Properties of Bott towers in Toric Topology

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Toric Manifold

- Toric Variety : a normal complex algebraic variety with $(\mathbb{C}^*)^n$ action having a dense orbit.
- Toric manifold : a compact non-singular toric variety.

We regard the compact torus T^n as the standard subgroup in $(\mathbb{C}^*)^n$, i.e., $T^n = \{(z_1, \ldots, z_n) \in (\mathbb{C}^*)^n : |z_i| = 1\} \cong (S^1)^n$.

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The *standard* action T^n on $\mathbb{C}^n$
$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n)$. The orbit space of this action is the positive cone $\mathbb{R}^n_+$.
Globally, the orbit space for a locally standard T^n actions on M^{2n} is an n-dimensional manifold with corners.

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1. The action of T^n on a toric manifold is *locally standard*.
2. The orbit space of a toric manifold with T^n can be identified with the simple polytope

Quasitoric manifold

By Davis and Januszkiewicz, we have the notion of a topological generalization by taking these two properties

- Quasitoric Manifold : a closed smooth manifold M of dim. $2n$ with a smooth $(S^1)^n$ action such that
  - the action is *locally standard*,
  - the orbit space $M/(S^1)^n$ is a simple convex polytope.

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Example

- $\mathbb{C}P^n$ with the standard T^n-action

$$(t_1, \dots, t_n) \cdot [z_0; z_1; \cdots ; z_n] = [t_0; t_1 z_1; \cdots ; t_n z_n]$$

  is a quasitoric manifold over the n-simplex Δ^n.
- $\prod \mathbb{C}P^{n_i}$ is a quasitoric manifold over $\prod \Delta^{n_i}$.

Characteristic Function

- P : simple polytope of dim n,
- $\mathfrak{F}(P) = \{F_1, \ldots, F_m\}$: the set of facets of P,
- M : a quasitoric manifold over P,

$$\begin{tikzcd} M \arrow[d, "\pi"] \\ P \end{tikzcd}$$

Note that $\pi^{-1}(F_i)$ is the connected component of the space fixed by certain circle subgroup of T^n. Thus, we have

- $\lambda : \mathfrak{F}(P) \to H_2(BT) = \text{Hom}(S^1, T^n) = \mathbb{Z}^n$: Characteristic Function with

$$\cap F_i \text{ is a vertex} \Rightarrow \{\lambda(F_i)\} \text{ is a basis of } \mathbb{Z}^n$$

Construction

- $F = \cap_j F_j$: face of P
- $T_F \subset T^n$: the torus subgroup generated by $\lambda(F_j)$

$$M(\lambda) = P \times T^n / \sim$$

Here

$$(p, g) \sim (q, h) \Leftrightarrow p = q \text{ and } g^{-1}h \in T_{F(p)}$$

where $F(p)$ is the face which contains p in its interior.

- There is a T^n-action on $M(\lambda)$

$$(t_1, \ldots, t_n) \cdot (p, (g_1, \ldots, g_n)) \mapsto (p, (t_1 g_1, \ldots, t_n g_n))$$

- $M(\lambda)$ is a quasitoric manifold over P.

(Equivariant) Cohomology ring

- $T \curvearrowright M$: A quasitoric manifold with characteristic map λ
- $P := M/T$: Simple polytope as an orbit space

We have a fibration $M \longrightarrow ET \times_T M \xrightarrow{\pi} BT$

$$\text{(known) } H_T^*(M) := H^*(ET \times_T M) \overset{\sim}{=} \mathbb{Z}(P) : \text{face ring}$$

Through $\pi^* : H^*(BT) \to H_T^*(M)$,

$$H_T^*(M) \text{ is an algebra over } H^*(BT) = \mathbb{Z}[t_1, \ldots, t_n]$$

Moreover the *Leray-Serre spectral sequence* collapses at the E_2 term. Thus,

$$H_T^*(M) = H^*(M) \otimes H^*(BT)$$

Assume $\lambda(F_i) = (\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{ni})^t \in \mathbb{Z}^n$.

$$\Lambda := \begin{pmatrix} \lambda(F_1) & \cdots & \lambda(F_m) \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2m} \\ \ldots & \ldots & \ddots & \ldots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nm} \end{pmatrix}$$

Define linear forms

$$\theta_i := \lambda_{i1} v_1 + \cdots + \lambda_{im} v_m \in \mathbb{Z}[v_1, \ldots, v_m]$$

where $1 \leq i \leq n$. In fact, $\theta_i = \pi^*(t_i)$.

(known) $H^*(M) := H_T^*(M)/J \cong \mathbb{Z}(P)/J$ as rings,

where J is the ideal generated by $\theta_1, \ldots, \theta_n$.

Outline

Introduction

- Toric manifold
- Cohomology ring of toric manifolds

Bott towers and Bott manifolds

Twist number of Bott manifolds

Bott Tower

$$B_n \xrightarrow{\pi_m} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 \ = \{\text{a point}\},$$

where $B_i = P(\eta_i \oplus \underline{\mathbb{C}})$ for $i = 1, \ldots, n$ and η_i is the $\mathbb{C}$-line bundle and $\underline{\mathbb{C}}$ is the trivial line bundle over B_{i-1}.

- γ : be the canonical line-bundle over $\mathbb{C}P^1$
- γ_i : the pullback of γ by the projection onto the i-th factor.

Note that $\eta_i = \bigotimes\limits_{j<i} \gamma_j^{a_{ji}}$ and the Bott tower structure is completely determined by η_i.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_{12} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 1 \end{pmatrix}$$

Bott manifold

We call each B_j a Bott manifold.
Note that a Bott manifold carries a natural torus action turning it into a quasitoric manifold over a cube with

$$\Lambda = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & -a_{12} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{1n} & -a_{2n} & \cdots & -1 \end{array}\right)$$

Cohomology ring of Bott manifold

A graded algebra S over $\mathbb{Z}$ generated by $v_1, \ldots, v_n$ of degree 2 is called a Bott quadratic algebra (BQ-algebra) over $\mathbb{Z}$ of rank n if

1. $v_k^2 = \sum_{i<k} a_{ik} v_i v_k$ where $c_{ik} \in \mathbb{Z}$ for $1 \leq k \leq n$. (In particular $v_i^2 = 0$.)
2. $\prod_{i=1}^n v_i \neq 0$.

M.Masuda, T.Panov(2007) and C, T.Panov and D.Y.Suh(2008)

- M : a quasitoric manifold over P.

$$H^*(M : \mathbb{Z}) \text{ is a BQ-algebra } \Longrightarrow P \approx I^n.$$

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$$I_P = \{v_k v_{n+k} : k = 1, \dots, n\}$$
$$J = \{v_k + v_{n+k} = \sum_{i<k} a_{ik} v_{n+i} : k = 1, \dots, n\}$$

$$H^*(M) = \mathbb{Z}[v_1, \dots, v_{2n}]/I_P + J$$

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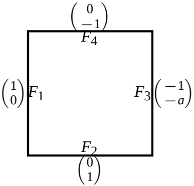
Hirzebruch Surface

- γ : the canonical line bundle over $\mathbb{C}P^1$
- $\underline{\mathbb{C}}$: the trivial line bundle over $\mathbb{C}P^1$

$M_a := P(\underline{\mathbb{C}} \oplus \gamma^{\otimes a})$ is a Hirzebruch surface. It indeed is a quasitoric manifold over a cube I^2 with the characteristic function



$$\Lambda = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -a & -1 \end{pmatrix}$$

$$\begin{aligned} H^*(M_a) &= \mathbb{Z}[v_1, v_2, v_3, v_4]/(v_1v_3, v_2v_4, v_1 - v_3, v_2 - av_3 - v_4) \\ &\cong \mathbb{Z}[v_3, v_4]/(v_3^2, v_4^2 + av_3v_4) \end{aligned}$$

A quasitoric manifold M of dim. $2n$ is equivalent to a Bott manifold B_n if

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr] & & B_n \arrow[dl] \\
& I^n &
\end{tikzcd}$$

where f is a weak equivariant homeomorphism.

Not all quasitoric manifolds over a cube are Bott manifolds.

Example
$\mathbb{C}P^2 \sharp \mathbb{C}P^2$ is a quasitoric manifold over I^2 with

$$\Lambda_* = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \end{pmatrix}.$$

Since it does not admit a complex structure, it is not a Bott manifold.

Quasitoric manifold with Bott tower structure

M.Masuda and T.Panov (2007) and N.Dobrinskaya(2001)

- M: quasitoric manifold over I^n
- $\Lambda = (E_n|\Lambda_*)$: characteristic matrix of M.

TFAE

1. M is equivalent to a Bott manifold;
2. all principal minors of $-\Lambda_*$ are 1;
3. M has a T^n-equivariant almost complex structure.

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C and D.Y. Suh

4. $H^*(M)$ is a BQ-algebra.

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- Twist number of Bott manifolds

Twist Number

A Bott tower

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

is called t-twisted if only t of the fibrations are non-trivial.

- M : a Bott manifold

M is called t-twisted if M is homeomorphic to B_m whose Bott tower structure is t-twisted. The minimal number of t is called twist number of M.

Lemma

A t-twisted Bott manifold M has the Bott tower structure whose only last t stages are twisted.

Cohomological complexity

Recall BQ-algebra

$$S = \mathbb{Z}[v_1, \ldots, v_m]/v_k(v_k + f_k) = 0 \text{ for } k = 1, \ldots, m,$$

where $f_k = \sum_{i<k} a_{ik} v_i$.
The number of nonzero f_k's is called the cohomological index of S. By up to graded ring isomorphism, the cohomological index can be reduced. The minimum is called the cohomological complexity of S.

Note that

the cohomological complexity of $H^*(M) \leq$ the twist number of M,

where M is a Bott manifold.

Theorem

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- M : a Bott manifold

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Corollary 1 (agree with the result of M.Masuda and T.Panov(2008))

- M : a quasitoric manifold

$$H^*(M) \cong H^*((\mathbb{C}P^1)^m) \Longleftrightarrow M \cong (\mathbb{C}P^1)^m$$

C, M.Masuda and D.Y.Suh(unpublished)

- B_1, B_2 : 1-twisted Bott manifolds

$$H^*(B_1) \cong H^*(B_2) \Longleftrightarrow B_1 \underset{\text{diffeo.}}{\cong} B_2$$

Corollary 2 Quasitoric manifolds whose cohomology ring is BQ-algebra with complexity 1 can be distinguished by their cohomology ring.

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Cohomological rigidity problem

M_1, M_2 : (quasi) toric manifolds

$$H^*(M_1) \cong H^*(M_2) \text{ as a ring} \Longrightarrow M_1 \cong M_2$$

up to diffeomorphism (or homeomorphism).

Proof

Let $\{B_m\}$ be a t-twisted Bott structure of M. We may assume that fibration $B_j \to b_{j-1}$ is trivial for $j = 1, \dots, m-t$. Let s be a cohomological complexity of M. Indeed, $t \geq s$. Suppose that $t > s$. We have

$$H^*(B_m) = \mathbb{Z}[x_1, \dots, x_m]/\{x_i^2 + f_i x_i = 0\}$$

$$\text{where } f_i = \begin{cases} 0 & \text{for } 1 \leq i \leq m-t \\ \sum_{j=1}^{i-1} c_{ij}x_j & \text{otherwise} \end{cases}$$

Since the cohomological complexity of the Bott tower is s, there is an isomorphism ψ such that

$$\psi : H^*(B_m) \to \mathbb{Z}[y_1, \dots, y_m]/\{y_i^2 + g_i y_i = 0\}$$

$$\text{where } g_i = \begin{cases} 0 & \text{for } 1 \leq i \leq m-s \\ \sum_{j=1}^{i-1} d_{ij}x_j & \text{otherwise} \end{cases} .$$

Claim

$\exists\, n$ ($m - t < n < m$) s.t. $f_n \equiv 0 (\text{mod } 2)$ and $f_n^2 = 0 \in H^*(B_{n-1})$.

Hence we can write as $f_n + 2w = 0$. Consider the line bundle $\underline{\mathbb{C}} \oplus \gamma^{f_n}$ over B_{n-1}. Then $c(\gamma^w \oplus \gamma^{f_n+w}) = (1+w)(1+f_n+w) = 1 + (f_n+2w) - \frac{f_n^2}{4} = 1.$

Lemma

A sum of two line bundles over any Bott manifold is trivial if and only if the total Chern class is 1.

Hence $\underline{\mathbb{C}} \oplus \gamma^{f_n}$ is trivial. Since $P(\underline{\mathbb{C}} \oplus \gamma^{f_n}) \cong P(\gamma^w \oplus \gamma^{f_n+w})$, $B_n \to B_{n-1}$ is trivial fibration. Thus we can reduce twist number to $t-1$. It is contradiction to the minimality of twist number. $\square$

Further Works

Generalized Bott tower

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 \ = \{\text{a point}\},$$

where B_i for $i = 1, \ldots, m$ is the projectivization of the Whitney sum of $n_i + 1$ $\mathbb{C}$-line bundles over B_{i-1}. In particular, if $n_i = 1$ for all i, then it is called a Bott tower. Each B_i is called generalized Bott manifolds.

Example

- $\prod_{j=1}^{m} \mathbb{C}P^{n_j}$ is a generalized Bott tower over $\prod_{j=1}^{m} \Delta^{n_j}$.
- Hirzebruch surface is a Bott tower over I^2.

Bundle Structure of generalized Bott tower

C, M.Masuda and D.Suh (2008)

- M: quasitoric manifold over $\prod_{j=1}^{m} \Delta^{n_j}$
- Λ_* : associated vector matrix with M.

TFAE

1. M is equivalent to a generalized Bott tower;
2. all principal minors of $-\Lambda_*$ are 1;
3. M is equivalent to a quasitoric manifold which admits an invariant almost complex structure under the action.

Partial Results

with S. Park

- $n_1 \leq n_2 \leq \cdots \leq n_m$
- B : generalized Bott tower

$$H^*(M) = H^*(B) \Longrightarrow M \underset{\text{homeo.}}{\cong} B',$$

where B' is some generalized Bott manifold.

C, M.Masuda and D.Suh (2008)

- M: quasitoric manifold

$$H^*(M) = H^*(\prod_{j=1}^{m} \mathbb{C}P^{n_j}) \Longleftrightarrow M \cong \prod_{j=1}^{m} \mathbb{C}P^{n_j}$$

Thank you for listening!