Properties of Bott towers in Toric Topology

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Toric Manifold

- **Toric Variety**: a normal complex algebraic variety with $(\mathbb{C}^*)^n$ action having a dense orbit.
- **Toric manifold**: a compact non-singular toric variety.

We regard the compact torus $T^n$ as the standard subgroup in $(\mathbb{C}^*)^n$, i.e., $T^n = \{(z_1, \ldots, z_n) \in (\mathbb{C}^*)^n : |z_i| = 1\} \cong (S^1)^n$.

The action of $T^n$ on a toric manifold is *locally standard*. 
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The *standard* action $T^n$ on $\mathbb{C}^n$

$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1z_1, \ldots, t_nz_n)$. The orbit space of this action is the positive cone $\mathbb{R}_+^n$.

Globally, the orbit space for a locally standard $T^n$ actions on $M^{2n}$ is an $n$-dimensional manifold with corners.
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1. The action of \(T^n\) on a toric manifold is *locally standard*.

2. The orbit space of a toric manifold with \(T^n\) can be identified with the simple polytope
Quasitoric manifold

By Davis and Januszkiewicz, we have the notion of a topological generalization by taking these two properties

- **Quasitoric Manifold**: a closed smooth manifold $M$ of dim. $2n$ with a smooth $(S^1)^n$ action such that
  - the action is *locally standard*,
  - the orbit space $M/(S^1)^n$ is a simple convex polytope.
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**Example**

- $\mathbb{C}P^n$ with the standard $T^n$-action

  $$(t_1, \ldots, t_n) \cdot [z_0; z_1; \cdots; z_n] = [t_0; t_1z_1; \cdots; t_nz_n]$$

  is a quasitoric manifold over the $n$-simplex $\Delta^n$.

- $\prod \mathbb{C}P^{n_i}$ is a quasitoric manifold over $\prod \Delta^{n_i}$. 
Characteristic Function

- $P$: simple polytope of dim $n$,
- $\mathcal{F}(P) = \{F_1, \ldots, F_m\}$: the set of facets of $P$,
- $M$: a quasitoric manifold over $P$,

$$
\begin{array}{c}
M \\
\downarrow \pi \\
\downarrow \\
P
\end{array}
$$

Note that $\pi^{-1}(F_i)$ is the connected component of the space fixed by certain circle subgroup of $T^n$. Thus, we have

- $\lambda: \mathcal{F}(P) \to H_2(BT) = \text{Hom}(S^1, T^n) = \mathbb{Z}^n$: Characteristic Function with

$\cap F_i$ is a vertex $\Rightarrow \{\lambda(F_i)\}$ is a basis of $\mathbb{Z}^n$
Construction

- $F = \cap_j F_j$ : face of $P$
- $T_F \subset T^n$ : the torus subgroup generated by $\lambda(F_j)$

$$M(\lambda) = P \times T^n \sslash \sim$$

Here

$$(p, g) \sim (q, h) \iff p = q \text{ and } g^{-1}h \in T_{F(p)}$$

where $F(p)$ is the face which contains $p$ in its interior.

- There is a $T^n$-action on $M(\lambda)$

$$ (t_1, \ldots, t_n) \cdot (p, (g_1, \ldots, g_n)) \mapsto (p, (t_1g_1, \ldots, t_ng_n))$$

- $M(\lambda)$ is a quasitoric manifold over $P$. 
(Equivariant) Cohomology ring

- $T \acts M$: A quasitoric manifold with characteristic map $\lambda$
- $P := M/T$: Simple polytope as an orbit space

We have a fibration $\xrightarrow{\pi} \xrightarrow{\pi} \xrightarrow{\pi}$

\[ \xrightarrow{\pi} \xrightarrow{\pi} \xrightarrow{\pi} \xrightarrow{\pi} \]

\[ (\text{known}) \quad H_T^*(M) := H^*(ET \times_T M) \cong \mathbb{Z}(P) : \text{face ring} \]

Through $\pi^* : H^*(BT) \rightarrow H_T^*(M)$,

\[ H_T^*(M) \text{ is an algebra over } H^*(BT) = \mathbb{Z}[t_1, \ldots, t_n] \]

Moreover the *Leray-Serre spectral sequence* collapses at the $E_2$ term. Thus,

\[ H_T^*(M) = H^*(M) \otimes H^*(BT) \]
Assume $\lambda(F_i) = (\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{ni})^t \in \mathbb{Z}^n$.

$$\Lambda := \begin{pmatrix} \lambda(F_1) & \cdots & \lambda(F_m) \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nm} \end{pmatrix}$$

Define linear forms

$$\theta_i := \lambda_{i1}v_1 + \cdots + \lambda_{im}v_m \in \mathbb{Z}[v_1, \ldots, v_m]$$

where $1 \leq i \leq n$. In fact, $\theta_i = \pi^*(t_i)$.

(known) $H^*(M) := H^*_T(M)/J \cong \mathbb{Z}(P)/J$ as rings, where $J$ is the ideal generated by $\theta_1, \ldots, \theta_n$. 
Outline

Introduction
Toric manifold
Cohomology ring of toric manifolds

Bott towers and Bott manifolds

Twist number of Bott manifolds
Bott Tower

\[ B_n \xrightarrow{\pi_m} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{ \text{a point} \}, \]

where \( B_i = P(\eta_i \oplus \mathbb{C}) \) for \( i = 1, \ldots, n \) and \( \eta_i \) is the \( \mathbb{C} \)-line bundle and \( \mathbb{C} \) is the trivial line bundle over \( B_{i-1} \).

- \( \gamma \): be the canonical line-bundle over \( \mathbb{C}P^1 \)
- \( \gamma_i \): the pullback of \( \gamma \) by the projection onto the \( i \)-th factor.

Note that \( \eta_i = \bigotimes_{j < i} \gamma_j^{a_{ji}} \) and the Bott tower structure is completely determined by \( \eta_i \).

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_1 & 1 & \cdots & 0 \\
a_1 & a_2 & \cdots & 1
\end{pmatrix}
\]
We call each $B_j$ a Bott manifold. Note that a Bott manifold carries a natural torus action turning it into a quasitoric manifold over a cube with

$$\Lambda = \begin{pmatrix}
1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & -a_{12} & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -a_{1n} & -a_{2n} & \cdots & -1
\end{pmatrix}$$
Cohomology ring of Bott manifold

A graded algebra $S$ over $\mathbb{Z}$ generated by $v_1, \ldots, v_n$ of degree 2 is called a Bott quadratic algebra (BQ-algebra) over $\mathbb{Z}$ of rank $n$ if

1. $v_k^2 = \sum_{i<k} a_{ik} v_i v_k$ where $c_{ik} \in \mathbb{Z}$ for $1 \leq k \leq n$. (In particular $v_i^2 = 0$.)

2. $\prod_{i=1}^n v_i \neq 0$.


- $M$ : a quasitoric manifold over $P$.

$H^*(M : \mathbb{Z})$ is a BQ-algebra $\implies P \approx I^n$. 

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$I_P = \{v_k v_{n+k} : k = 1, \ldots, n\}$

$J = \{v_k + v_{n+k} = \sum_{i<k} a_{ik} v_{n+i} : k = 1, \ldots, n\}$

\[H^*(M) = \mathbb{Z}[v_1, \ldots, v_{2n}] / I_P + J\]


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Hirzebruch Surface

- $\gamma$: the canonical line bundle over $\mathbb{C}P^1$
- $\mathbb{C}$: the trivial line bundle over $\mathbb{C}P^1$

$M_a := P(\mathbb{C} \oplus \gamma ^a)$ is a Hirzebruch surface. It indeed is a quasitoric manifold over a cube $I^2$ with the characteristic function

$$\Lambda = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -a & -1 \end{pmatrix}$$

$$H^*(M_a) = \mathbb{Z}[v_1, v_2, v_3, v_4]/(v_1 v_3, v_2 v_4, v_1 - v_3, v_2 - av_3 - v_4)$$

$$\cong \mathbb{Z}[v_3, v_4]/(v_3^2, v_4^2 + av_3 v_4)$$
A quasitoric manifold $M$ of dim. $2n$ is equivalent to a Bott manifold $B_n$ if

\[
\begin{array}{ccc}
M & \xrightarrow{f} & B_n \\
\downarrow & & \downarrow \\
I^n & & \\
\end{array}
\]

where $f$ is a weak equivariant homeomorphism.
Not all quasitoric manifolds over a cube are Bott manifolds.

Example

$\mathbb{C}P^2 \# \mathbb{C}P^2$ is a quasitoric manifold over $I^2$ with

$$\Lambda_* = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \end{pmatrix}.$$  

Since it does not admit a complex structure, it is not a Bott manifold.
Quasitoric manifold with Bott tower structure

M. Masuda and T. Panov (2007) and N. Dobrinskaya (2001)

- $M$: quasitoric manifold over $I^n$
- $\Lambda = (E_n | \Lambda_*)$: characteristic matrix of $M$.

TFAE

1. $M$ is equivalent to a Bott manifold;
2. all principal minors of $-\Lambda_*$ are 1;
3. $M$ has a $T^n$-equivariant almost complex structure.
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G and D.Y. Suh

4. $H^*(M)$ is a BQ-algebra.
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Twist Number

A Bott tower

\[ B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{ \text{a point} \}, \]

is called \textit{t-twisted} if only \( t \) of the fibrations are non-trivial.

\[ M : \text{a Bott manifold} \]

\( M \) is called \textit{t-twisted} if \( M \) is homeomorphic to \( B_m \) whose Bott tower structure is \( t \)-twisted. The minimal number of \( t \) is called \textit{twist number} of \( M \).

Lemma

A \( t \)-twisted Bott manifold \( M \) has the Bott tower structure whose only last \( t \) stages are twisted.
Cohomological complexity

Recall BQ-algebra

\[ S = \mathbb{Z}[v_1, \ldots, v_m]/v_k(v_k + f_k) = 0 \text{ for } k = 1, \ldots, m, \]

where \( f_k = \sum_{i < k} a_{ik}v_i. \)

The number of nonzero \( f_k \)'s is called the cohomological index of \( S \). By up to graded ring isomorphism, the cohomological index can be reduced. The minimum is called the cohomological complexity of \( S \).

Note that

the cohomological complexity of \( H^*(M) \) \( \leq \) the twist number of \( M \),

where \( M \) is a Bott manifold.
Theorem

- \( M \): a Bott manifold

the cohomological complexity of \( H^*(M) \) = the twist number of \( M \),
Theorem

\begin{itemize}
\item $M$ : a Bott manifold
\end{itemize}

the cohomological complexity of $H^*(M) =$ the twist number of $M$,

Corollary 1 (agree with the result of M.Masuda and T.Panov (2008))

\begin{itemize}
\item $M$ : a quasitoric manifold
\end{itemize}

$$H^*(M) \cong H^*((\mathbb{C}P^1)^m) \iff M \cong (\mathbb{C}P^1)^m$$
Corollary 2 Quasitoric manifolds whose cohomology ring is BQ-algebra with complexity 1 can be distinguished by their cohomology ring.
C, M. Masuda and D. Y. Suh (unpublished)

- $B_1, B_2$: 1-twisted Bott manifolds

\[ H^*(B_1) \cong H^*(B_2) \iff B_1 \cong B_2 \text{ diffeo.} \]

**Corollary 2** Quasitoric manifolds whose cohomology ring is BQ-algebra with complexity 1 can be distinguished by their cohomology ring.

**Cohomological rigidity problem**

$M_1, M_2$: (quasi) toric manifolds

\[ H^*(M_1) \cong H^*(M_2) \text{ as a ring} \implies M_1 \cong M_2 \]

up to diffeomorphism (or homeomorphism).
Proof

Let \( \{B_m\} \) be a \( t \)-twisted Bott structure of \( M \). We may assume that fibration \( B_j \rightarrow b_{j-1} \) is trivial for \( j = 1, \ldots, m - t \). Let \( s \) be a cohomological complexity of \( M \). Indeed, \( t \geq s \). Suppose that \( t > s \). We have

\[
H^*(B_m) = \mathbb{Z}[x_1, \ldots, x_m]/\{x_i^2 + f_i x_i = 0\}
\]

where \( f_i = \begin{cases} 0 & \text{for } 1 \leq i \leq m - t \\ \sum_{j=1}^{i-1} c_{ij} x_j & \text{otherwise} \end{cases} \)

Since the cohomological complexity of the Bott tower is \( s \), there is an isomorphism \( \psi \) such that

\[
\psi : H^*(B_m) \rightarrow \mathbb{Z}[y_1, \ldots, y_m]/\{y_i^2 + g_i y_i = 0\}
\]

where \( g_i = \begin{cases} 0 & \text{for } 1 \leq i \leq m - s \\ \sum_{j=1}^{i-1} d_{ij} x_j & \text{otherwise} \end{cases} \).
Claim
\[ \exists \, n \, (m - t < n < m) \text{ s.t. } f_n \equiv 0 \pmod{2} \text{ and } f_n^2 = 0 \in H^*(B_{n-1}). \]

Hence we can write as \( f_n + 2w = 0 \). Consider the line bundle \( \mathbb{C} \oplus \gamma f_n \) over \( B_{n-1} \). Then
\[
c(\gamma^w \oplus \gamma^{f_n+w}) = (1 + w)(1 + f_n + w) = 1 + (f_n + 2w) - \frac{f_n^2}{4} = 1.
\]

Lemma
A sum of two line bundles over any Bott manifold is trivial if and only if the total Chern class is 1.

Hence \( \mathbb{C} \oplus \gamma f_n \) is trivial. Since \( P(\mathbb{C} \oplus \gamma f_n) \cong P(\gamma^w \oplus \gamma^{f_n+w}) \), \( B_n \to B_{n-1} \) is trivial fibration. Thus we can reduce twist number to \( t - 1 \). It is contradiction to the minimality of twist number. \( \square \)
Further Works

**Generalized Bott tower**

\[ B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{ \text{a point} \}, \]

where \( B_i \) for \( i = 1, \ldots, m \) is the projectivization of the Whitney sum of \( n_i + 1 \) \( \mathbb{C} \)-line bundles over \( B_{i-1} \). In particular, if \( n_i = 1 \) for all \( i \), then it is called a **Bott tower**. Each \( B_i \) is called **generalized Bott manifolds**.

**Example**

- \( \prod_{j=1}^{m} \mathbb{C}P^{n_j} \) is a generalized Bott tower over \( \prod_{j=1}^{m} \Delta^{n_j} \).
- Hirzebruch surface is a Bott tower over \( I^2 \).
Bundle Structure of generalized Bott tower

**C, M.Masuda and D.Suh (2008)**

- $M$: quasitoric manifold over $\prod_{j=1}^{m} \Delta^{n_j}$
- $\Lambda_*$: associated vector matrix with $M$.

TFAE

1. $M$ is equivalent to a generalized Bott tower;
2. all principal minors of $-\Lambda_*$ are 1;
3. $M$ is equivalent to a quasitoric manifold which admits an invariant almost complex structure under the action.
Partial Results
with S. Park

- $n_1 \leq n_2 \leq \cdots \leq n_m$
- $B$: generalized Bott tower

\[ H^*(M) = H^*(B) \implies M \cong B', \text{homeo.} \]

where $B'$ is some generalized Bott manifold.

C, M.Masuda and D.Suh (2008)

- $M$: quasitoric manifold

\[ H^*(M) = H^*\left(\prod_{j=1}^{m} \mathbb{C}P^{n_j}\right) \iff M \cong \prod_{j=1}^{m} \mathbb{C}P^{n_j} \]
Thank you for listening!