

Twisting Segal's K -Homology Theory

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How to Twist Cohomology Theories

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Donovan and Karoubi tried to introduce local coefficients in K -theory. Their construction can be thought of as one of the origins of the twisted K -theory.
J. Rosenberg (1989) independently considered twistings of K -theory of C^* -algebras.

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- Witten (1998).
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Twisted K -theory can be used to study D-branes.
- Atiyah and Segal (2004).
- More general twisted cohomology theories.

Twisting of representable functors

Suppose a functor

$$F : \mathbf{Spaces}^{op} \longrightarrow \mathbf{Abelian\ Groups}$$

is represented by a space BF

$$F(X) \cong [X, BF] = \pi_0(\mathrm{Map}(X, BF)) = \pi_0(\Gamma(X \times BF \rightarrow X)).$$

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Idea

$BF \rightarrow X \times BF \rightarrow X$: the trivial bundle

\Downarrow

$BF \rightarrow E \rightarrow X$: more general fiber bundles.

The definition of twisting of a representable functor

Data:

- G : a topological group.
- $\tau : G \rightarrow \text{Homeo}(BF)$: an action of G on BF .
- $\varphi : X \rightarrow BG$.

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Make a bundle:

- $E_\varphi = \varphi^*(EG) \rightarrow X$: the principal G -bundle corresponding to φ .
- $p_\varphi : E_\varphi \times_G BF \rightarrow X$: the associated BF -bundle.

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$$\tau F(\varphi) = \pi_0(\Gamma(p_\varphi)).$$

$$\tau F : (\mathbf{Spaces} \downarrow BG)^{op} \longrightarrow \mathbf{Abelian Groups}.$$

Twisting cohomology theories

Let $\tilde{E}^*(-)$ be a cohomology theory represented by a spectrum E .

$$\tilde{E}^n : \mathbf{Spaces}_* \longrightarrow \mathbf{Abelian\ Groups} .$$

- We need a based version of twisting of representable functors.
- Do we need modern fancy spectra?
 - S -modules, symmetric spectra, orthogonal spectra, \dots

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 $E = \{E_n, \varepsilon_n\}$

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- There are ways to work in the categories of fancy spectra. For example,
 - Waldmüller (arXiv:math/0611225).

Twisting cohomology theories

Data:

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- An action of G on E .
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An action of G on E is a sequence of based actions

$$\tau_n : G \longrightarrow \text{Homeo}_*(E_n)$$

making the following diagram commutative:

$$\begin{array}{ccc} \text{Homeo}_*(E_n) & \longrightarrow & \text{Homeo}_*(\Omega E_{n+1}) \\ \uparrow \tau_n & & \uparrow \\ G & \xrightarrow{\tau_{n+1}} & \text{Homeo}_*(E_{n+1}) \end{array}$$

Twisting of spectra

- $p_{G,E_n} : EG \times_G E_n \longrightarrow BG$.
- $s_{\tau_n} : BG \longrightarrow EG \times_G E_n$: the zero section.
- (p_{G,E_n}, s_{τ_n}) defines an object in the category \mathbf{Spaces}_{BG} of ex-spaces over BG .
- The structure maps of E make the sequence (p_{G,E_n}, s_{τ_n}) into a spectrum in \mathbf{Spaces}_{BG} .
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Twisting is ...

$$\left. \begin{array}{l} E : \text{a spectrum} \\ \tau : \text{an action of } G \text{ on } E \end{array} \right\} \rightsquigarrow {}_{\tau}E : \text{a spectrum in } \mathbf{Spaces}_{BG}.$$

Twisting of cohomology and homology theories

$${}_{\tau}\tilde{E}^n(\varphi, s) = \operatorname{colim}_k [(\varphi, s), \Omega_{BG}^k({}_{\tau}E)_{n+k}]_{BG}.$$

${}_{\tau}\tilde{E}^* : \mathbf{Spaces}_{BG}^{op} \longrightarrow \mathbf{Graded\ Abelian\ Groups} .$

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$$\Omega^{\infty}({}_{\tau}E \wedge (\varphi, s)) = \operatorname{colim}_k \Omega^k(\varphi^*(EG) \times_G E_{n+k} / (s_{\tau_n}(X) \cup \{*\} \times E_{n+k})).$$

$$\begin{array}{ccc}
 \mathbf{Spaces}_{BG} & \xrightarrow{\Omega^{\infty}({}_{\tau}E \wedge (-))} & \mathbf{Spaces}_* \\
 & \searrow \tilde{E}_* & \downarrow \pi_* \\
 & & \mathbf{Graded Abelian Groups}
 \end{array}$$

Atiyah-Segal Twisting of K -Theory

Representing the complex K -theory

When we try to twist a representable functor $F \dots$

- There are many choices for the representing space BF . Although they are unique up to homotopy.
- We need a topological group G acting on BF .

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A choice by Atiyah and Segal

H : a separable infinite dimensional Hilbert space over \mathbb{C} .

$$K(X) \cong [X, \text{Fred}(H)],$$

$$G = PU(H) = U(H)/U(1).$$

$PU(H)$

- $PU(H)$ acts on $\text{Fred}(H)$ by conjugation

$$\tau_{AS} : PU(H) \times \text{Fred}(H) \longrightarrow \text{Fred}(H).$$

- This action gives rise to a twisting of K -theory

$$\tau_{AS}K : (\mathbf{Spaces} \downarrow BPU(H))^{op} \longrightarrow \mathbf{Abelian Groups}.$$

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(Atiyah-Segal twisted K -theory)

- $U(H) \simeq *$.
- $PU(H) \simeq BS^1 \simeq K(\mathbb{Z}, 2)$.
- $BPU(H) \simeq K(\mathbb{Z}, 3)$.
- $\tau_{AS}K : (\mathbf{Spaces} \downarrow K(\mathbb{Z}, 3))^{op} \longrightarrow \mathbf{Abelian Groups}$.
- For $\alpha = [\varphi] \in H^3(X; \mathbb{Z})$, $\tau_{AS}K(\varphi)$ is often denoted by $K_\alpha(X)$.
- $\tau_{AS}K$ can be made into a cohomology theory.

Segal's Connective K -Homology

Segal's functor

Segal constructed a functor $ko(-)$ in 1977:

$$\begin{array}{ccc} \mathbf{Spaces}_* & \xrightarrow{\widetilde{ko}_n(-)} & \mathbf{Abelian\ Groups} \\ & \searrow^{ko(-)} & \nearrow^{\pi_n} \\ & \mathbf{Spaces}_* & \end{array}$$

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This is an analogue of the Dold-Thom theorem:

$$\begin{array}{ccc} \mathbf{Spaces}_* & \xrightarrow{\widetilde{H}_n(-;\mathbb{Z})} & \mathbf{Abelian\ Groups} \\ & \searrow SP^\infty(-) & \nearrow \pi_n \\ & \mathbf{Spaces}_* & \end{array}$$

Segal's definition

H : a separable infinite dimensional Hilbert space over \mathbb{R} .

$$ko(X) = \left\{ [V_{x_1}, V_{x_2}, \dots, V_{x_j}] \left| \begin{array}{l} x_i \in X, \\ V_{x_i} \subset H : \text{finite dim.}, \\ x_i \neq x_{i'} \text{ if } i \neq i', \\ V_{x_i} \perp V_{x_{i'}} \text{ if } i \neq i'. \end{array} \right. \right\}$$

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- How to topologize?

When x_i gets close to x_j , V_{x_i} and V_{x_j} are summed up $V_{x_i} \oplus V_{x_j}$.

- Segal proved $ko : \mathbf{Spaces}_* \longrightarrow \mathbf{Spaces}_*$ is a linear functor. (It transforms cofibrations into quasifibrations.)
- The proof is a modification of Dold-Thom's idea.
- We have a straightforward translation to a complex version $ku(-)$.

Linear isometries operad

- H : an inner product space over \mathbb{C} with a countable basis.
- $\mathcal{L}(0; H) = \{*\}$.
- $\mathcal{L}(j; H) = \text{Iso}(\underbrace{H \oplus \cdots \oplus H}_j, H)$.
- $\mathcal{L}(H) = \{\mathcal{L}(j; H)\}$ is a complex version of the linear isometries operad used by Elmendorf, Kriz, Mandell, and May in their construction of the symmetric monoidal category of S -modules.
- When H is infinite dimensional, $\mathcal{L}(j; H) \simeq *$.
- X, Y : $\mathcal{L}(1; H)$ -spaces

$$X \times_{\mathcal{L}(H)} Y = \mathcal{L}(2; H) \times_{\mathcal{L}(1; H) \times \mathcal{L}(1; H)} (X \times Y).$$

- $(X \times_{\mathcal{L}(H)} Y) \times_{\mathcal{L}(H)} Z \cong X \times_{\mathcal{L}(H)} (Y \times_{\mathcal{L}(H)} Z)$

Projective spaces

H : a Hilbert space.

- $\mathbb{P}(H) = \{\ell \subset H \mid \dim \ell = 1\}$.
- $\mathbb{P}_0(H) = \{*\}$.
- $\mathbb{P}_j(H) = \underbrace{\mathbb{P}(H) \times_{\mathcal{L}(H)} \cdots \times_{\mathcal{L}(H)} \mathbb{P}(H)}_j \cong \mathcal{L}(j; H) \times_{\mathcal{L}(1; H)^j} \mathbb{P}(H)^j$.

$ku(X; H)$ can be defined by using these spaces.

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$ku(X; H)$ can be defined by using these spaces.

X : a based space.

$$ku(X; H) = \left(\prod_{j=0}^{\infty} \mathbb{P}_j(H) \times_{\Sigma_j} X^j \right) / \underset{*}{\sim}, \underset{\text{Gr}}{\sim}.$$

Equivalence relations

The equivalence relation $\underset{*}{\sim}$ is generated by:

$$[\varphi, \ell_1, \dots, \ell_j; x_1, \dots, x_j] \underset{*}{\sim} [s_i(\varphi), \ell_1, \dots, \ell_{i-1}, \ell_{i+1}, \dots, \ell_j; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_j],$$

where s_i is

$$\mathcal{L}(j) \cong \mathcal{L}(j) \times \mathcal{L}(0) \xrightarrow{\circ_i} \mathcal{L}(j-1).$$

Equivalence relations

The equivalence relation $\underset{\text{Gr}}{\sim}$ is generated by: in the nondegenerate form
(all base points are removed)

$$[\varphi; \ell_1, \dots, \ell_j; x_1, \dots, x_j] \underset{\text{Gr}}{\sim} [\varphi'; \ell'_1, \dots, \ell'_j; x'_1, \dots, x'_j]$$

if and only if, under an appropriate permutation,

1. $x_i = x'_i$ for all i ,
2. $\varphi(\ell_{i_1} \oplus \dots \oplus \ell_{i_k}) = \varphi'(\ell'_{i_1} \oplus \dots \oplus \ell'_{i_k})$ if $x_{i_1} = \dots = x_{i_k}$.

Twisting

- Atiyah-Segal twisting is given by the conjugation action

$$\tau_{AS} : PU(H) \times \text{Fred}(H) \longrightarrow \text{Fred}(H).$$

- $PU(H)$ acts on $\mathbb{P}(H)$.
- $\varphi : X \rightarrow BPU(H) \leftrightarrow \mathbb{P}(H)$ -bundle over X (with structure group $PU(H)$).

$$\mathbf{Spaces} \downarrow BPU(H) \cong \{\mathbb{P}(H)\text{-bundles}\}.$$

- Atiyah-Segal twisted K -theory can be interpreted as

$$\tau_{AS}K : \{\mathbb{P}(H)\text{-bundles}\}^{op} \longrightarrow \mathbf{Abelian Groups}.$$

Twisting

The definition of $ku(X; H)$ contains trivial $\mathbb{P}(H)$ -bundles.

$$\begin{aligned} ku(X; H) &= \left(\prod_{j=0}^{\infty} \mathbb{P}_j(H) \times_{\Sigma_j} X^j \right) / \underset{*}{\sim}, \underset{\text{Gr}}{\sim} \\ &= \left(\prod_{j=0}^{\infty} (\mathcal{L}(j; H) \times_{\mathcal{L}(1; H)^j} (\mathbb{P}(H) \times X)^j) / \Sigma_j \right) / \underset{*}{\sim}, \underset{\text{Gr}}{\sim}. \end{aligned}$$

Twisting

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Replace the trivial bundle $\mathbb{P}(H) \times X$ by the bundle classified by $\varphi : X \rightarrow BPU(H)$.

$\mathcal{L}(H; 1)$ -spaces

We need to take actions of $\mathcal{L}(H; 1)$ into account.

- $\mathcal{L} \mathbf{Spaces}_*$: the category of spaces with actions of $\mathcal{L}(1; H)$.
- Define $\mathcal{L} : \mathbf{Spaces}_* \rightarrow \mathcal{L} \mathbf{Spaces}_*$ by

$$\mathcal{L}(X) = (\mathcal{L}(H; 1)_+) \wedge X.$$

- $\mathcal{L}(1; H)$ acts on $BPU(H)$.
- $\varphi : X \rightarrow BPU(H)$: a morphism in $\mathcal{L} \mathbf{Spaces}_*$. The associated $\mathbb{P}(H)$ -bundle is denoted by

$$p_\varphi : E_\varphi(\mathbb{P}(H)) \rightarrow X.$$

Twisted Segal K -homology

(φ, s) : an object in $\mathbf{Spaces}_{BPU(H)}$.

$$ku^{AS}(\varphi, s; H) = \left(\coprod_{j=0}^{\infty} (\mathcal{L}(j; H) \times_{\mathcal{L}(1; H)^j} E_{\varphi}(\mathbb{P}(H))^j) / \Sigma_j \right) / \underset{*}{\sim}, \underset{\text{Gr}}{\sim}.$$

- Equivalence relations $\underset{*}{\sim}, \underset{\text{Gr}}{\sim}$ are analogous to the untwisted case.
- We need to use Grassmannian bundles for $\underset{\text{Gr}}{\sim}$.

Twisted Segal K -homology

Theorem. ku^{AS} has the following properties:

- *It is a linear functor.*
- *For a based space X , if $\varphi = * \vee 1_{BPU(H)} : X \vee BPU(H) \rightarrow BPU(H)$,*

$$ku^{AS}(* \vee 1_{BPU(H)}, s; H) = ku(X; H).$$

Bar Constructions on Partial Monoids

Connective homology theories and linear functors

- Let C be a model category with a 0-object $*$. A functor

$$F : C \longrightarrow \mathbf{Spaces}_*$$

is called linear if

- $F(*) \underset{w}{\simeq} *$.
- F converts sums into products

$$\prod_{\alpha \in A} F(X_\alpha) \underset{w}{\simeq} F\left(\bigvee_{\alpha \in A} X_\alpha\right).$$

- F converts a cofibration $A \rightarrow X \rightarrow X/A$ into a quasifibration

$$F(A) \rightarrow F(X) \rightarrow F(X/A).$$

- If F is linear, $X \rightarrow \pi_*(F(X))$ is a homology theory.

Quasifibrations

How can we show a functor is linear?

- The first two conditions are easy.
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- The first two conditions are easy.
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How can we show a map is a quasifibration?

- The Dold-Thom criterion for quasifibrations.
- Shimakawa's result (for functors obtained from partial Abelian monoids).
- What else?

The case of infinite symmetric products

For $X \xleftarrow{f} Y \xrightarrow{g} Z$

- $SP^\infty(M_{f,g}) \cong |B_*(SP^\infty(X), SP^\infty(Y), SP^\infty(Z))|$,
- $SP^\infty(Y \cup_f CY) \cong |B_*(SP^\infty(X), SP^\infty(Y), *)|$
- The sequence

$$SP^\infty(Z) \rightarrow SP^\infty(M_{f,g}) \rightarrow SP^\infty(X \cup_f CY)$$

can be identified with

$$\begin{aligned} SP^\infty(Z) &\rightarrow |B_*(SP^\infty(X), SP^\infty(Y), SP^\infty(Z))| \\ &\rightarrow |B_*(SP^\infty(X), SP^\infty(Y), *)|. \end{aligned}$$

- Milgram and May proved that for reasonably good topological monoids, the bar construction as above gives rise to a quasifibration.

The case of Segal's K -homology

For based spaces X and Y , define

$$ku(X; H) \overset{\perp}{\times} ku(Y; H) \subset ku(X; H) \times ku(Y; H)$$

as follows: If $([\mathbf{V}; \mathbf{x}], [\mathbf{W}; \mathbf{y}]) \in ku(X; H) \times ku(X; H)$ is represented by

$$\begin{aligned}(\mathbf{V}; \mathbf{x}) &= (V_1, \dots, V_j; x_1, \dots, x_j) \\ (\mathbf{W}; \mathbf{y}) &= (W_1, \dots, W_k; y_1, \dots, y_k),\end{aligned}$$

$$([\mathbf{V}; \mathbf{x}], [\mathbf{W}; \mathbf{y}]) \in ku(X; H) \overset{\perp}{\times} ku(X; H) \iff V_i \perp W_k \text{ for all } i, k.$$

Topological partial monoids

- We can define an analogue of the two-sided bar construction

$$B_*^\perp(ku(X; H), ku(Y; H), ku(Z; H))$$

for $X \xleftarrow{f} Y \xrightarrow{g} Z$ by using \times^\perp .

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- More generally, for a topological partial monoid M and spaces X and Y on which M acts, we define

$$B_n(X, M, Y) \subset X \times M^n \times Y$$

to be the set of “sequences of elements, any two of which can be multiplied”.

Topological partial monoids

Theorem. *Let $C_M(Y)$ be the set of pairs of elements in $M \times Y$ for which the action is defined.*

If the inclusions

$$B_n(X, M, Y) \hookrightarrow X \times M^n \times Y$$

$$B_n(X, M, *) \hookrightarrow X \times M^n$$

$$C_M(Y) \hookrightarrow M \times Y$$

are weak equivalences for all n , and if the action of M on Y induces a weak equivalence

$$Y_m = \{y \in Y \mid (m, y) \in C_M(Y)\} \xrightarrow{m \cdot} Y,$$

then

$$p^Y : |B_*(X, M, Y)| \longrightarrow |B_*(X, M, *)|$$

is a quasifibration with fiber Y .