On the homotopy type of the complement of complex lines arrangements

Nguyen Viet Dung

Department of Geometry-Topology
Institute of Mathematics, VAST, Hanoi

Singapore December 15, 2008
Let $\mathcal{A}$ be an $\ell$-arrangement, that is a finite set of (affine) hyperplanes in $\mathbb{C}^\ell$. Each hyperplane is defined by a linear form. If all hyperplanes of $\mathcal{A}$ are defined by real defining forms, we have a real arrangement. The complement of $\mathcal{A}$ is the open $2\ell$-manifold $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$.


P. Orlik [*J. of Alg. Geom.* **1** (1992), pp. 147-156] proved the existence of such a complex for an arbitrary arrangement of subspaces, the most general case. However, the cell structure of the Orlik’s complex can not be described explicitly.
For the case of complexification of arrangement of real lines, M.Falk [Invent. Math. 111 (1993), pp. 139-150] suggested another complex which is homotopy equivalent to the complement.

We want to deal with the case of arrangement of complex lines in $\mathbb{C}^2$.

Our motivations are
- The above result of M. Falk;
- A result of A. Ligober, J. Fur Die Reine Und Ang. Math., 367 (1986), pp 103-114], saying that the canonical 2-complex associated to the braid monodromy presentation for the fundamental group of the complement of a plane curve is homotopy to its complement;
- Our previous result, [Kodai Math. J. Volume 22, Number 1 (1999), 46-55], determining the braid monodromy presentation for the fundamental group of the complement of an arrangement of complex lines in $\mathbb{C}^2$. 
Let \( C = \{ f(x, y) = 0 \} \in \mathbb{C}^2 \) be a plane curve and \( pr_1 : \mathbb{C}^2 \to \mathbb{C}^1 \) the projection onto the x-axis. Let \( S(C) = \{ \alpha \in C; \partial f(\alpha)/\partial y = 0 \} \) and \( D(C) = pr_1(S(C)) \). For a point \( \tilde{x} \) of the x-plane \( \mathbb{C}^1 \) let \( \mathbb{C}_{\tilde{x}} = \{ (x, y) \in \mathbb{C}^2 ; x = \tilde{x} \} \).

Obviously, outside the set \( S(C) \) the restriction \( pr_1|_C \) is a trivial bundle. Given a path \( \gamma : I \to \mathbb{C}^1 \setminus D(C) \) on the x-coordinate \( \mathbb{C}^1 \). A trivialization of \( (pr_1, pr_1|_C) \) will induce a homeomorphisms

\[
(pr_1^{-1}(\gamma(0)), pr_1^{-1}(\gamma(0)) \cap C) \longrightarrow (pr_1^{-1}(\gamma(t)), pr_1^{-1}(\gamma(t)) \cap C),
\]

\( t \in [0, 1] \), called the braid homeomorphism defined over the path \( \gamma \), or simply the braid defined over \( \gamma \).

Fix a base point \( x_0 \) of the x-axis, \( x_0 \in \mathbb{C}^1 \setminus D(C) \). When \( \gamma \) is a loop based at \( x_0 \), we obtain a homeomorphism

\[
(\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap C) \longrightarrow (\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap C).
\]
This defines a homomorphism
\[ \theta : \pi_1(\mathbb{C}^1 \setminus D(C); x_0) \rightarrow B[C_{x_0}, C_{x_0} \cap C]. \]

Here \( B[P, K] \) denotes the group of isotopy classes of compact support homeomorphisms of a 2-plane \( P \) preserving a fixed finite subset \( K \subset P \).

**Definition**

The homomorphism \( \theta \) is called the braid monodromy of the curve \( C \).

The braid monodromy is computed in two steps.
1. Suppose that \( D(C) = \{ x_1, \ldots, x_N \} \). For a point \( x_k \in D(C) \), denote \( D^\epsilon_{x_k} \) a small disk of radius \( \epsilon \), centered at \( x_k \) and fix a point \( x^\epsilon_k \) on its boundary \( \partial D^\epsilon_{x_k} \).
   Moving the fiber \( C_{x_k}^\epsilon \) over this point \( x^\epsilon_k \) counterclockwise along the boundary of the disk \( D^\epsilon_{x_k} \) we obtain a homeomorphism of \( C_{x_k}^\epsilon \) into itself, preserving \( C_{x_k}^\epsilon \cap C \).
   It gives rise an element of the braid group \( B[C_{x_k}^\epsilon, C_{x_k}^\epsilon \cap C] \), called the local braid monodromy of \( C \) at \( x_k \).
2. Let $\Gamma_1, \ldots, \Gamma_N$ be a system of simple paths in $\mathbb{C}^1 \setminus D(C)$ going from $x_0$ to $x_i^\epsilon$ respectively. Let $\theta(\Gamma_i)$ be the braid homeomorphism defined over the path $\Gamma_i$. Let $\gamma_i \in \pi_1(\mathbb{C}^1 \setminus D(C))$ be represented by $\Gamma_i.\partial D_\epsilon x_i.\Gamma_i^{-1}$. The set of all those $\gamma_i$’s is a system of generators of $\pi_1(\mathbb{C}^1 \setminus D(C))$. To determine the braid monodromy $\theta$ means to find all $\theta(\gamma_i)$, $1 \leq i \leq N$. Then it is clear that $\theta(\gamma_i)$ can be completely determined by the local braid monodromy at $x_i$ and the braid $\theta(\Gamma_i)$. 
We now consider the case when $\mathcal{C}$ is an arrangement $\mathcal{A}$ of complex lines in $\mathbb{C}^2$.

**Definition**

A multiple point $P$ of the arrangement $\mathcal{A}$ is the intersection of two or more lines of $\mathcal{A}$

$$P = \bigcap_{j=1}^{r} H_{ij},$$

where $H_{ij} \in \mathcal{A}$.

The points of $S(\mathcal{C})$ are multiple points of the arrangement $\mathcal{A}$.

Without loss of generality we can assume that the multiple points of the arrangement $\mathcal{A}$ are distinct by their $x$-coordinates. In other words, the images of multiple points of $\mathcal{A}$ on the $x$-plane $\mathbb{C}^1$ are pairwise distinct.
Suppose that each line $H_i \in \mathcal{A}$ is defined by an equation $y = \alpha_i(x)$, where $\alpha_i$ is a linear function $\alpha_i : \mathbb{C} \rightarrow \mathbb{C}$.

Let $R_i(x) = \Re(\alpha_i(x))$.

For any $1 \leq i < j \leq n$, the subset $L_{i,j}$ of the $x$-axis $\mathbb{C}^1$, defined by

$$L_{i,j} = \{x \in \mathbb{C}^1; R_i(x) = R_j(x)\},$$

is a (real) line in $\mathbb{C}^1$.

**Definition**

We call the set

$$\mathcal{L}(\mathcal{A}) = \{L_{i,j} ; 1 \leq i < j \leq n\}$$

the *labyrinth* of the arrangement $\mathcal{A}$. 
Suppose that $P_k = \cap_{j=1}^{r} H_{ij}$ is a multiple point of $A$ and $x_k \in C^1$ its image under the projection $pr_1$.

It is clear that $x_k$ belongs to the lines $L_{is,it}, 1 \leq s < t \leq r$ of the labyrinth $\mathcal{L}(A)$.

However, there might be another line $L \in \mathcal{L}(A)$, which does not belong to $\{L_{is,it}; 1 \leq s < t \leq r\}$, going through this point $x_k$.

**Remark**

After a suitable change of coordinates we can assume that

(i) For any multiple point $P_k = \cap_{j=1}^{r} H_{ij}$ of $A$ there is not any line of $\mathcal{L}(A)$ except $L_{is,it}; 1 \leq s < t \leq r$, going through $x_k = pr_1(P_k)$.

(ii) Each (real) line of $\mathcal{L}(A)$ is determined by exactly two lines $H_{is}$ and $H_{it}$ of $A$.

Now we show how we can determine the braid monodromy of an arrangement $A$ from its labyrinth.
The intersection \( \mathbb{C}_x \cap \left( \bigcup_{i=1}^n H_i \right) \) of \( \mathbb{C}_x \), the fiber over the point \( \tilde{x} \in \mathbb{C}^1 \setminus D(A) \), with lines of \( A \), consists of \( n \) distinct points.

When we move the point \( \tilde{x} \) along a path in \( \mathbb{C}^1 \setminus D(A) \), the fiber \( \mathbb{C}_x \) will move correspondingly.

These \( n \) points form a braid on \( n \) strings. We will call the string corresponding to the hyperplane \( H_i \) the \( i^{th} \) string.

In general, these points have distinct real parts. A braiding will occur when the path intersects a line of the labyrinth \( \mathcal{L}(A) \).

Suppose that \( \mathcal{P} = \{ P_1, \ldots, P_N \} \) denotes the set of all multiple points of \( A \). For each multiple point \( P_k \), let \( l_k \) be its local index.

Choose a system of simple paths \( \Gamma_k \), \( k = 1, \ldots, N \) as above. It gives us a system of generators \( \{ \gamma_1, \ldots, \gamma_N \} \) of \( \pi_1(\mathbb{C}^1 \setminus D(A)) \).
The local braid monodromy at $P_k$ has been determined by many authors.

The braid over the path $\Gamma_k$ is determined as follows.

- We move $\mathbb{C}_{\tilde{x}}$ from $x_0$ to $x_0^\epsilon$ along the path $\Gamma_k$.

- Suppose $\Gamma_k$ intersects $L_{i,j}$ of the labyrinth $\mathcal{L}(A)$ first. Then we will obtain a braiding of the $i^{th}$ string and $j^{th}$ string.

- The received braid is determined up to sign.

- The sign of this braid depends on the fact that which of these strings moves over the other one. This can also be determined by the labyrinth $\mathcal{L}(A)$.

- Recording successively all these braids when the fiber moves along the path $\Gamma_k$, we will get a braid denoted by $\beta_k$. 
Now the braid monodromy of an arrangement $\mathcal{A}$ of complex lines is determined in the following theorem.

**Theorem**

The braid monodromy of $\mathcal{A}$ is determined by

$$\theta(\gamma_k) = \beta_k \cdot A_{l_k} \cdot \beta_k^{-1},$$

$1 \leq k \leq N$, where $A_{l_k}$ is the full twist on $l_k$, $\beta_k$ is a braid which can be read off from the labyrinth $\mathcal{L}(\mathcal{A})$.

As a consequence we get the braid monodromy presentation of the fundamental group $\pi_1(\mathbb{C}^2 \setminus \bigcup_{H \in \mathcal{A}} H)$ of the complement of arrangement $\mathcal{A}$.

The set $\mathbb{C}_{x_0} \setminus (\mathbb{C}_{x_0} \cap (\bigcup_{H \in \mathcal{A}} H))$ is a punctured complex line with $n$ removed points. Let $g_1, \ldots, g_n$ denote the generators of the free group $\pi_1(\mathbb{C}_{x_0} \setminus (\mathbb{C}_{x_0} \cap (\bigcup_{H \in \mathcal{A}} H)))$. 
Identify these generators with their images in $\mathbb{C}^\ell \setminus \bigcup_{H \in A} H$ via the embedding $\mathbb{C}_{x_0} \setminus (\mathbb{C}_{x_0} \cap (\bigcup_{H \in A} H)) \subset \mathbb{C}^\ell \setminus \bigcup_{H \in A} H$.

The braid group $B[\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap (\bigcup_{H \in A} H)]$ can be considered naturally as a group of automorphisms of $\pi_1(\mathbb{C}^\ell \setminus \bigcup_{H \in A} H)$.

For each multiple point $P_k$, $1 \leq k \leq N$, we denote by $I_k$ the set of indices of all lines of $A$ going through $P_k$. Then

Corollary

The fundamental group of the complement to the arrangement $A$, $\pi_1(\mathbb{C}^\ell \setminus \bigcup_{H \in A} H)$, is generated by elements $g_1, \ldots, g_n$, with the defining relations

$$g_i = \beta_k A_{I_k} \beta_k^{-1} g_i,$$

$i \in I_k$, $k = 1, \ldots, N$. 
Let $x_0$ be a fixed base point in $\mathbb{C}^1 \setminus D(\mathcal{A})$.

First, associate with $x_0$ a cellular complex $C_0 = S^1 \vee \cdots \vee S^1$ and denote each copy of $S^1$ by $e_i; i = 1, \ldots, n$ respectively.

Now, for each multiple point $P_k, 1 \leq k \leq N$, associate a complex $C_k$ as follows.

Remind that $\mathcal{I}_k = \{i_1, \ldots, i_r\}$ is the set of all indices of those lines of $\mathcal{A}$ passing through the multiple point $P_k$. Denote by $\mathcal{A}_k$ the arrangement of hyperplanes $H_{i_s}; s = 1, \ldots r$.

Observe that, locally at the point $P_k$, after a suitable isotopy, we can consider that we have a real arrangement.

Denote by $R_k$ the so-called simplified Randell’s complex corresponding to the arrangement $\mathcal{A}_k$ as defined by M. Falk in [Invent. Math. 111 (1993)].
Then we set $C_k = R_k \lor (\mathbb{S}^1 \lor \ldots \lor \mathbb{S}^1)$. We denote the 1-cells of $R_k$ by $e_{i_s}^{(k)}; s = 1, \ldots, r$ respectively, and each copy of $\mathbb{S}^1$ by $e_i^{(k)}, i \in \{1, \ldots, n\} \setminus \mathcal{I}_k$.

Next we have to attach these complexes $C_i$'s to each other. In fact, each $C_k$ will be attached to $C_0$ and the attachment will be done along the chosen simple path $\Gamma_k$ connecting $x_0$ to $x_k^\epsilon$.

We have two cases.

**Case I:** If the path $\Gamma_k$ does not cut any line of the labyrinth $L(A)$, we will attach a 1-cell $l_k$ connecting the 0-cell of $C_0$ to the 0-cell of $C_k$. Then for each $i = 1, \ldots, n$ we attach a 2-cell having the boundary $e_i.l_k.(e_i^{(k)})^{-1}l_k^{-1}$. 
Case II: If the path \( \Gamma_k \) cuts some lines of the labyrinth \( \mathcal{L}(\mathcal{A}) \), the attachment will be done in several steps.

Step 1: Suppose that beginning from \( x_0 \), \( \Gamma_k \) cuts the line \( L_{m,n} \) first. We take a new copy of complex \( C_0 \), denote it by \( C_0^{(1)} \) and denote its 1-cells by \( g_i^{(1)}, i = 1, ..., n \), respectively. We attach \( C_0^{(1)} \) to \( C_0 \) first by attaching a 1-cell connecting two 0-cells of these complexes. Then, for each \( i \in \{1, ..., n\} \setminus \{m, n\} \) we attach a 2-cell as in the case I. Finally, we attach a new 2-cell having the boundary \( e_m.e_n.(g_m^{(1)})^{-1}.(g_n^{(1)})^{-1} \).

Step 2: Suppose that after \( L_{m,n} \), the path \( \Gamma_k \) will cut next another line \( L_{s,t} \) of \( \mathcal{L}(\mathcal{A}) \). Then we will repeat the step 1, using the the complex \( C_0^{(1)} \) instead of the complex \( C_0 \), the 1-cells \( g_i^{(1)}, i = 1, ..., n \), instead of 1-cells \( e_i; i = 1, ..., n \), and the indices \( s, t \) instead of indices \( m, n \). Continuing this way, after a finite number of steps we will come to a complex \( C_0^{(h)} \) and from here...
the path $\Gamma_k$ will go to the point $x^c_k$, without cutting any other line of the labyrinth. We denote the 1-cells of this complex by $e_i^{(h)}$, $i = 1, \ldots, n$.

Step 3: Now, the complex $C_0^{(h)}$ is attached to the complex $C_k$ in the same way as it has been done in the Case I.

By this attachment procedure, we receive a CW-complex, denoted by $C(\mathcal{A})$.

**Theorem**

*The CW-complex $C(\mathcal{A})$ is homotopy equivalent to the complement of the arrangement $\mathcal{A}$ in $\mathbb{C}^2$.***
\textit{Proof}

The proof is quite elementary and can be deduced from the construction of this CW-complex.

If the arrangement $\mathcal{A}$ is central, as noted above, after a suitable isotopy we can consider it to be a real arrangement. Then, the complex $C(\mathcal{A})$ is the simplified Randell’s complex. And the theorem follows from Falk’s result.

Suppose that the arrangement $\mathcal{A}$ has more than one multiple points. From the complex $C(\mathcal{A})$ we first collapse the new 2-cells occurring in the attachment. By this way, we have identified the 1-cells $e_i^{(k)}, \ i = 1, \ldots, n, \ k = 1, \ldots, N$ to $e_i$ respectively, modulo a conjugation.

Note that according to the construction of the complex $C(\mathcal{A})$ these conjugations are the same conjugations appearing in the determining of braid monodromy as in the above theorem.
Observe that the union of all new 1-cells appear in the construction of $C(A)$ is a copy of the union of all $\Gamma_k$, $k = 1, \ldots, N$. Because $\Gamma_k$, $k = 1, \ldots, N$, is a good system of simple paths, this union is contractible. So, we can collapse all new 1-cells to the only 0-cell of $C_0$.

The resulting complex is the canonical 2-complex associated to the braid monodromy presentation of the fundamental group of the complement given in previous corollary. According to Libgober’s result, it is homotopy equivalent to the complement $M(A)$ of the arrangement $A$. 