

On the homotopy type of the complement of complex lines arrangements

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- Let \mathcal{A} be an ℓ -arrangement, that is a finite set of (affine) hyperplanes in \mathbb{C}^ℓ . Each hyperplane is defined by a linear form. If all hyperplanes of \mathcal{A} are defined by real defining forms, we have a real arrangement. The complement of \mathcal{A} is the open 2ℓ -manifold $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$.
- M.Salvetti [*Invent. Math.***88** (1987), pp. 603-618] constructed a regular CW-complex which is a strong deformation of the complement for the case of real arrangements.
- P.Orlik [*J. of Alg. Geom.***1** (1992), pp.147-156] proved the existence of such a complex for an arbitrary arrangement of subspaces, the most general case. However, the cell structure of the Orlik's complex can not be described explicitly.

- For the case of complexification of arrangement of real lines, M.Falk [*Invent. Math.* **111**(1993), pp. 139-150] suggested another complex which is homotopy equivalent to the complement.
- We want to deal with the case of arrangement of complex lines in \mathbb{C}^2 .
- Our motivations are
 - The above result of M. Falk;
 - A result of A. Ligober, *J. Fur Die Reine Und Ang. Math.*, **367**(1986), pp 103-114], saying that the canonical 2-complex associated to the braid monodromy presentation for the fundamental group of the complement of a plane curve is homotopy to its complement;
 - Our previous result, [*Kodai Math. J. Volume 22, Number 1* (1999), 46-55], determining the braid monodromy presentation for the fundamental group of the complement of an arrangement of complex lines in \mathbb{C}^2 .

1. Braid monodromy

Let $\mathcal{C} = \{f(x, y) = 0\} \in \mathbb{C}^2$ be a plane curve and $pr_1 : \mathbb{C}^2 \rightarrow \mathbb{C}^1$ the projection onto the x -axis.

Let $S(\mathcal{C}) = \{\alpha \in \mathcal{C}; \partial f(\alpha)/\partial y = 0\}$ and $D(\mathcal{C}) = pr_1(S(\mathcal{C}))$. For a point \tilde{x} of the x -plane \mathbb{C}^1 let $\mathbb{C}_{\tilde{x}} = \{(x, y) \in \mathbb{C}^2; x = \tilde{x}\}$.

Obviously, outside the set $S(\mathcal{C})$ the restriction $pr_1|_{\mathcal{C}}$ is a trivial bundle. Given a path $\gamma : I \rightarrow \mathbb{C}^1 \setminus D(\mathcal{C})$ on the x -coordinate \mathbb{C}^1 . A trivialization of $(pr_1, pr_1|_{\mathcal{C}})$ will induce a homeomorphism

$$(pr_1^{-1}(\gamma(0)), pr_1^{-1}(\gamma(0)) \cap \mathcal{C}) \longrightarrow (pr_1^{-1}(\gamma(t)), pr_1^{-1}(\gamma(t)) \cap \mathcal{C}),$$

$t \in [0, 1]$, called the braid homeomorphism defined over the path γ , or simply the braid defined over γ .

Fix a base point x_0 of the x -axis, $x_0 \in \mathbb{C}^1 \setminus D(\mathcal{C})$. When γ is a loop based at x_0 , we obtain a homeomorphism

$$(\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap \mathcal{C}) \longrightarrow (\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap \mathcal{C}).$$

This defines a homomorphism

$$\theta : \pi_1(\mathbb{C}^1 \setminus D(\mathcal{C}); x_0) \longrightarrow B[\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap \mathcal{C}].$$

Here $B[P, K]$ denotes the group of isotopy classes of compact support homeomorphisms of a 2-plane P preserving a fixed finite subset $K \subset P$.

Definition

The homomorphism θ is called the braid monodromy of the curve \mathcal{C}

The braid monodromy is computed in two steps.

1. Suppose that $D(\mathcal{C}) = \{x_1, \dots, x_N\}$. For a point $x_k \in D(\mathcal{C})$, denote $D_{x_k}^\epsilon$ a small disk of radius ϵ , centered at x_k and fix a point x_k^ϵ on its boundary $\partial D_{x_k}^\epsilon$.

Moving the fiber $\mathbb{C}_{x_k^\epsilon}$ over this point x_k^ϵ counterclockwise along the boundary of the disk $D_{x_k}^\epsilon$ we obtain a homeomorphism of $\mathbb{C}_{x_k^\epsilon}$ into itself, preserving $\mathbb{C}_{x_k^\epsilon} \cap \mathcal{C}$.

It gives rise an element of the braid group $B[\mathbb{C}_{x_k^\epsilon}, \mathbb{C}_{x_k^\epsilon} \cap \mathcal{C}]$, called the local braid monodromy of \mathcal{C} at x_k .

2. Let $\Gamma_1, \dots, \Gamma_N$ be a system of simple paths in $\mathbb{C}^1 \setminus D(\mathcal{C})$ going from x_0 to x_j^ϵ respectively.

Let $\theta(\Gamma_i)$ be the braid homeomorphism defined over the path Γ_i . Let $\gamma_i \in \pi_1(\mathbb{C}^1 \setminus D(\mathcal{C}))$ be represented by $\Gamma_i \cdot \partial D_{x_j^\epsilon} \cdot \Gamma_i^{-1}$. The set of all those γ_i 's is a system of generators of $\pi_1(\mathbb{C}^1 \setminus D(\mathcal{C}))$. To determine the braid monodromy θ means to find all $\theta(\gamma_i)$, $1 \leq i \leq N$.

Then it is clear that $\theta(\gamma_i)$ can be completely determined by the local braid monodromy at x_j and the braid $\theta(\Gamma_i)$.

2. Braid monodromy of arrangement of complex lines

We now consider the case when \mathcal{C} is an arrangement \mathcal{A} of complex lines in \mathbb{C}^2 .

Definition

A multiple point P of the arrangement \mathcal{A} is the intersection of two or more lines of \mathcal{A}

$$P = \cap_{j=1}^r H_{i_j},$$

where $H_{i_j} \in \mathcal{A}$.

The points of $S(\mathcal{C})$ are multiple points of the arrangement \mathcal{A} .

Without loss of generality we can assume that the multiple points of the arrangement \mathcal{A} are distinct by their x -coordinates. In other words, the images of multiple points of \mathcal{A} on the x -plane \mathbb{C}^1 are pairwise distinct.

Suppose that each line $H_i \in \mathcal{A}$ is defined by an equation $y = \alpha_i(x)$, where α_i is a linear function $\alpha_i : \mathbb{C} \rightarrow \mathbb{C}$.

Let $R_i(x) = \operatorname{Re}(\alpha_i(x))$.

For any $1 \leq i < j \leq n$, the subset $L_{i,j}$ of the x -axis \mathbb{C}^1 , defined by

$$L_{i,j} = \{x \in \mathbb{C}^1; R_i(x) = R_j(x)\},$$

is a (*real*) line in \mathbb{C}^1 .

Definition

We call the set

$$\mathcal{L}(\mathcal{A}) = \{L_{i,j}; 1 \leq i < j \leq n\}$$

the *labyrinth* of the arrangement \mathcal{A} .

Suppose that $P_k = \bigcap_{j=1}^r H_j$ is a multiple point of \mathcal{A} and $x_k \in \mathbb{C}^1$ its image under the projection pr_1 .

It is clear that x_k belongs to the lines L_{i_s, i_t} , $1 \leq s < t \leq r$ of the labyrinth $\mathcal{L}(\mathcal{A})$.

However, there might be another line $L \in \mathcal{L}(\mathcal{A})$, which does not belong to $\{L_{i_s, i_t}; 1 \leq s < t \leq r\}$, going through this point x_k .

Remark

After a suitable change of coordinates we can assume that

- (i) For any multiple point $P_k = \bigcap_{j=1}^r H_j$ of \mathcal{A} there is not any line of $\mathcal{L}(\mathcal{A})$ except L_{i_s, i_t} ; $1 \leq s < t \leq r$, going through $x_k = pr_1(P_k)$.*
- (ii) Each (real) line of $\mathcal{L}(\mathcal{A})$ is determined by exactly two lines H_{i_s} and H_{i_t} of \mathcal{A} .*

Now we show how we can determine the braid monodromy of an arrangement \mathcal{A} from its labyrinth.

The intersection $\mathbb{C}_{\tilde{x}} \cap (\bigcup_{i=1}^n H_i)$ of $\mathbb{C}_{\tilde{x}}$, the fiber over the point $\tilde{x} \in \mathbb{C}^1 \setminus D(\mathcal{A})$, with lines of \mathcal{A} , consists of n distinct points.

When we move the point \tilde{x} along a path in $\mathbb{C}^1 \setminus D(\mathcal{A})$, the fiber $\mathbb{C}_{\tilde{x}}$ will move correspondingly.

These n points form a braid on n strings. We will call the string corresponding to the hyperplane H_i the i^{th} string.

In general, these points have distinct real parts. A braiding will occur when the path intersects a line of the labyrinth $\mathcal{L}(\mathcal{A})$.

Suppose that $\mathcal{P} = \{P_1, \dots, P_N\}$ denotes the set of all multiple points of \mathcal{A} . For each multiple point P_k , let l_k be its local index.

Choose a system of simple paths Γ_k , $k = 1, \dots, N$ as above. It gives us a system of generators $\{\gamma_1, \dots, \gamma_N\}$ of $\pi_1(\mathbb{C}^1 \setminus D(\mathcal{A}))$.

The local braid monodromy at P_k has been determined by many authors.

The braid over the path Γ_k is determined as follows.

- We move $\mathbb{C}_{\tilde{x}}$ from x_0 to x_k^ϵ along the path Γ_k .
- Suppose Γ_k intersects $L_{i,j}$ of the labyrinth $\mathcal{L}(\mathcal{A})$ first. Then we will obtain a braiding of the i^{th} string and j^{th} string.
- The received braid is determined up to sign.
- The sign of this braid depends on the fact that which of these strings moves over the other one. This can also be determined by the labyrinth $\mathcal{L}(\mathcal{A})$.
- Recording successively all these braids when the fiber moves along the path Γ_k , we will get a braid denoted by β_k .

Now the braid monodromy of an arrangement \mathcal{A} of complex lines is determined in the following theorem.

Theorem

The braid monodromy of \mathcal{A} is determined by

$$\theta(\gamma_k) = \beta_k \cdot A_{l_k} \cdot \beta_k^{-1},$$

$1 \leq k \leq N$, where A_{l_k} is the full twist on l_k , β_k is a braid which can be read off from the labyrinth $\mathcal{L}(\mathcal{A})$.

As a consequence we get the braid monodromy presentation of the fundamental group $\pi_1(\mathbb{C}^2 \setminus \bigcup_{H \in \mathcal{A}} H)$ of the complement of arrangement \mathcal{A} .

The set $\mathbb{C}_{x_0} \setminus (\mathbb{C}_{x_0} \cap (\bigcup_{H \in \mathcal{A}} H))$ is a punctured complex line with n removed points. Let g_1, \dots, g_n denote the generators of the free group $\pi_1(\mathbb{C}_{x_0} \setminus (\mathbb{C}_{x_0} \cap (\bigcup_{H \in \mathcal{A}} H)))$.

Identify these generators with their images in $\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ via the embedding $\mathbb{C}_{x_0} \setminus (\mathbb{C}_{x_0} \cap (\bigcup_{H \in \mathcal{A}} H)) \subset \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$.

The braid group $B[\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap (\bigcup_{H \in \mathcal{A}} H)]$ can be considered naturally as a group of automorphisms of $\pi_1(\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H)$.

For each multiple point P_k , $1 \leq k \leq N$, we denote by \mathcal{I}_k the set of indices of all lines of \mathcal{A} going through P_k . Then

Corollary

The fundamental group of the complement to the arrangement \mathcal{A} , $\pi_1(\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H)$, is generated by elements g_1, \dots, g_n , with the defining relations

$$g_i = \beta_k \cdot A_{l_k} \cdot \beta_k^{-1} \cdot g_i,$$

$$i \in \mathcal{I}_k, k = 1, \dots, N.$$

3. A model for the homotopy type of the complement

Let x_0 be a fixed base point in $\mathbb{C}^1 \setminus D(\mathcal{A})$.

First, associate with x_0 a cellular complex $C_0 = \underbrace{\mathbb{S}^1 \vee \dots \vee \mathbb{S}^1}_n$

and denote each copy of \mathbb{S}^1 by e_i ; $i = 1, \dots, n$ respectively.

Now, for each multiple point P_k , $1 \leq k \leq N$, associate a complex C_k as follows.

Remind that $\mathcal{I}_k = \{i_1, \dots, i_r\}$ is the set of all indices of those lines of \mathcal{A} passing through the multiple point P_k . Denote by \mathcal{A}_k the arrangement of hyperplanes H_{i_s} ; $s = 1, \dots, r$.

Observe that, locally at the point P_k , after a suitable isotopy, we can consider that we have a real arrangement.

Denote by R_k the so-called simplified Randell's complex corresponding to the arrangement \mathcal{A}_k as defined by M. Falk in [*Invent. Math.* **111**(1993)].

Then we set $C_k = R_k \vee \underbrace{(\mathbb{S}^1 \vee \dots \vee \mathbb{S}^1)}_{n-r}$. We denote the 1-cells of

R_k by $e_{i_s}^{(k)}$; $s = 1, \dots, r$ respectively, and each copy of \mathbb{S}^1 by $e_i^{(k)}$, $i \in \{1, \dots, n\} \setminus \mathcal{I}_k$.

Next we have to attach these complexes C_i 's to each other. In fact, each C_k will be attached to C_0 and the attachment will be done along the chosen simple path Γ_k connecting x_0 to x_k^ϵ . We have two cases.

Case 1: If the path Γ_k does not cut any line of the labyrinth $\mathcal{L}(\mathcal{A})$, we will attach a 1-cell l_k connecting the 0-cell of C_0 to the 0-cell of C_k . Then for each $i = 1, \dots, n$ we attach a 2-cell having the boundary $e_i \cdot l_k \cdot (e_i^{(k)})^{-1} l_k^{-1}$.

Case II: If the path Γ_k cuts some lines of the labyrinth $\mathcal{L}(\mathcal{A})$, the attachment will be done in several steps.

Step 1: Suppose that beginning from x_0 , Γ_k cuts the line $L_{m,n}$ first. We take a new copy of complex C_0 , denote it by $C_0^{(1)}$ and denote its 1-cells by $g_i^{(1)}$, $i = 1, \dots, n$, respectively. We attach $C_0^{(1)}$ to C_0 first by attaching a 1-cell connecting two 0-cells of these complexes. Then, for each $i \in \{1, \dots, n\} \setminus \{m, n\}$ we attach a 2-cell as in the case I. Finally, we attach a new 2-cell having the boundary $e_m \cdot e_n \cdot (g_m^{(1)})^{-1} \cdot (g_n^{(1)})^{-1}$.

Step 2: Suppose that after $L_{m,n}$, the path Γ_k will cut next another line $L_{s,t}$ of $\mathcal{L}(\mathcal{A})$. Then we will repeat the step 1, using the the complex $C_0^{(1)}$ instead of the complex C_0 , the 1-cells $g_i^{(1)}$, $i = 1, \dots, n$, instead of 1-cells e_i ; $i = 1, \dots, n$, and the indices s , t instead of indices m , n . Continuing this way, after a finite number of steps we will come to a complex $C_0^{(h)}$ and from here

the path Γ_k will go to the point x_k^ϵ , without cutting any other line of the labyrinth. We denote the 1-cells of this complex by $e_i^{(h)}, i = 1, \dots, n$.

Step 3: Now, the complex $C_0^{(h)}$ is attached to the complex C_k in the same way as it has been done in the Case I.

By this attachment procedure, we receive a CW-complex, denoted by $C(\mathcal{A})$.

Theorem

The CW-complex $C(\mathcal{A})$ is homotopy equivalent to the complement of the arrangement \mathcal{A} in \mathbb{C}^2 .

Proof

The proof is quite elementary and can be deduced from the construction of this CW-complex.

If the arrangement \mathcal{A} is central, as noted above, after a suitable isotopy we can consider it to be a real arrangement. Then, the complex $C(\mathcal{A})$ is the simplified Randell's complex. And the theorem follows from Falk's result.

Suppose that the arrangement \mathcal{A} has more than one multiple points. From the complex $C(\mathcal{A})$ we first collapse the new 2-cells occurring in the attachment. By this way, we have identified the 1-cells $e_i^{(k)}$, $i = 1, \dots, n$, $k = 1, \dots, N$ to e_i respectively, modulo a conjugation.

Note that according to the construction of the complex $C(\mathcal{A})$ these conjugations are the same conjugations appearing in the determining of braid monodromy as in the above theorem.

Observe that the union of all new 1-cells appear in the construction of $C(\mathcal{A})$ is a copy of the union of all $\Gamma_k, k = 1, \dots, N$. Because $\Gamma_k, k = 1, \dots, N$, is a good system of simple paths, this union is contractible. So, we can collapse all new 1-cells to the only 0-cell of C_0 .

The resulting complex is the canonical 2-complex associated to the braid monodromy presentation of the fundamental group of the complement given in previous corollary. According to Libgober's result, it is homotopy equivalent to the complement $M(\mathcal{A})$ of the arrangement \mathcal{A} .