Group actions on 4-manifolds

Ximin Liu
(South China University of Technology)

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Abstract

In this talk, some topics and results around nonsmoothable group actions on 4-manifolds are given. Especially, I will explain some recent results about nonsmoothable group actions on 4-manifolds, which are related to the Seiberg-Witten theory.

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1. Introduction and overview

Let X be an oriented topological manifold and G a finite group.

A topological G-action on X is called locally linear if $\forall x \in X$, $\exists V_x$: a G_x-invariant neighborhood of X, (G_x: isotropy subgroup of x) such that

(1) $V_x \cong R^n$ (homeomorphism),

(2) G_x acts on $R^n \cong V_x$ in a linear orthogonal way.

In general, smooth action is locally linear, but locally linear action is not necessary to be smooth.

If X admits a smooth structure and a smooth structure σ is specified, then we write the manifold with the smooth structure σ by X_σ.

Let $LL(G, X)$ be the set of equivalence classes of orientation-preserving locally linear G-actions on X. Here, the equivalence of two locally linear actions is given via a homeomorphism f of X. Similarly, let $C^\infty(G, X_\sigma)$ be the set of equivalence classes of orientation-preserving smooth G-actions on X_σ. Here, the equivalence of two smooth actions is given via a diffeomorphism of X_σ. Then we have a forgetful map $\Phi_\sigma : C^\infty(G, X_\sigma) \to LL(G, X)$ forgetting the smooth structure.

Definition 1.1 A locally linear action is called nonsmoothable with respect to the smooth structure σ on X if its class is not contained in the image of Φ_σ.

Facts:

If $n = \dim X \leq 3$, then no nonsmoothable locally linear action.

(i) It is a classical result that every finite group action on a surface is equivalent to a smooth one.

(ii) Kawasik and Lee (1988) proved that in dimension 3 a finite group action is smoothable if and only if it is locally linear action.

For $n = 4$: many examples of nonsmoothable locally linear actions are known.

1. [Kwasik-Lee, 1988] $G = \mathbb{Z}_2 \curvearrowright X$: a closed smooth 4-manifold.

2. [Kwasik-Lawson, 1993] $G = \mathbb{Z}_p \curvearrowright X$: contractible such that $\partial X = \Sigma(a, b, c)$: Brieskorn sphere, where p is a prime.

3. [Hambleton-Lee, 1995] $G = \mathbb{Z}_5 \curvearrowright X = CP^2 \sharp CP^2$.

4. [Liu-Nakamura, 2005-2007] $G = \mathbb{Z}_p \curvearrowright X = E(n)$, $p = 3, 5, 7$.

5. [Chen-Kwasik, 2007] $\exists$ family of symplectic exotic $K3$ such that $\forall$ nontrivial odd order cyclic group locally linear actions are nonsmoothable.

6. [Nakamura, 2007] $G = \mathbb{Z}_2 \curvearrowright X = K3 \sharp K3$.

7. [Kiyono, 2008] $G = \mathbb{Z}_p \curvearrowright X$, p prime, lage enough. X any simply connected, spin 4-manifold except S^4 and $S^2 \times S^2$.

How to construct a nonsmoothable action?

Usually two steps:

(1) Existence: To construct loc. lin. actions concretely.

(2) Nonsmoothable: Prove the above action is nonsmoothable.

When $G = \mathbb{Z}_p$, p is a prime, X is a closed simply connected 4-manifold, general construction for (1) is established by Edmonds-Ewing.

(i) fixed point data

(ii) G-action on the intersection form

with certain condition $\Rightarrow \exists$ locally linear action realizing (i) (ii).

For (2) various techniques are used.

1. [Kwasik-Lee, 1988]

(i) Existence $\leftarrow$ Surgery theory

(ii) Nonsmoothable $\leftarrow$ Kirby-Siebenmann invariants

2. [Kwasik-Lawson, 1993]

(i) Existence $\leftarrow$ [Edmonds 1987: Equivariant handle construction]

(ii) Nonsmoothable $\leftarrow$ Donaldson theory

3. [Hambleton-Lee, 1995]

(i) Existence $\leftarrow$ [Edmonds-Ewing 1992]

(ii) Nonsmoothable ← Donaldson theory

4. [Liu-Nakamura, 2005-2007]

(i) Existence ← [Edmonds-Ewing 1992]

(ii) Nonsmoothable ← Seiberg-Witten invariants (Mod p vanishing theorem under Z_p-actions)

5. [Chen-Kwasik, 2007]

(i) Existence ← [Edmonds-Ewing 1992]

(ii) Nonsmoothable ← Seiberg-Witten theory (SW $\longrightarrow$ Gr) etc.

6. [Nakamura, 2007]

(i) Existence ← [Edmonds-Ewing 1992]

(ii) Nonsmoothable ← Bauer-Furuta invariants (a vanishing theorem under Z_2-actions)

7. [Kiyono, 2008]

(i) Existence ← [Edmonds-Ewing 1992]

(ii) Nonsmoothable ← Seiberg-Witten theory (G-equiv 10/8-inequality)

$G \curvearrowright X \Rightarrow X/G$ should satisfy 10/8-inequality.

Intersection forms of 4-manifolds

Let X be a closed oriented smooth 4-manifold, we know it admits a fundamental class $[X] \in H_4(X; Z)$.

Definition 1.2 The symmetric bilinear form

$$Q_X : H^2(X; Z) \times H^2(X; Z) \to Z$$

defined by $Q_X(a, b) = < a \cup b, [X] > = a \cdot b \in Z$ is called the intersection form of X.

Remark Note that for the definition of Q_X, we only need the topological structure of X. Since by Poincaré duality $H_2(X; Z) \cong H^2(X; Z)$, Q_X is defined on $H_2(X; Z) \times H_2(X; Z)$ as well.

Definition 1.3 For a given symmetric bilinear form Q on a finitely generated free abelian group A, the rank, signature and parity of Q are defined in the following way: The rank $\mathrm{rk}(Q)$ of Q is the dimension of A. Extend and diagonal Q over $A \otimes_Z R$. The number of $+1$'s (-1's resp.) on the diagonal is denoted by b_2^+ (resp. b_2^-); the difference $b_2^+ - b_2^-$ is the signature $\sigma(Q)$. Finally Q is even is $Q(a,a) \equiv 0(\text{mod}2)$ for every $a \in A$; Q is odd otherwise.

Definition 1.4 (a) Q is positive (negative) definite if $\text{rk}(Q) = \sigma(Q)$ ($\text{rk}(Q) = -\sigma(Q)$ resp.). Q is indefinite otherwise.

(b) Q is called unimodular if $\det Q = \pm 1$.

Remark By poincaré duality, if X is a closed 4-manifold, then I_X is unimodular.

Consider the following 8×8 intersection form matrix E_8:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

As a matrix of a bilinear form Q on Z^8, E_8 gives a definite, even, unimodular form with $\sigma(Q) = 8$.

Recall that H is the hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Using E_8 and H as building blocks, for every pair $(\sigma, r) \in Z \times N$, one can build up an indefinite unimodular form $Q = aE_8 \oplus bH$ with $\sigma = \sigma(Q)$ and $r = \text{rk}(Q)$, where $a = \frac{\sigma}{8}$ and $b = \frac{\text{rk}(\sigma) - |\sigma|}{2}$.

Theorem 1.1 *Suppose that Q is an indefinite, unimodular form. If Q is odd, then it is isomorphic to $b_2^+(1) \oplus b_2^-(-1)$; if Q is even, then it is isomorphic to $\frac{\sigma(Q)}{8} E_8 \oplus \frac{\mathrm{rk}(\sigma)-|\sigma(Q)|}{2} H$.*

Theorem 1.2 (Freedman) *For every unimodular symmetric bilinear form Q there exists a simply connected, closed topological 4-manifolds X such that $Q_X \cong Q$. If Q is even, this manifold is unique (up to homeomorphism). If Q is odd, there are exactly two different homeomorphism types of 4-manifolds with the given intersection form. At most one of these homeomorphism types carries a smooth structure. Consequently, simply connected, smooth 4-manifolds are determined up to homeomorphism by their intersection forms.*

The special case is the topological 4-dimensional Ponicaré conjecture:

Corollary 1.1 *If a topological 4-manifold X is homotopy equivalent to S^4, then X is homeomorphic to the 4-sphere.*

Theorem 1.3 (Rohlin) *If the intersection form Q_X of a smooth, simply connected, closed 4-manifold X is even, then the signature $\sigma(X)$ is divisible by 16.*

Theorem 1.4 (Donaldson) *If the intersection form Q_X of a smooth, simply connected, closed 4-manifold X is negative definite, then Q_X is equivalent to $n(-1)$.*

Let X be a smooth, closed, connected spin 4-manifold. We denote by $b_2(X)$ the second Betti number and denote by $\sigma(X)$ the signature of X.

$\frac{11}{8}$-conjecture:

$$b_2(X) \geq \frac{11}{8}|\sigma(X)|. \tag{1.1}$$

From the classification of unimodular even integral quadratic forms and the Rochlin's theorem, for the choice of orientation with non-positive signature the intersection form of a closed spin 4-manifold X

is

$$-2kE_8 \oplus mH, \qquad k \geq 0.$$

Thus, $m = b_2^+(X)$ and $k = -\frac{\sigma(X)}{16}$ and so the $\frac{11}{8}$-conjecture is equivalent to $3k \leq m$.

In 1995, by using the finite dimensional approximation of the Seiberg-Witten equations, Furuta proved:

$\frac{10}{8}$-inequality:

$$b_2(X) \geq \frac{10}{8}(=\frac{5}{4})|\sigma(X)| + 2. \tag{1.2}$$

It is equivalent to the following:

Theorem 1.5 (Furuta) *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ with non-positive signature. If $m \neq 0$, then*

$$2k + 1 \leq m.$$

2. The SW invariants and Cyclic group actions

In this section, we introduce the SW invariants under cyclic group actions.

(1) **The Seiberg-Witten invariants**

Let X be an oriented closed 4-manifold. Fixing a Riemannian metric on X determines an SO(4)-frame bundle $\mathrm{P_{SO}}$ on X. Assume that $b^+(M) > 1$. A Spin^c structure on X is a lift of SO(4)-frame bundle $\mathrm{P_{SO}}$ to a Spin^c-bundle on X. Each Spin^c structure gives a pair of unitary C^2 bundles S^+ and S^-, that is the bundles of positive and negative spinors.

For a given Spin^c structure c, let L be the determinant bundle, $\mathcal{A}(L)$ be the space of $U(1)$-connections on L, and $\Gamma(S^+)$ be the space of sections of S^+.

The Seberg-Witten equations are a system of equations for $(A, \phi) \in \mathcal{A}(L) \times \Gamma(S^+)$

$$\begin{cases} D_A\phi = 0 \\ F_A^+ = q(\phi), \end{cases}$$

where D_A denotes the Dirac operator associated to the Riemannian metric and the connection A, F_A^+ denote the self-dual part of the curvature F_A, and $q(\phi)$ is the trace free part of the endomorphism $\phi \otimes \phi^*$ of S^+ and this endomorphism is identified with an imaginary-valued self-dual 2-form via the Clifford multiplication.

The gauge transformation group $\mathcal{G}_X$ can be identified with the space of maps from X to S^1, and $\mathcal{G}_X$ acts on (A, ϕ) as follows: for $u \in \mathcal{G}_X$, $u(A, \phi) = (A - 2u^{-1}du, u\phi)$. This group acts freely at (A, ϕ) where ϕ is not identically zero. (A, ϕ) with $\phi \equiv 0$ is called reducible, otherwise is called irreducible. Denote by $\mathcal{C}_{X,c} = \mathcal{A}(L) \times \Gamma(S^+)$ and put $\mathcal{B}_{X,c} = \mathcal{C}_{X,c}/\mathcal{G}_X$.

The Seiberg-Witten equations are $\mathcal{G}_X$-equivariant. The quotient space of solutions by $\mathcal{G}_X$ is denoted by $\mathcal{M}_{X,c}$. The space $\mathcal{M}_{X,c}$ is

Back
Close

called the muduli space. Let $\mathcal{C}^*_{X,c} \subset \mathcal{C}_{X,c}$ be the set of irreducibles, and set $\mathcal{B}^*_{X,c} = \mathcal{C}^*_{X,c}/\mathcal{G}_X$.

We could perturb the Seiberg-Witten equations to make the moduli space $\mathcal{M}_{X,c}$ to be a compact smooth manifold whose dimension is

$$\frac{1}{4}[c_1(L)^2 - \mathrm{Sign(X)}] - (1 - b_1 + b^+).$$

For a base point x_0 in X, let $\mathcal{G}^0_X$ be the subgroup of $\mathcal{G}_X$ which consists of maps which map the base point to 1. Let $\tilde{\mathcal{B}}^*_{X,c} = \mathcal{C}^*_{X,c}/\mathcal{G}^0_X$. Then the projection $\tilde{\mathcal{B}}^*_{X,c} \to \mathcal{B}^*_{X,c}$ defines an S^1-bundle.

Definition 2.1 Let X be a compact, oriented 4-manifold with $b_2^+ \geq 1$, and c be a Spin^c-structure on X. The Seiberg-Witten invariant of (X, c) is

$$SW_{(X,c)} = < c_1^d, [\mathcal{M}_{X,c}] >,$$

where $d = \frac{1}{2}\mathrm{dim}\mathcal{M}_{X,c}$, c_1 is the first Chern class of the complex line bundle associated to the S^1-bundle $\tilde{\mathcal{B}}^*_{X,c} \to \mathcal{B}^*_{X,c}$.

Remark. When $b_2^+ \geq 2$, $SW_{(X,c)}$ does not depend on the choice of metric and perturbation, i.e., it is a diffeomorphism invariant of X. When $b_2^+ = 1$, $SW_{(X,c)}$ depends on the choice of metric and perturbation.

(2) **Mod p equality theorem for free actions**

Let p be an odd prime, and suppose that $G = \mathbb{Z}/p$ acts on a smooth 4-manifold X *smoothly.*

Theorem 2.1 Let G be the cyclic group of prime order p, and X be a closed oriented 4-manifold with $b_+ \geq 2$. Suppose that G acts on X freely, and $b_+^G = b_+(X/G) \geq 2$. Let $\bar{X} = X/G$, and $\pi : X \to \bar{X}$ is the projection. When $p = 2$, suppose further that the G action is orientation-preserving and X has simple type in that only the dimension zero SW invariants are non-zero. For a Spin^c-structure

$\bar{c}$ and let c be its lift to X. Then

$$SW_X(c) \equiv \sum_{\bar{a} \in \ker \pi^*} SW_{\bar{X}}(\bar{c} + \bar{a}) \pmod{p},$$

where $\ker\pi^*$ is the kernel of $\pi^* : H^2(\bar{X}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$, and $\bar{c} + \bar{a}$ is the Spin^c-structure on $\bar{X}$ of which the difference from $\bar{c}$ is represented by the cohomology class $\bar{a}$.

Theorem 2.1 is proved by several authors independently. Y. Yuan and S. Wang (2000) proves the theorem in the case when $p = 2$, and generalized to the case to 2-fold branched covering. M. Szymik proves the theorem in the case when $b_1 = 0$ and general p by using the G-equivariant version of the stable homotopy SW invariant (Barer-Furuta invariant). The general case is proved by N. Nakamura (2002).

(3) **Mod p vanishing theorem**

Let p be an odd prime, and suppose that $G = \mathbb{Z}/p$ acts on a smooth 4-manifold X smoothly. When $p = 2$, we assume that the G-action is orientation-preserving. Fixing a G-invariant metric on X, we have a G-action on the frame bundle P_{SO}. We say that a Spin^c-structure c is *G-equivariant* if the G-action on P_{SO} lifts to a G-action on the Spin^c-bundle P_{Spin^c} of c.

Suppose that a G-equivariant Spin^c-structure c is given. Fix a G-invariant connection A_0 on the determinant line bundle L of c. Then the Dirac operator D_{A_0} associated to A_0 is G-equivariant, and the G-index of D_{A_0} can be written as $\mathrm{ind}_G D_{A_0} = \sum_{j=0}^{p-1} k_j \mathbb{C}_j \in R(G) \cong \mathbb{Z}[t]/(t^p - 1)$, where $\mathbb{C}_j$ is the complex 1-dimensional weight j representation of G and $R(G)$ is the representation ring of G.

In such a situation, the following theorem is proved.

Theorem 2.2[Fang] *Let G be the cyclic group of prime order p, and X be a smooth closed oriented 4-dimensional G-manifold with $b_1 = 0$ and $b_+ \geq 2$. Let c be a G-equivariant* Spin^c*-structure. Suppose that $b_+^G = b_+$, if $2k_j < 1 + b_+^G$ for $j = 0, 1, \ldots, p-1$, then the SW invariant* $\mathrm{SW}_X(c)$ *for c satisfies*

$$\mathrm{SW}_X(c) \equiv 0 \mod p.$$

Fang suppose that $b_+^G = b_+$. Nakamura weakened this condition, and generalized to the case when $b_1 \geq 1$.

(4) **Bauer-Furuta invariant**

On the other hand, Bauer and Furuta defined a stable cohomotopy refinement of the Seiberg-Witten invariants. In general, Bauer-Furuta invariants may be non-trivial even if ordinary SW invariants are trivial. Nakamura proved a similar vanishing theorem for Bauer-Furuta invariants for 4-manifolds with smooth $\mathbb{Z}_2$-actions. As an application, he constructed a locally linear $\mathbb{Z}_2$-action on $X = K3\sharp K3$ which can not be smooth.

To proved the vanishing theorem for Bauer-Furuta invariants, Nakamura used the equivariant obstruction theory on Bredon cohomology, in fact in some situations, equivariant Bauer-Furuta invariants can be written as equivariant obstruction classes. Nakamura calculated Bredon cohomology groups explicitly, from that he got the vanishing theorem for Bauer-Furuta invariants under $\mathbb{Z}_2$-actions.

3. Recent results on nonsmoothable group actions

Let G be the cyclic group of order 3, 5 or 7, and suppose that G acts locally linearly and pseudofreely on a smooth 4-manifold X.

Theorem 3.1(Liu-Nakamura) *Let $X = E(n)$ be the minimal simply connected elliptic surface which has the Euler number $12n$. Suppose n is even and $n \geq 2$, and let*

$$c_{n-2} := \binom{n-2}{\frac{n-2}{2}}. \tag{3.1}$$

If $c_{n-2} \not\equiv 0 \mod p$, then there exists a locally linear G-action on X which can not be smooth with respect to infinitely many smooth structures on X.

Remark. An action is called *pseudofree* if it is free on the compliment of a discrete subset.

To prove above main theorem, we should do two things:

(1) The existence of locally linear actions. We will use a remarkable realization theorem by Edmonds and Ewing.

(2) The constraint on smooth actions. We use the mod p vanishing theorem of SW invariants.

Theorem 3.2(Nakamura) *There exists a locally linear $\mathbb{Z}_2$-action on $X = K3\sharp K3$ which can not be smooth with respect to any smooth structure on X.*

The idea to prove Theorem 3.2 is similar to Theorem 3.1.

(1) Using the realization theorem due to Edmonds and Ewing to construct a locally linear action.

(2) To obtain constraint on smooth actions, Nakamura uses the vanishing theorem for Bauer-Furuta invariants together with the non-vanishing result of the Bauer-Furuta invariant of $K3\sharp K3$ by Furuta, Kametani and Minami.

Recently, Kiyono proved the following result.

Theorem 3.3(Kiyono) *Let X be a closed, simply connected, spin topological 4-manifold not homeomorphic to either S^4 or $S^2 \times S^2$. Then for any sufficiently large prime number p, then there exists a homologically trivial, pseudofree, locally linear $\mathbb{Z}_p$-action on X which is nonsmoothable.*

Let $NS(X)$ be the set of prime number p for which X admits a homologically trivial, pseudofree, nonsmoothable locally linear $\mathbb{Z}_p$-action. Kiyono proves that $NS(X)$ contains all the prime numbers p satisfying

$$p \geq 12\left[\frac{\max\{b_2^+(X), b_2^-(X)\}+1}{2}\right] - 5,$$

here $[x]$ is the maximum integer less than or equal to x.

Kiyono still uses the realization theorem of Edmonds and Ewing to construct a locally linear action. For nonsmoothable, Kiyono applied the following $\frac{10}{8}$-type inequality for the quotient V-manifold $X/\mathbb{Z}_p$:

$$(3.2) \qquad \dim(\mathrm{ind}_G D)^G < \dim_{\mathbb{R}} H^2_+(X;\mathbb{R})^G.$$

Now we explain the case of $\mathbb{Z}/3$-actions more precisely. In particular, we will give the outline of the proof for $\mathbb{Z}/3$-actions on $K3$ surface as a model case.

When we fix a generator g of $G = \mathbb{Z}/3$, the representation at a fixed point can be described by a pair of nonzero integers (a, b) modulo 3 which is well-defined up to order and changing the sign of both together. Hence, there are two types of fixed points.

- The type $(+)$: $(1, 2) = (2, 1)$.
- The type $(-)$: $(1, 1) = (2, 2)$.

Let m_+ be the number of fixed points of the type $(+)$, and m_- be the number of fixed points of the type $(-)$.

The Euler number of a 4-manifold X is denoted by $\chi(X)$, and the signature by $\mathrm{Sign}(X)$. For any G-space V, let V^G be the fixed point set of the G-action. Let b_i be the i-th Betti number of X, and b_+ (resp. b_-) be the rank of a maximal positive (resp. negative) definite subspace $H^+(X;\mathbb{R})$ (resp. $H^-(X;\mathbb{R})$) of $H^2(X;\mathbb{R})$.

Theorem 3.4 *Let G be the cyclic group of order 3. For locally linear pseudofree G-actions on a $K3$ surface X, we have the following:*

(1) *Every locally linear pseudofree G-action on X belongs to one of four types. Furthermore, each of four types can be actually realized by a locally linear pseudofree G-action on X.*

(2) *The type A_1 can not be realized by a smooth action on infinitely many smooth structures on X.*

Remark: There exists a smooth G-action of the type A_0 on the $K3$ surface of Fermat type which is defined by the equation $\sum_{i=0}^{3} z_i^4 = 0$ in $\mathbb{CP}^3$. By the symmetry of the defining equation, the symmetric

Table 1. The classification of actions

Type	$\#X^G$	m_+	m_-	b_2^G	b_+^G	b_-^G	$\mathrm{Sign}(X/G)$
A_0	6	6	0	10	3	7	-4
A_1	9	3	6	12	3	9	-6
A_2	12	0	12	14	3	11	-8
B	3	0	3	8	1	7	-6

group of degree 4 acts on X as permutations of variables. Therefore G acts smoothly on X via this action. We can easily check that the G-action is pseudofree, and belongs to the type A_0.

4. Outline of the proof

In this section, we give the idea to prove Theorem 3.4. First, we give some preliminaries.

(1) The G-index theorem

Let G be the cyclic group of prime order p, and g be a generator. Suppose G acts on a 4-manifold X pseudofreely, and the fixed point data for the generator g is given as $\{(a_i, b_i)\}_{i=1}^N$.

Then the *G-signature formula* is

(4.1) $$\mathrm{Sign}(g, X) = \sum_{i=1}^{N} s_{a_i b_i}$$

where

(4.2) $$s_{xy} = \frac{(\zeta^x + 1)(\zeta^y + 1)}{(\zeta^x - 1)(\zeta^y - 1)},$$

and $\zeta = \exp(2\pi\sqrt{-1}/p)$.

Suppose further that X is *spin* and G-action is a *spin* action. Let D_X be the G-equivariant Dirac operator. Then the G-*spin theorem* is

$$\operatorname{ind}_g D_X = \sum_{i=1}^{N} p_{a_i b_i}, \tag{4.3}$$

where

$$p_{xy} = \frac{1}{(\zeta^x)^{1/2} - (\zeta^x)^{-1/2}} \frac{1}{(\zeta^y)^{1/2} - (\zeta^y)^{-1/2}}, \tag{4.4}$$

and signs of $(\zeta^x)^{1/2}$ and $(\zeta^y)^{1/2}$ are determined by the rule

$$\left\{(\zeta^x)^{1/2}\right\}^p = \left\{(\zeta^y)^{1/2}\right\}^p = 1.$$

(This is because, in our case, the g-action on the Spin-structure generates a G-action on the Spin-structure.)

(2) The realization theorem by Edmonds and Ewing

We summarize the realization theorem of locally linear pseudofree actions by A. L. Edmonds and J. H. Ewing in the very special case when $G = \mathbb{Z}/3$.

Theorem 4.1 [Edmonds and Ewing (1992)] *Let G be the cyclic group of order p, where $p = 3$. Suppose that one is given a fixed point data*

$$D = \{(a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n), (a_{n+1}, b_{n+1})\},$$

where $a_i, b_i \in \mathbb{Z}/p \setminus \{0\}$, and a G-invariant symmetric unimodular form

$$\Phi\colon V \times V \to \mathbb{Z},$$

where V be a finitely generated $\mathbb{Z}$-free $\mathbb{Z}[G]$-module. Then the data D and the form (V, Φ) are realizable by a locally linear, pseudofree, G-action on a closed, simply-connected, topological 4-manifold if and only if they satisfy the following two conditions:

(1) *The condition REP: As a* $\mathbb{Z}[G]$*-module,* V *splits into* $F \oplus T$*, where* F *is free and* T *is a trivial* $\mathbb{Z}[G]$*-module with* $\operatorname{rank}_{\mathbb{Z}} T = n$*.*

(2) *The condition GSF: The G-Signature Formula is satisfied:*

$$\operatorname{Sign}(g,(V,\Phi)) = \sum_{i=0}^{n+1} \frac{(\zeta^{a_i}+1)(\zeta^{b_i}+1)}{(\zeta^{a_i}-1)(\zeta^{b_i}-1)}, \tag{4.5}$$

where $\zeta = \exp(2\pi\sqrt{-1}/p)$*.*

Remark For general p, the third condition *TOR* which is related to the Reidemeister torsion should be satisfied. However, when p is a prime less than 23, the condition *TOR* is is not necessary.

(3) Outline of the proof

Now, let us begin the proof of the assertion (1) of Theorem 3.4.

By the ordinary Lefschetz formula we have: $L(g, X) = 2 + \mathrm{tr}(g|_{H^2(X)}) = \# X^G$. Noting that $\# X^G = m_+ + m_-$ and $2 + \mathrm{tr}(g|_{H^2(X)}) \leq 24$, we obtain

$$m_+ + m_- \leq 24.$$

Note that

$$\chi(X/G) = \frac{1}{3}\{24 + 2(m_+ + m_-)\}.$$

By Theorem 4.1, the G-Signature Formula should hold:

$$\mathrm{Sign}(g, X) = \mathrm{Sign}(g^2, X) = \frac{1}{3}(m_+ - m_-),$$

$$\mathrm{Sign}(X/G) = \frac{1}{3}\left\{-16 + \frac{2}{3}(m_+ - m_-)\right\}.$$

From the inequality $-24 \leq m_+ - m_- \leq 24$, we could get

$$m_+ - m_- = -21, -12, -3, 6, 15, 24. \tag{4.6}$$

Then we can calculate b_+^G and b_-^G from $\chi(X/G)$ and $\mathrm{Sign}(X/G)$.

- When $b_+^G = 1$, $2m_+ + m_- = 3$.
- When $b_+^G = 3$, $2m_+ + m_- = 12$.

By these equations, we obtain Table 1.

Next we will prove the existence of actions. First, we construct a smooth G-action of type A_0 on the Fermat quartic surface.

Proposition 4.1 *There exists a smooth G-action of the type A_0 on the Fermat quartic surface X which is defined by the equation $\sum_{i=0}^{3} z_i^4 = 0$ in $\mathbb{C}\mathrm{P}^3$.*

Proof By the symmetry of the defining equation, the symmetric group of degree 4 acts on X as permutations of variables. Therefore G acts smoothly on X via this action. We can easily check that the G-action is pseudofree, and belongs to the type A_0.

To prove the existence of actions of other types, we use Theorem 4.1. We could construct G-actions on the intersection form. Let (V_{K3}, Φ_{K3}) be the intersection form of the $K3$ surface, which is even and indefinite. Since an even indefinite form is completely characterized by its rank and signature, (V_{K3}, Φ_{K3}) is isomorphic to $3H \oplus \Gamma_{16}$, where H is the hyperbolic form, and Γ_{16} is a negative definite even form of rank 16. We constructed G-actions on $3H$ and Γ_{16} separately.

Lemma 4.1 *For each integer k which satisfies $0 \leq k \leq 5$, there is a G-action on Γ_{16} such that*

$$\Gamma_{16} \cong (16 - 3k)\mathbb{Z} \oplus k\mathbb{Z}[G] \text{ as a } \mathbb{Z}[G]\text{-module.}$$

Lemma 4.2 *There is a G-action on $3H$ such that $3H \cong \mathbb{Z}[G] \oplus \mathbb{Z}[G]$ as a $\mathbb{Z}[G]$-module, and G-fixed parts of a maximal positive definite subspace and a negative one of $3H \otimes \mathbb{R}$ both have rank 1.*

With Lemma 4.1 and Lemma 4.2 understood, for each of A_1, A_2 and B, the corresponding G-action on (V_{K3}, Φ_{K3}) can be constructed. That is,

- for A_1, $3H \cong 6\mathbb{Z}$ and $\Gamma_{16} \cong \mathbb{Z} \oplus 5\mathbb{Z}[G]$,
- for A_2, $3H \cong 6\mathbb{Z}$ and $\Gamma_{16} \cong 4\mathbb{Z} \oplus 4\mathbb{Z}[G]$,
- for B, $3H \cong \mathbb{Z}[G] \oplus \mathbb{Z}[G]$ and $\Gamma_{16} \cong \mathbb{Z} \oplus 5\mathbb{Z}[G]$.

Now the conditions *REP* and *GSF* are satisfied. Therefore we have a locally linear pseudofree G-action on a closed simply-connected 4-manifold X whose intersection form is just (V_{K3}, Φ_{K3}) by Theorem 4.1. Since X is simply-connected and its intersection form is even, we see that X is homeomorphic to the $K3$ surface by Freedman's theorem. Thus the assertion (1) is proved.

Remark We can also construct a locally linear pseudofree action of the type A_0 by Theorem 4.1.

Next let us begin the proof of the assertion (2) of Theorem 3.4. Consider X as the smooth $K3$ surface. Suppose now that a smooth action of the type A_1 exists. We use the canonical Spin^c-structure c_0 whose determinant line bundle L is trivial in the case of $K3$ surface X.

Since X is simply-connected and L is trivial, we can see that every $G = \mathbb{Z}/3$-action on X lifts to a G-action on the Spin^c-structure c_0.

Then, the G-index of the Dirac operator D_X can be written as $\operatorname{ind}_G D_X = \sum_{j=0}^{2} k_j \mathbb{C}_j \in R(G) \cong \mathbb{Z}[t]/(t^3 = 1)$.

On the other hand, it is well-known that $\mathrm{SW}_X(c_0) = \pm 1$ for the standard $K3$ surface X. Therefore, in the case when G acts on (X, c_0), we have $k_j > 1$ for some j by Theorem 2.2.

Coefficients k_j are calculated by the G-spin theorem. For the fixed generator $g \in G$, G-index $\operatorname{ind}_g D_X$ is calculated by the formula

$$\operatorname{ind}_g D_X = \sum_{j=0}^{2} \zeta^j k_j,$$

where $\zeta = \exp(2\pi\sqrt{-1}/3)$.

By G-spin theorem, it is easy to get

$$\begin{aligned} \operatorname{ind}_g D_X &= k_0 + \zeta k_1 + \zeta^2 k_2 = \frac{1}{3}(m_+ - m_-), \\ \operatorname{ind}_{g^2} D_X &= k_0 + \zeta^2 k_1 + \zeta k_2 = \frac{1}{3}(m_+ - m_-), \\ \operatorname{ind}_1 D_X &= k_0 + k_1 + k_2 = 2. \end{aligned}$$

Solving these equations, we have

$$\begin{aligned} k_0 &= \frac{1}{9}\left\{6 + 2(m_+ - m_-)\right\}, \\ k_1 = k_2 &= \frac{1}{9}\left\{6 - (m_+ - m_-)\right\}. \end{aligned}$$

In the case of an action of type A_1, $m_+ = 3$ and $m_- = 6$. Hence, we have $k_0 = 0$ and $k_1 = k_2 = 1$. Therefore there is no j so that $k_j > 1$. This is a contradiction.

Theorem 3.4 is true for the smooth structure such that the SW invariant for the Spin^c-structure with trivial determinant line bundle is not congruent to 0 modulo 3. Let us examine elliptic surfaces which are homeomorphic to $K3$. Consider minimal regular elliptic surfaces with at most two multiple fibers whose Euler number is 24. Let p and q be the multiplicities of multiple fibers, and let us write such elliptic surface as $E(2)_{p,q}$. (We assume that p and q may be 1.) The following are known about $E(2)_{p,q}$.

(1) $E(2)_{1,1}$ (no multiple fiber) is diffeomorphic to the standard $K3$.
(2) $E(2)_{p,q}$ is homeomorphic to the $K3$ surface if and only if $\gcd(p,q) = 1$.
(3) $E(2)_{p,q}$ is not diffeomorphic to $E(2)_{p',q'}$ if $pq \neq p'q'$.
(4) Let c_0 be the Spin^c-structure with trivial determinant line bundle. If p and q are odd, then $\mathrm{SW}_{E(2)_{p,q}}(c_0) = \pm 1$.

Thus we see that the type A_1 can not be realized by a smooth action on $E(2)_{p,q}$ such that $\gcd(p,q) = 1$ and p and q are odd. Note that there are infinitely many (p,q) which give different smooth structures.

At last, we give some remarks.

Kiyono's actions are nonsmoothable with respect to any smooth structure on spin 4-manifolds. Because $\frac{10}{8}$-inequality is an inequality about b_2 and $\text{sign}(X)$, does not depend on smooth structures.

Our actions use SW-invariants, so we need to check the value of SW invariant for each smooth structure in order to judge the nonsmoothability. This fact suggests that our examples could be subtle in that the smoothability of each action might depend on smooth structure.

So the following might happen:

$\exists\ \sigma,\ \sigma'$: two distinct smooth structures such that the action is smooth with respect to σ, nonsmoothable with respect to σ'.

Problem: Is there such a σ'?

[Chen-Kwasik]: ∃ locally linear action on $K3$, which is smoothable with respect to the standard smooth structure, nonsmoothable with respect to (∃) exotic smooth structure.

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