwise.} \end{cases}$$

But

$$H^j(X_G) = \begin{cases} E_\infty^{0,j} & \text{if } j \text{ even} \\ E_\infty^{1,j-1} & \text{if } j \text{ odd.} \end{cases}$$

Therefore

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq j \leq 2m-1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = \rho^*(t) \in E_\infty^{1,0}$ be determined by $t \otimes 1 \in E_2^{1,0}$. Note that $x^2 \in E_\infty^{2,0} = 0$. The element $1 \otimes a^2 \in E_2^{0,2}$ is a permanent cocycle and determines an element $y \in E_\infty^{0,2} = H^2(X_G)$. Also $i^*(y) = a^2$ and $y^m = 0$. Since the multiplication

$$x \cup (-) : H^k(X_G) \to H^{k+1}(X_G)$$

is an isomorphism for $0 \leq k \leq 2m-2$, we have $xy^r \neq 0$ for $0 \leq r \leq m-1$. Therefore we get

$$H^*(X_G) \cong \mathbb{Z}_2[x,y]/\langle x^2, y^m \rangle.$$

As the action of G is free, $H^*(X/G) \cong H^*(X_G)$.

Proof of Theorem B. Analogous.

Characteristic polynomials for $(X \simeq_2 \mathbb{R}P^n, E, \pi, B)$

Before we state our main results, we define characteristic polynomals associated to bundles. We deal the real and the complex case separately. Throughout Čech cohomology with $\mathbb{Z}_2$ coefficients will be used.

Let $(X \simeq_2 \mathbb{R}P^n, E, \pi, B)$ be a fiber bundle with a fiberwise free $\mathbb{Z}_2$-action such that the quotient bundle $(X/G, \overline{E}, \overline{\pi}, B)$ has a cohomology extension of the fiber, that is,
there is a $\mathbb{Z}_2$-module homomorphism

$$\theta : H^*(X/G; \mathbb{Z}_2) \to H^*(\overline{E}; \mathbb{Z}_2)$$

of degree zero such that for any $b \in B$, the composition

$$H^*(X/G; \mathbb{Z}_2) \xrightarrow{\theta} H^*(\overline{E}; \mathbb{Z}_2) \xrightarrow{i_b^*} H^*((X/G)_b; \mathbb{Z}_2)$$

is an isomorphism, where $i_b : (X/G)_b \hookrightarrow \overline{E}$ is the inclusion of the fiber over b.

Since G acts freely on $X \simeq_2 \mathbb{R}P^n$, n is odd and by Theorem A, $H^*(X/G;\mathbb{Z}_2)$ is a free graded algebra generated by the elements

$$1,\ u,\ v,\ uv,\ v^2, ..., uv^{\frac{n-3}{2}},\ v^{\frac{n-1}{2}},\ uv^{\frac{n-1}{2}},$$

subject to the relations $u^2 = 0$ and $v^{\frac{n+1}{2}} = 0$, where $u \in H^1(X/G;\mathbb{Z}_2)$ and $v \in H^2(X/G;\mathbb{Z}_2)$.
By the Leray-Hirsch theorem, there exist elements $a \in H^1(\overline{E})$ and $b \in H^2(\overline{E})$ such that the restriction to a typical fiber $j^* : H^*(\overline{E}) \to H^*(X/G)$ maps $a \mapsto u$ and $b \mapsto v$. Note that $H^*(\overline{E})$ is a $H^*(B)$-module and is generated by the basis

$$1,\ a,\ b,\ ab,\ b^2, ..., ab^{\frac{n-3}{2}},\ b^{\frac{n-1}{2}},\ ab^{\frac{n-1}{2}}.$$

Express the element $b^{\frac{n+1}{2}} \in H^{n+1}(\overline{E})$ in terms of the basis. Therefore,

$$b^{\frac{n+1}{2}} = w_{n+1} + w_n a + w_{n-1} b + \cdots + w_2 b^{\frac{n-1}{2}} + w_1 a b^{\frac{n-1}{2}}$$

where $w_i \in H^i(B)$ are unique elements. Similarly, express the element $a^2 \in H^2(\overline{E})$ as

$$a^2 = \nu_2 + \nu_1 a + \alpha b,$$

where $\nu_i \in H^i(B)$ and $\alpha \in \mathbb{Z}_2$ are unique elements.
Let $H^*(B)[x, y]$ be the polynomial ring over $H^*(B)$ in the indeterminates x and y. The characteristic polynomials in the indeterminates x and y, of degrees respectively 1 and 2, associated to the fiber bundle $(X \simeq_2 \mathbb{R}P^n, E, \pi, B)$ are defined by

$$W_1(x,y) = w_{n+1} + w_n x + w_{n-1} y + \cdots + w_2 y^{\frac{n-1}{2}} + w_1 x y^{\frac{n-1}{2}} + y^{\frac{n+1}{2}}$$

$$\text{and} \quad W_2(x,y) = \nu_2 + \nu_1 x + \alpha y + x^2.$$

On substituting the values for the indeterminates x and y, we obtain the homomorphism of $H^*(B)$-algebras

$$\sigma : H^*(B)[x, y] \to H^*(\overline{E})$$

given by $(x, y) \mapsto (a, b)$ with $Ker(\sigma)$ as the ideal generated by the polynomials $W_1(x, y)$ and $W_2(x, y)$ and hence

$$H^*(B)[x, y]/\langle W_1(x, y), W_2(x, y)\rangle \cong H^*(\overline{E}). \quad (1)$$

We now define the characteristic polynomial associated to the k-dimensional vector bundle $\pi' : E^{'} \to B$ equipped with a fiberwise $\mathbb{Z}_2$-action on $E^{'}$ which is free on $E^{'} - 0$. Let $SE^{'}$ denote the total space of sphere bundle of $\pi' : E^{'} \to B$. Since the action is free on $SE^{'}$, we obtain the projective space bundle $(\mathbb{R}P^{k-1}, \overline{SE'}, \overline{\pi}^{'}, B)$ and the principal $\mathbb{Z}_2$-bundle $SE^{'} \to \overline{SE'}$. We know that

$$H^*(\mathbb{R}P^{k-1}; \mathbb{Z}_2) \cong \mathbb{Z}_2[u^{'}]/\langle {u^{'}}^{k} \rangle,$$

where $u^{'} = g^*(s)$, $s \in H^1(B_G)$ and $g : \mathbb{R}P^{k-1} \to B_G$ is a classifying map for the principal $\mathbb{Z}_2$-bundle $S^{k-1} \to \mathbb{R}P^{k-1}$.

If $h : \overline{SE'} \to B_G$ is a classifying map for the principal $\mathbb{Z}_2$-bundle $SE^{'} \to \overline{SE'}$ and $a^{'} = h^*(s) \in H^1(\overline{SE'})$, then the $\mathbb{Z}_2$-module homomorphism $\theta^{'} : H^*(\mathbb{R}P^{k-1}) \to H^*(\overline{SE'})$ given by $u^{'} \mapsto a^{'}$ is a cohomology extension of the fiber. Again, by the Leray-Hirsch theorem $H^*(\overline{SE'})$ is generated as a $H^*(B)$-module by the basis

$$1, a^{'}, a^{'2}, ..., a^{'k-1}.$$

We write $a^{'k} \in H^k(\overline{SE'})$ as

$$a^{'k} = w_k^{'} + w_{k-1}^{'} a^{'} + \cdots + w_1^{'} a^{'k-1},$$

where $w_i^{'} \in H^i(B)$ are unique elements.

Now the characteristic polynomial in the indeterminate x of degree 1, associated to the vector bundle $\pi' : E^{'} \to B$ is defined as

$$W^{'}(x) = w_k^{'} + w_{k-1}^{'}x + \cdots + w_1^{'}x^{k-1} + x^k.$$

By similar arguements as used above, we have the following isomorphism of $H^*(B)$-algebras

$$H^*(B)[x]/\langle W^{'}(x)\rangle \cong H^*(\overline{SE'})$$

given by $x \mapsto a^{'}$.

Statements of results for $(X \simeq_2 \mathbb{R}P^n, E, \pi, B)$

For a fiber preserving $\mathbb{Z}_2$-equivariant map $f : E \to E'$, we define

$$Z_f = \{x \in E \mid f(x) = 0\}$$

and $\overline{Z_f} = Z_f/\mathbb{Z}_2$ to be the quotient by the free $\mathbb{Z}_2$-action induced on Z_f. Since,

$$H^*(B)[x,y]/\langle W_1(x,y), W_2(x,y)\rangle \cong H^*(\overline{E}),$$

each polynomial $q(x,y)$ in $H^*(B)[x,y]$ defines an element of $H^*(\overline{E})$, which we denote by $q(x,y)|_{\overline{E}}$. We denote by $q(x,y)|_{\overline{Z_f}}$ the image of $q(x,y)|_{\overline{E}}$ by the $H^*(B)$-homomorphism $i^* : H^*(\overline{E}) \to H^*(\overline{Z_f})$, where i^* is the map induced by the inclusion $i : \overline{Z_f} \hookrightarrow \overline{E}$.

Under the above hypothesis and notations, we obtain the following results for the real case:

Theorem 1. *Let $X \simeq_2 \mathbb{R}P^n$. If $q(x,y)$ in $H^*(B)[x,y]$ is a polynomial such that $q(x,y)|_{\overline{Z_f}} = 0$, then there are polynomials $r_1(x,y)$ and $r_2(x,y)$ in $H^*(B)[x,y]$ such that*

$$q(x,y)W^{'}(x) = r_1(x,y)W_1(x,y) + r_2(x,y)W_2(x,y)$$

in the ring $H^*(B)[x,y]$, where $W^{'}(x)$, $W_1(x,y)$ and $W_2(x,y)$ are the characteristic polynomials.

As a corollary we have the following parametrized version of the Borsuk-Ulam theorem.

Corollary 2. *Let* $X \simeq_2 \mathbb{R}P^n$. *If the fiber dimension of* $E' \to B$ *is* k, *then* $q(x,y)|_{\overline{Z_f}} \neq 0$ *for all nonzero polynomials* $q(x,y)$ *in* $H^*(B)[x,y]$, *whose degree in* x *and* y *is less than* $(n-k+1)$. *Equivalently, the* $H^*(B)$*-homomorphism*

$$\sum_{i+j=0}^{n-k} H^*(B)x^i y^j \to H^*(\overline{Z_f})$$

given by $x^i \to x^i|_{\overline{Z_f}}$ *and* $y^j \to y^j|_{\overline{Z_f}}$ *is a monomorphism. As a result, if* $n \geq k$, *then*

$$cohom.dim(Z_f) \geq cohom.dim(B) + (n-k).$$

As an application we prove the following:
Let $(X \simeq_2 \mathbb{R}P^n, E, \pi, B)$ be a fiber bundle with the above hypothesis and let $E^{''} \to B$ be a k-dimensional vector bundle. Let $f : E \to E^{''}$ be a fiber preserving map. Here we do not assume that $E^{''}$ has an involution. Even if $E^{''}$ has an involution, f is not assumed to be $\mathbb{Z}_2$-equivariant. If $T : E \to E$ is a generator of the $\mathbb{Z}_2$ action, then the $\mathbb{Z}_2$-coincidence set of f is defined as

$$A(f) = \{x \in E \mid f(x) = f(T(x))\}.$$

With above hypothesis, we have:
Theorem 3. *If $X \simeq_2 \mathbb{R}P^n$, then* $cohom.dim A(f) \geq cohom.dim(B) + (n - k)$.

Proofs of results for $(X \simeq_2 \mathbb{R}P^n, E, \pi, B)$

Proof of Theorem 1. Let $q(x,y) \in H^*(B)[x,y]$ be such that $q(x,y)|_{\overline{Z_f}} = 0$. By continuity property of Čech cohomology theory, there is an open subset $V \subset \overline{E}$ such that $\overline{Z_f} \subset V$ and $q(x,y)|_V = 0$. Consider the exact cohomology sequence for $(\overline{E}, V)$,

$$\cdots \to H^*(\overline{E}, V) \xrightarrow{j_1^*} H^*(\overline{E}) \to H^*(V) \to H^*(\overline{E}, V) \to \cdots .$$

By exactness, there exist $\mu \in H^*(\overline{E}, V)$ such that $j_1^*(\mu) = q(x,y)|_{\overline{E}}$, where $j_1 : \overline{E} \to (\overline{E}, V)$ is the natural inclusion. The $\mathbb{Z}_2$-equivariant map $f : E \to E^{'}$ gives the map $\overline{f} : \overline{E} - \overline{Z_f} \to \overline{E^{'}} - 0$. The induced map $\overline{f}^* : H^*(\overline{E^{'}} - 0) \to H^*(\overline{E} - \overline{Z_f})$ is a $H^*(B)$-homomorphism. Also we have $W^{'}(a^{'}) = 0$. Therefore,

$$W^{'}(x)|_{\overline{E} - \overline{Z_f}} = W^{'}(a) = W^{'}\left(\overline{f}^*(a^{'})\right) = \overline{f}^*\left(W^{'}(a^{'})\right) = 0.$$

Now consider the exact cohomology sequence for $(\overline{E}, \overline{E} - \overline{Z_f})$,

$$\cdots \to H^*(\overline{E}, \overline{E} - \overline{Z_f}) \xrightarrow{j_2^*} H^*(\overline{E}) \to H^*(\overline{E} - \overline{Z_f}) \to H^*(\overline{E}, \overline{E} - \overline{Z_f}) \to \cdots .$$

Again by exactness, there exist $\lambda \in H^*(\overline{E}, \overline{E} - \overline{Z_f})$ such that $j_2^*(\lambda) = W^{'}(x)|_{\overline{E}}$, where $j_2 : \overline{E} \to (\overline{E}, \overline{E} - \overline{Z_f})$ is the natural inclusion. Thus,

$$q(x,y)W^{'}(x)|_{\overline{E}} = j_1^*(\mu) j_2^*(\lambda) = j^*(\mu \cup \lambda)$$

by the naturality of the cup product. But, $\mu \cup \lambda \in H^*(\overline{E}, V \bigcup (\overline{E} - \overline{Z_f})) = H^*(\overline{E}, \overline{E}) = 0$ and hence $q(x,y)W^{'}(x)|_{\overline{E}} = 0$. Therefore, by (1), there exist polynomials $r_1(x,y)$ and $r_2(x,y)$ in $H^*(B)[x,y]$ such that

$$q(x,y)W^{'}(x) = r_1(x,y)W_1(x,y) + r_2(x,y)W_2(x,y)$$

in $H^*(B)[x,y]$. □

Proof of Corollary 2. Let $q(x, y) \in H^*(B)[x, y]$ be a non zero polynomial such that deg$(q(x, y)) < (n - k + 1)$. If $q(x, y)|_{\overline{Z_f}} = 0$, then by Theorem 1, $q(x, y)W^{'}(x) = r(x, y)W_1(x, y)$ in the ring $H^*(B)[x, y]/\langle W_2(x, y)\rangle$ for some $r(x, y)$ in $H^*(B)[x, y]$. Since deg$(W^{'}(x)) = k$ and deg$(W_1(x, y)) = n + 1$, we have $\deg(q(x, y)) + k = \deg(r(x, y)) + (n + 1)$. This gives deg$(q(x, y)) \geq (n - k + 1)$, which is a contradiction. Hence $q(x, y)|_{\overline{Z_f}} \neq 0$.

Equivalently, the $H^*(B)$-homomorphism

$$\sum_{i+j=0}^{n-k} H^*(B)x^i y^j \to H^*(\overline{Z_f})$$

given by $x^i \to x^i|_{\overline{Z_f}}$ and $y^j \to y^j|_{\overline{Z_f}}$ is a monomorphism. Hence, for $n \geq k$, we have

$$cohom.dim(Z_f) \geq cohom.dim(B) + (n-k),$$

since $cohom.dim(Z_f) \geq cohom.dim(\overline{Z_f})$. $\square$

Proof of Theorem 3. Let $f : E \to E^{''}$ be a fiber preserving map. Take $V = E^{''} \oplus E^{''}$ the Whitney sum of two copies of $E^{''} \to B$. Then $\mathbb{Z}_2$ acts on V by permuting the coordinates and has the diagonal D in V as the fixed point set. Note that D is a k-dimensional subbundle of V and the orthogonal complement $D^{\perp}$ of D is also a k-dimensional subbundle of V. Also note that $D^{\perp}$ is $\mathbb{Z}_2$ invariant and has a $\mathbb{Z}_2$ action which is free outside the zero section. The map $f^{'} : E \to V$ given by

$$f^{'}(x) = \big(f(x), f(T(x))\big)$$

is $\mathbb{Z}_2$-equivariant.

Also the linear projection along the diagonal defines a $\mathbb{Z}_2$-equivariant fiber preserving map $g : V \to D^{\perp}$ such that $g(V - D) \subset D^{\perp} - 0$, where 0 is the zero section of $D^{\perp}$. Let $h = g \circ f^{'}$ be the composition

$$(E, E - A(f)) \to (V, V - D) \to (D^{\perp}, D^{\perp} - 0).$$

Note that $Z_h = h^{-1}(0) = {f^{'}}^{-1}(D) = A(f)$ and $h : E \to D^{\perp}$ is fiber preserving $\mathbb{Z}_2$-equivariant map. Applying Corollary 2 to h, we have $cohom.dim A(f) \geq cohom.dim(B) + (n - k)$. □

Characteristic polynomials for $(X \simeq_2 \mathbb{C}P^n, E, \pi, B)$

Under the same hypothesis on bundles as in the real case and using Theorem B, the characteristic polynomials in the indeterminates x and y, of degrees respectively 1 and 4, associated to the fiber bundle $(X \simeq_2 \mathbb{C}P^n, E, \pi, B)$ are given by

$$W_1(x, y) = w_{2n+2} + w_{2n+1}x + w_{2n}x^2 + \cdots + w_2 x^2 y^{\frac{n-1}{2}} + y^{\frac{n+1}{2}}$$

$$\text{and} \quad W_2(x) = \nu_3 + \nu_2 x + \nu_1 x^2 + x^3.$$

This gives a homomorphism of $H^*(B)$-algebras

$$\sigma : H^*(B)[x, y] \to H^*(\overline{E})$$

given by $(x, y) \mapsto (a, b)$ and with $Ker(\sigma)$ as the ideal generated by the polynomials $W_1(x, y)$ and $W_2(x)$. Hence

$$H^*(B)[x, y]/\langle W_1(x, y), W_2(x)\rangle \cong H^*(\overline{E}).$$

The characteristic polynomial associated to the k-dimensional vector bundle $\pi' : E^{'} \to B$ remains $W^{'}(x)$ as in the real case.

For the complex case, we prove:

Theorem 4. *Let* $X \simeq_2 \mathbb{C}P^n$. *If* $q(x, y)$ *in* $H^*(B)[x, y]$ *is a polynomial such that* $q(x, y)|_{\overline{Z_f}} = 0$, *then there are polynomials* $r_1(x, y)$ *and* $r_2(x, y)$ *in* $H^*(B)[x, y]$ *such that*

$$q(x, y)W^{'}(x) = r_1(x, y)W_1(x, y) + r_2(x, y)W_2(x)$$

in the ring $H^*(B)[x, y]$, *where* $W^{'}(x)$, $W_1(x, y)$ *and* $W_2(x)$ *are the characteristic polynomials.*

Corollary 5. *Let $X \simeq_2 \mathbb{C}P^n$. If the fiber dimension of $E' \to B$ is k, then $q(x,y)|_{\overline{Z_f}} \neq 0$ for all nonzero polynomials $q(x,y)$ in $H^*(B)[x,y]$, whose degree in x and y is less than $(2n-k+2)$. Equivalently, the $H^*(B)$-homomorphism*

$$\sum_{i+j=0}^{2n-k+1} H^*(B)x^i y^j \to H^*(\overline{Z_f})$$

given by $x^i \to x^i|_{\overline{Z_f}}$ and $y^j \to y^j|_{\overline{Z_f}}$ is a monomorphism. As a result, if $2n \geq k$, then

$$cohom.dim(Z_f) \geq cohom.dim(B) + (2n-k+1).$$

Let $f : E \to E^{''}$ be a fiber preserving map. Here we do not assume that $E^{''}$ has an involution. Even if $E^{''}$ has an involution, f is not assumed to be $\mathbb{Z}_2$-equivariant. If $T : E \to E$ is a generator of the $\mathbb{Z}_2$ action, then the $\mathbb{Z}_2$-coincidence set of f is defined as

$$A(f) = \{x \in E \mid f(x) = f(T(x))\}.$$

With the above hypothesis, we have:
Theorem 6. If $X \simeq_2 \mathbb{C}P^n$, then
$cohom.dim A(f) \geq cohom.dim(B) + (2n - k + 1).$

Proofs of results for $(X \simeq_2 \mathbb{C}P^n, E, \pi, B)$

The proofs are analogous to the real case.

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