On mapping class groups of non-orientable surfaces

Miguel A. Xicoténcatl

Center for Research and Advanced Studies
MEXICO

Motivation

$$\Gamma(M_{g,n}^k) \;=\; \pi_0\, \mathsf{Diff}(M_{g,n}^k)$$

Motivation

$$\Gamma(M^k_{g,n}) \;=\; \pi_0 \,\mathsf{Diff}(M^k_{g,n})$$

- Classification of surface bundles (characteristic classes)

Motivation

$$\Gamma(M^k_{g,n}) \;=\; \pi_0\,\mathrm{Diff}(M^k_{g,n})$$

- Classification of surface bundles (characteristic classes)
- N. Wahl: Homology stability of $\Gamma(M_{g,n})$, non-oriented version of the Mumford conjecture

Motivation

$$\Gamma(M^k_{g,n}) \;=\; \pi_0\,\mathsf{Diff}(M^k_{g,n})$$

- Classification of surface bundles (characteristic classes)
- N. Wahl: Homology stability of $\Gamma(M_{g,n})$, non-oriented version of the Mumford conjecture
- E. Hanbury: Homology stability $\Gamma(M^k_{g,n})$

Motivation

$$\Gamma(M^k_{g,n}) \;=\; \pi_0\,\mathrm{Diff}(M^k_{g,n})$$

- Classification of surface bundles (characteristic classes)
- N. Wahl: Homology stability of $\Gamma(M_{g,n})$, non-oriented version of the Mumford conjecture
- E. Hanbury: Homology stability $\Gamma(M^k_{g,n})$
- O. Randall-Williams: Homology of stable non-orientable MCG

Mapping class groups

$$\begin{aligned} M &= \text{compact, connected surface} \\ \mathcal{D}(M) &= \text{group of diffeos } f : M \to M \\ \mathcal{D}_0(M) &= \text{diffeos isotopic to } 1_M \end{aligned}$$

Mapping class groups

$$
\begin{aligned}
M &= \text{compact, connected surface} \\
\mathcal{D}(M) &= \text{group of diffeos } f : M \to M \\
\mathcal{D}_0(M) &= \text{diffeos isotopic to } 1_M
\end{aligned}
$$

Definition

(Extended) mapping class group

$$\Gamma(M) := \pi_0\,\mathcal{D}(M) = \frac{\mathcal{D}(M)}{\mathcal{D}_0(M)}$$

Mapping class groups

$$\begin{aligned} M &= \text{compact, connected surface} \\ \mathcal{D}(M) &= \text{group of diffeos } f : M \to M \\ \mathcal{D}_0(M) &= \text{diffeos isotopic to } 1_M \end{aligned}$$

Definition

(Extended) mapping class group

$$\Gamma(M) \;:=\; \pi_0\, \mathcal{D}(M) \;=\; \frac{\mathcal{D}(M)}{\mathcal{D}_0(M)}$$

For M orientable, usually consider $\Gamma^+(M) := \pi_0\, \mathcal{D}^+(M)$

Mapping class groups

$$\begin{aligned} M &= \text{compact, connected surface} \\ \mathcal{D}(M) &= \text{group of diffeos } f : M \to M \\ \mathcal{D}_0(M) &= \text{diffeos isotopic to } 1_M \end{aligned}$$

Definition

(Extended) mapping class group

$$\Gamma(M) \; := \; \pi_0 \, \mathcal{D}(M) \; = \; \frac{\mathcal{D}(M)}{\mathcal{D}_0(M)}$$

For M orientable, usually consider $\Gamma^+(M) := \pi_0 \, \mathcal{D}^+(M)$

Variations: MCG with marked points $\quad \mathbf{x_0} = \{x_1, \ldots, x_k\}$

$\mathcal{D}^k(M) =$ diffeos leaving $\mathbf{x_0} = \{x_1, \ldots, x_k\}$ fixed

$\mathcal{D}^k(M) =$ diffeos leaving $\mathbf{x_0} = \{x_1, \ldots, x_k\}$ fixed

- MCG with k marked points:

$$\Gamma^k(M) \;=\; \pi_0\, \mathcal{D}^k(M) \;=\; \frac{\mathcal{D}^k(M)}{[\mathcal{D}^k(M)]_0}$$

$\mathcal{D}^k(M) =$ diffeos leaving $\mathbf{x_0} = \{x_1, \ldots, x_k\}$ fixed

- MCG with k marked points:

$$\Gamma^k(M) \; = \; \pi_0\, \mathcal{D}^k(M) \; = \; \frac{\mathcal{D}^k(M)}{[\mathcal{D}^k(M)]_0}$$

- Reduced MCG with k marked points:

$$\widetilde{\Gamma}^k(M) \; = \; \pi_0 \; [\mathcal{D}^k(M) \cap \mathcal{D}_0(M)]$$

$$= \; \frac{\mathcal{D}^k(M) \cap \mathcal{D}_0(M)}{[\mathcal{D}^k(M)]_0}$$

Theorem

For every closed surface M there is a S.E.S.

$$1 \to \widetilde{\Gamma}^k(M) \to \Gamma^k(M) \to \Gamma(M) \to 1$$

Theorem

For every closed surface M there is a S.E.S.

$$1 \to \widetilde{\Gamma}^k(M) \to \Gamma^k(M) \to \Gamma(M) \to 1$$

Proof:

$$\frac{\mathcal{D}^k \cap \mathcal{D}_0}{[\mathcal{D}^k]_0} \to \frac{\mathcal{D}^k}{[\mathcal{D}^k]_0} \to \frac{\mathcal{D}^k}{\mathcal{D}^k \cap \mathcal{D}_0} \cong \frac{\mathcal{D}}{\mathcal{D}_0}$$

Theorem

For every closed surface M there is a S.E.S.

$$1 \to \widetilde{\Gamma}^k(M) \to \Gamma^k(M) \to \Gamma(M) \to 1$$

Proof:

$$\frac{\mathcal{D}^k \cap \mathcal{D}_0}{[\mathcal{D}^k]_0} \to \frac{\mathcal{D}^k}{[\mathcal{D}^k]_0} \to \frac{\mathcal{D}^k}{\mathcal{D}^k \cap \mathcal{D}_0} \cong \frac{\mathcal{D}}{\mathcal{D}_0}$$

Example

Case $M = S^2$

$$1 \to \widetilde{\Gamma}^k(S^2) \to \Gamma^k(S^2) \to \mathbb{Z}/2 \to 1$$

Example

$N_g = \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2 \qquad (g - \text{summands})$

Case $g = 1$

$$1 \to \widetilde{\Gamma}^k(\mathbb{R}P^2) \xrightarrow{\cong} \Gamma^k(\mathbb{R}P^2) \to \{1\} \to 1$$

Example

$N_g = \mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$ (g – summands)

Case $g = 1$

$$1 \to \widetilde{\Gamma}^k(\mathbb{R}P^2) \xrightarrow{\cong} \Gamma^k(\mathbb{R}P^2) \to \{1\} \to 1$$

Case $g = 2$

$$1 \to \widetilde{\Gamma}^k(K) \to \Gamma^k(K) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 1$$

Example

$N_g = \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$ (g – summands)

Case $g = 1$

$$1 \to \widetilde{\Gamma}^k(\mathbb{R}P^2) \xrightarrow{\cong} \Gamma^k(\mathbb{R}P^2) \to \{1\} \to 1$$

Case $g = 2$

$$1 \to \widetilde{\Gamma}^k(K) \to \Gamma^k(K) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 1$$

Case $g = 3$

$$1 \to \widetilde{\Gamma}^k(N_3) \to \Gamma^k(N_3) \to SL(2, \mathbb{Z}) \to 1$$

Configuration spaces

Definition

M = surface (manifold, top. space)

$$F_k(M) = \{(m_1, \ldots, m_k) \in M^k \mid m_i \neq m_j, \text{ if } i \neq j\}$$

Configuration spaces

Definition

M = surface (manifold, top. space)

$$F_k(M) = \{(m_1, \ldots, m_k) \in M^k \mid m_i \neq m_j, \text{ if } i \neq j\}$$

Notice that:

- $\mathcal{D}(M)$ acts transitively on $F_k(M)/\Sigma_k$

Configuration spaces

Definition

M = surface (manifold, top. space)

$$F_k(M) = \{(m_1, \ldots, m_k) \in M^k \mid m_i \neq m_j, \text{ if } i \neq j\}$$

Notice that:

- $\mathcal{D}(M)$ acts transitively on $F_k(M)/\Sigma_k$
- $\mathcal{D}^k(S)$ = stabilizer of $\mathbf{x_0} = \{x_1, \ldots, x_k\}$

Configuration spaces

Definition

M = surface (manifold, top. space)

$$F_k(M) = \{(m_1, \ldots, m_k) \in M^k \mid m_i \neq m_j, \text{ if } i \neq j\}$$

Notice that:

- $\mathcal{D}(M)$ acts transitively on $F_k(M)/\Sigma_k$
- $\mathcal{D}^k(S)$ = stabilizer of $\mathbf{x_0} = \{x_1, \ldots, x_k\}$
- $F_k(M)/\Sigma_k \approx \mathcal{D}(M)/\mathcal{D}^k(M)$

Lemma

If G acts transitively on X, and $x_0 \in X$, there is an equivalence

$$EG \underset{G}{\times} X \simeq BG_{x_0}$$

Lemma

If G acts transitively on X, and $x_0 \in X$, there is an equivalence

$$EG \underset{G}{\times} X \simeq BG_{x_0}$$

Therefore:

$$E\,\mathcal{D}(M) \underset{\mathcal{D}(M)}{\times} F_k(M)/\Sigma_k \simeq B\,\mathcal{D}^k(M)$$

Lemma

If G acts transitively on X, and $x_0 \in X$, there is an equivalence

$$EG \underset{G}{\times} X \simeq BG_{x_0}$$

Therefore:

$$E\,\mathcal{D}(M) \underset{\mathcal{D}(M)}{\times} F_k(M)/\Sigma_k \simeq B\,\mathcal{D}^k(M)$$

$$\pi_1 \left(E\,\mathcal{D}(M) \underset{\mathcal{D}(M)}{\times} F_k(M)/\Sigma_k \right) \cong \pi_0\,\mathcal{D}^k(M)$$

$$\cong \Gamma^k(M)$$

Theorem (Smale)

The inclusion $SO(3)$ in $\mathcal{D}^+(S^2)$ is a homotopy equivalence

Theorem (Smale)

The inclusion $SO(3)$ in $\mathcal{D}^+(S^2)$ is a homotopy equivalence

Therefore:

$$E\,\mathcal{D}^+(S^2) \underset{\mathcal{D}^+(S^2)}{\times} F_k(S)/\Sigma_k \quad \simeq \quad ESO(3) \underset{SO(3)}{\times} F_k(S^2)/\Sigma_k$$

Theorem (Smale)

The inclusion $SO(3)$ in $\mathcal{D}^+(S^2)$ is a homotopy equivalence

Therefore:

$$E\,\mathcal{D}^+(S^2) \underset{\mathcal{D}^+(S^2)}{\times} F_k(S)/\Sigma_k \quad \simeq \quad ESO(3) \underset{SO(3)}{\times} F_k(S^2)/\Sigma_k$$

Theorem (F. Cohen)

The previous space is a $K(\pi, 1)$

Corollary

- $ESO(3) \underset{SO(3)}{\times} F_k(S^2)/\Sigma_k$ is a $K(\Gamma_+^k(S^2), 1)$

Corollary

- $ESO(3) \underset{SO(3)}{\times} F_k(S^2)/\Sigma_k$ is a $K(\Gamma^k_+(S^2), 1)$
- $H^* \, \Gamma^k_+(S^2) \cong H^* \left(ESO(3) \underset{SO(3)}{\times} F_k(S^2)/\Sigma_k \right)$

Corollary

- $ESO(3) \underset{SO(3)}{\times} F_k(S^2)/\Sigma_k$ is a $K(\Gamma^k_+(S^2), 1)$
- $H^* \, \Gamma^k_+(S^2) \cong H^* \left(ESO(3) \underset{SO(3)}{\times} F_k(S^2)/\Sigma_k \right)$

Computed by C.F. Bödigheimer, F. Cohen, D. Peim

Case M non-orientable

Theorem (Earle–Ells, Gramain)

There are homotopy equivalences

- $\mathcal{D}_0(\mathbb{R}P^2) \simeq SO(3)$

Case M non-orientable

Theorem (Earle–Ells, Gramain)

There are homotopy equivalences

- $\mathcal{D}_0(\mathbb{R}P^2) \simeq SO(3)$
- $\mathcal{D}_0(K) \simeq SO(2)$

Case M non-orientable

Theorem (Earle–Ells, Gramain)

There are homotopy equivalences

- $\mathcal{D}_0(\mathbb{R}P^2) \simeq SO(3)$
- $\mathcal{D}_0(K) \simeq SO(2)$
- $\mathcal{D}_0(N_g) \simeq *$ for $g \geq 3$

Case M non-orientable

Theorem (Earle–Ells, Gramain)

There are homotopy equivalences

- $\mathcal{D}_0(\mathbb{R}P^2) \simeq SO(3)$
- $\mathcal{D}_0(K) \simeq SO(2)$
- $\mathcal{D}_0(N_g) \simeq *$ for $g \geq 3$

Thus:

$$E\,\mathcal{D}_0(\mathbb{R}P^2) \underset{\mathcal{D}_0(\mathbb{R}P^2)}{\times} F_k(\mathbb{R}P^2)/\Sigma_k \quad \simeq \quad ESO(3) \underset{SO(3)}{\times} F_k(\mathbb{R}P^2)/\Sigma_k$$

- $E\,\mathcal{D}_0(K) \underset{\mathcal{D}_0(K)}{\times} F_k(K)/\Sigma_k \quad \simeq \quad ESO(2) \underset{SO(2)}{\times} F_k(K)/\Sigma_k$

- $E\,\mathcal{D}_0(K) \underset{\mathcal{D}_0(K)}{\times} F_k(K)/\Sigma_k \quad \simeq \quad ESO(2) \underset{SO(2)}{\times} F_k(K)/\Sigma_k$

- $E\,\mathcal{D}_0(N_g) \underset{\mathcal{D}_0(N_g)}{\times} F_k(N_g)/\Sigma_k \quad \simeq \quad F_k(N_g)/\Sigma_k$

  for $g \geq 3$. Moreover:

- $E\mathcal{D}_0(K) \underset{\mathcal{D}_0(K)}{\times} F_k(K)/\Sigma_k \simeq ESO(2) \underset{SO(2)}{\times} F_k(K)/\Sigma_k$
- $E\mathcal{D}_0(N_g) \underset{\mathcal{D}_0(N_g)}{\times} F_k(N_g)/\Sigma_k \simeq F_k(N_g)/\Sigma_k$

  for $g \geq 3$. Moreover:

Theorem

- $ESO(3) \underset{SO(3)}{\times} F_k(\mathbb{R}P^2)/\Sigma_k$ is a $K(\widetilde{\Gamma}^k(\mathbb{R}P^2), 1)$

- $E\mathcal{D}_0(K) \underset{\mathcal{D}_0(K)}{\times} F_k(K)/\Sigma_k \simeq ESO(2) \underset{SO(2)}{\times} F_k(K)/\Sigma_k$

- $E\mathcal{D}_0(N_g) \underset{\mathcal{D}_0(N_g)}{\times} F_k(N_g)/\Sigma_k \simeq F_k(N_g)/\Sigma_k$

  for $g \geq 3$. Moreover:

Theorem

- $ESO(3) \underset{SO(3)}{\times} F_k(\mathbb{R}P^2)/\Sigma_k$ is a $K(\widetilde{\Gamma}^k(\mathbb{R}P^2), 1)$

- $ESO(2) \underset{SO(2)}{\times} F_k(K)/\Sigma_k$ is a $K(\widetilde{\Gamma}^k(K), 1)$

An application:

Lemma

Asumme $SO(3)$ acts on X and let S^3 act on X via the double cover $S^3 \to SO(3)$. If $ESO(3) \underset{SO(3)}{\times} X$ is a $K(\pi, 1)$ then $ES^3 \underset{S^3}{\times} X$ is a $K(\pi', 1)$, with $\pi' = \pi_1(X)$.

An application:

Lemma

Asumme $SO(3)$ acts on X and let S^3 act on X via the double cover $S^3 \to SO(3)$. If $ESO(3) \underset{SO(3)}{\times} X$ is a $K(\pi, 1)$ then $ES^3 \underset{S^3}{\times} X$ is a $K(\pi', 1)$, with $\pi' = \pi_1(X)$.

Corollary

$$ES^3 \underset{S^3}{\times} F_k(\mathbb{R}P^2)/\Sigma_k \quad \simeq \quad K(\, B_k(\mathbb{R}P^2)\,, 1)$$

$$1 \to \mathbb{Z}/2 \to \pi_1\left(ES^3 \underset{S^3}{\times} X\right) \to \pi_1\left(ESO(3) \underset{SO(3)}{\times} X\right) \to 1$$

$$1 \to \mathbb{Z}/2 \to \pi_1\left(ES^3 \underset{S^3}{\times} X\right) \to \pi_1\left(ESO(3) \underset{SO(3)}{\times} X\right) \to 1$$

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow B_k(\mathbb{R}P^2) \longrightarrow \widetilde{\Gamma}^k(\mathbb{R}P^2) \longrightarrow 1$$

$$1 \to \mathbb{Z}/2 \to \pi_1\left(ES^3 \underset{S^3}{\times} X\right) \to \pi_1\left(ESO(3) \underset{SO(3)}{\times} X\right) \to 1$$

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow B_k(\mathbb{R}P^2) \longrightarrow \widetilde{\Gamma}^k(\mathbb{R}P^2) \longrightarrow 1$$

$$\implies \qquad \widetilde{\Gamma}^k(\mathbb{R}P^2) \;\cong\; B_k(\mathbb{R}P^2)/\text{center}$$

Analyzing the fibrations

$$F_k(\mathbb{R}P^2)/\Sigma_k \;\to\; ESO(3) \underset{SO(3)}{\times} F_k(\mathbb{R}P^2)/\Sigma_k \;\to\; BSO(3)$$

$$F_k(K)/\Sigma_k \;\to\; ESO(2) \underset{SO(2)}{\times} F_k(K)/\Sigma_k \;\to\; BSO(2)$$

Analyzing the fibrations

$$F_k(\mathbb{R}P^2)/\Sigma_k \;\to\; ESO(3) \underset{SO(3)}{\times} F_k(\mathbb{R}P^2)/\Sigma_k \;\to\; BSO(3)$$

$$F_k(K)/\Sigma_k \;\to\; ESO(2) \underset{SO(2)}{\times} F_k(K)/\Sigma_k \;\to\; BSO(2)$$

have isomorphisms:

$$H^*\left(\widetilde{\Gamma}^k(\mathbb{R}P^2)\;;\;\mathbb{F}_2\right) \quad\cong\quad \mathbb{F}_2[w_2, w_3] \otimes H^*(F_k(\mathbb{R}P^2)/\Sigma_k\;;\;\mathbb{F}_2)$$

$$H^*\left(\widetilde{\Gamma}^k(K)\;;\;\mathbb{F}_2\right) \quad\cong\quad \mathbb{F}_2[w_2] \otimes H^*(F_k(K)/\Sigma_k\;;\;\mathbb{F}_2)$$

Labelled configuration space

Definition

$$C(M;X) \quad := \quad \left(\coprod_{k=0}^{\infty} F_k(M) \underset{\Sigma_k}{\times} X^k \right) \Big/ \approx$$

where:

$$[m_1, \ldots, m_k; x_1, \ldots, x_k] \approx [m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1}]$$

if $x_k = *$.

Labelled configuration space

Definition

$$C(M;X) \quad := \quad \left(\coprod_{k=0}^{\infty} F_k(M) \underset{\Sigma_k}{\times} X^k \right) \Big/ \approx$$

where:

$$[m_1, \ldots, m_k; x_1, \ldots, x_k] \approx [m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1}]$$

if $x_k = *$.

Theorem (May'72)

There is an equivalence $C(\mathbb{R}^n, X) \simeq \Omega^n \Sigma^n X$

Labelled configuration space

Definition

$$C(M;X) \quad := \quad \left(\coprod_{k=0}^{\infty} F_k(M) \underset{\Sigma_k}{\times} X^k \right) \Big/ \approx$$

where:

$$[m_1, \ldots, m_k; x_1, \ldots, x_k] \approx [m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1}]$$

if $x_k = *$.

Theorem (May'72)

There is an equivalence $\quad C(\mathbb{R}^n, X) \simeq \Omega^n \Sigma^n X$

Filtration:

$$* = C_0(M;X) \subset \cdots \subset C_k(M;X) \subset \cdots \subset C(M;X)$$

Define: $D_k(M;X) := C_k(M;X) \,/\, C_{k-1}(M;X)$

Define: $D_k(M;X) := C_k(M;X) \,/\, C_{k-1}(M;X)$

Properties (Snaith, Bödigheimer, Cohen, Taylor)

- $C(M;X) \underset{s}{\simeq} \bigvee_{k=1}^{\infty} D_k(M;X)$

Define: $D_k(M;X) := C_k(M;X) \,/\, C_{k-1}(M;X)$

Properties (Snaith, Bödigheimer, Cohen, Taylor)

- $C(M;X) \underset{s}{\simeq} \bigvee_{k=1}^{\infty} D_k(M;X)$
- $D_k(M;S^n) \simeq \Sigma^{kn} F_k(M)/\Sigma_k \vee S^{kn}$

Define: $D_k(M;X) := C_k(M;X) \,/\, C_{k-1}(M;X)$

Properties (Snaith, Bödigheimer, Cohen, Taylor)

- $C(M;X) \;\underset{s}{\simeq}\; \bigvee_{k=1}^{\infty} D_k(M;X)$
- $D_k(M;S^n) \;\simeq\; \Sigma^{kn}\, F_k(M)/\Sigma_k \,\vee\, S^{kn}$
- $H_*(C(M;S^n);\mathbb{F}) \;\cong\; \bigotimes_{q=0}^{m} H_*(\Omega^{m-q}S^{m+n};\mathbb{F})^{\otimes \beta_q}$

Case $M = N_g$

$$H_q(N_g; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & q = 0 \\ (\mathbb{F}_2)^g & q = 1 \\ \mathbb{F}_2 & q = 2 \end{cases}$$

Case $M = N_g$

$$H_q(N_g;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & q=0 \\ (\mathbb{F}_2)^g & q=1 \\ \mathbb{F}_2 & q=2 \end{cases}$$

$$\begin{aligned} H_*(C(N_g;S^n)\ ;\ \mathbb{F}_2) &\cong H_*(S^{n+2}) \otimes H_*(\Omega S^{n+2})^{\otimes g} \otimes H_*(\Omega^2 S^{n+2}) \\ &\cong E[a_{n+2}] \otimes \mathbb{F}_2[x_1,\dots,x_g] \otimes \mathbb{F}_2[y_{(n+1)2^i-1}] \end{aligned}$$

$$|x_i| = n+1$$