# How to see the saddle sets of Smale flows in 3-manifolds

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# Outline

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# Abstract

A smooth flow $\phi_t$ is said to be a Smale flow if the chain recurrent set of $\phi_t$ is hyperbolic and has zero or one dimension. A one-dimensional basic set $\Omega$ of a flow $\phi_t$ on a 3-manifold having index 0 or 2 must be a closed orbit. A basic set of index 1, i.e. a saddle set, may have very complicated structure. The famous Smale horse-shoe is a typical example. In this talk, we shall illustrate how to describe the complexity of a one-dimensional saddle set in 3-manifold by using topological invariant.

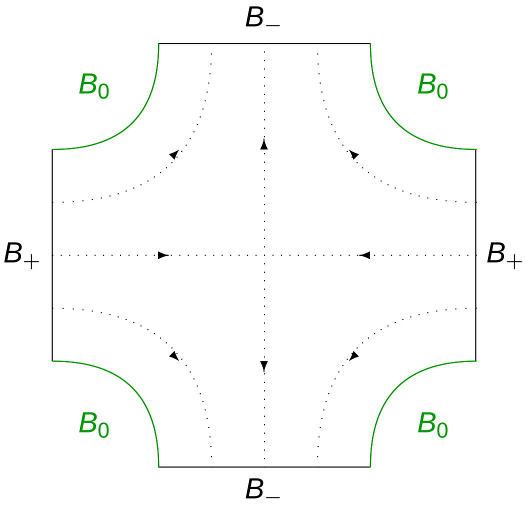
# Isolating block

## Definition

A sub-manifold $B$ with corners of $M$ is an *isolating block* for the flow $\phi_t$ if

- $\partial B = \partial_+ B \cup \partial_0 B \cup \partial_- B$ is a union of submanifolds with boundaries of codimension one.
- the vector field $\dfrac{d\phi_t}{dt}$ points inward on $\partial_+ B$ and points outward on $\partial_- B$.
- for any $x \in \partial_0 B$, there is an interval $[T_1, T_2]$ containing 0 such that $\phi_{[T_1,T_2]}(x) \subseteq \partial_0 B$, $\phi_{T_1}(x) \in \partial_+ B$ and $\phi_{T_2}(x) \in \partial_- B$.

# An isolating block for a singular point



# invariant set

- A subset $K$ of $M$ is said to be an invariant set of a flow $\phi_t$ if $\cup_{-\infty<t<\infty}\phi_t(K) = K$.
- The maximal invariant set of a given set $B$ is $\cap_{-\infty<t<\infty}\phi_t(B)$

**Theorem**

*(J. Montgomery 1973) If two isolating blocks B and $B'$ have the same maximal invariant set, then $(B, \partial\_B)$ and $(B', \partial\_B')$ have the same homotopy type.*

The homotopy type of pair $(B, \partial\_B)$ is the Conley index of the maximal invariant set $\cap_{-\infty<t<\infty}\phi_t(B)$.

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# Hyperbolic chain recurrent sets

A invariant set $K$ of a smooth flow $\phi_t$ is said to be hyperbolic if

- the tangent bundle $TM = E^s \oplus E^u \oplus E^c$ is a Whitney sum of the three sub bundles over $K$, each sub bundle is invariant under $\phi_t$,
- the vector field tangent to $\phi_t$ spans $E^c$,
- there are $C > 0$ and $\lambda > 0$ such that,

$$\|D\phi_t(v)\| \leq Ce^{-\lambda t}\|v\|, \ \|D\phi_t(w)\| \geq Ce^{\lambda t}\|w\|, \ t \geq 0, v \in E^s, w \in$$

The sub bundles $E^s$ and $E^u$ are respectively said to be stable and unstable bundles of $\phi_t$. The fiber dimension of unstable bundle $E^u$ is called the index of hyperbolic set $K$.

# Smale flow

## Definition

A smooth flow $\phi_t$ is said to be a Smale flow if the chain recurrent set of $\phi_t$ is hyperbolic and has zero or one dimension. A Smale flow said to be non-singular if it does not contain any singular point.

## Theorem

*(Smale 1967) If chain recurrent set $\mathfrak{R}$ is hyperbolic, then $\mathfrak{R}$ is a finite disjoint union of compact invariant sets $\Omega_1, \Omega_2, \ldots, \Omega_n$. Each $\Omega_i$, which said to be a* basic set *of flow $\phi_t$, contains an orbit of $\phi_t$ which is dense in it.*

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# Conley index for Smale flow

For a basic set $\Omega$ with dimension 1 of a Smale flow $\phi_t$ on a 3-manifold,

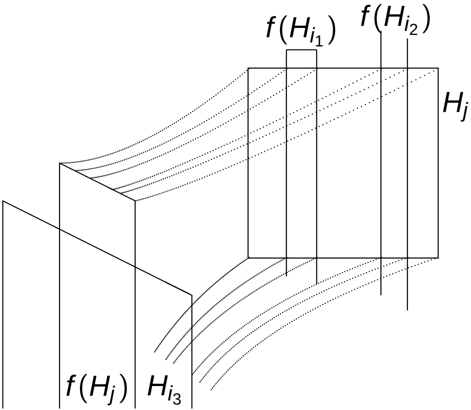
| hyperbolic index | $\Omega$ | Conley index |
|---|---|---|
| 0 | attracting closed orbit | $(S^1 \times D^2, \emptyset)$ |
| 1 | saddle set | ?? |
| 2 | repelling closed orbit | $(S^1 \times D^2, S^1 \times \partial D^2)$ |

A basic set $\Omega$ with hyperbolic index 1 may be a closed orbit. But this is not all. For example, it may be a suspension of Smale horse shoe.

# for a saddle set

## Proposition

*(Birman, Williams1983) Let $\Omega$ be a basic with index 1 of a flow on a 3-manifold $M$. Then there is a Poincaré section $Q$ to the basic set $\Omega$ with a return map $f : Q \to Q$. Moreover, there are finitely many disjoint rectangles $\{H_i\}$ in $Q$ such that*
*(1) $\cup_i (\Omega \cap H_i) = Q \cap \Omega$;*
*(2) as a rectangle, each $H_i$ is coordinated by local stable and unstable manifolds;*
*(3) if $f(H_i) \cap H_j \neq \emptyset$, then $f(H_i)$ stretches completely across $H_j$ in the unstable direction, and $f^{-1}(H_j)$ stretches completely across $H_i$ in the stable direction;*
*(4) $f(H_i) \cap H_j$ has at most one connected component.*


$f(H_{i_1})$
$f(H_{i_2})$
$H_j$
$f(H_j)$
$H_{i_3}$

# Structure matrix

Since each handle is very small, the map $Df_x : E^u_x \to E^u_{f(x)}$ will preserve (or reverses) the unstable orientation simultaneously for all $x \in H_i \cap f^{-1}(H_j)$.
One can define the structure matrix $A = (a_{ij})$, where $a_{ij}$ is defined to be 1 (or $-1$) if $Df_x$ will preserve (or reverses) the unstable orientation and if $f(H_i) \cap H_j \neq \emptyset$; to be 0 if $f(H_i) \cap H_j = \emptyset$.

**Example**

The structure matrix of suspension of “horse shoe” is

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

# Structure matrix

Let $A$ be a structure matrix of a basic set $\Omega$ of a non-singular Smale flow on some 3-manifold with hyperbolic index 1. Then

1. $\det(I - A)$ is an invariant of $\Omega$,
2. the Conley index of $\Omega$ is $(B, \partial_- B)$, where
   1. $B$ is a handlebody of genus $-1 + \sum |a_{ij}|$,
   2. and $\partial_- B$ is a 2-sphere with $\sum |a_{ij}|$ holes.

Note that $\partial B$ is a double of $\partial_- B$.

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# Realization results of Franks

**Theorem**

*(Franks) Suppose A is an irreducible integer matrix. Then there exists a non-singular Smale flow $\phi_t$ on some 3-manifold such that a basic set of $\phi_t$ has structure matrix A, and the other basic sets are closed orbits.*

**Theorem**

*(Franks) If L is link in $S^3$, then there exists a non-singular Smale flow $\phi_t$ such that the set of attractors is L and $\phi_t$ has a single unknotted repeller.*

# General question

Given a description (Conley index, structure matrix) of some saddle set, is there a non-singular Smale flow having one attractor, one repeller such that its saddle set matches the given description.

Such a flow is said to be a simple non-singular Smale flow.

One may ask for more information about the three basic sets. Are they knotted, linked, braided or twisted?

If the structure matrix of the saddle set for some simple non-singular Smale flow in $S^3$ is $A$ , then $\det(I - A)$ is the link number of the attractor and the repeller.

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# Recent result

**Theorem**

*(M. Sullivan 2002) Let $\phi_t$ be a non-singular simple Smale flow in $S^3$. If the the saddle set is modeled by Lorenz template (with structure matrix ($\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$)), then the link consisting of the attractor and the repeller is either a Hopf link, or a trefoil and meridian. In the latter case the saddle is unknotted and untwisted.*

# Our observation

## Theorem

*(working with Bin YU)*
*Let $\phi_t$ be a non-singular simple Smale flow in $S^3$. If the the saddle set is modeled by Lorenz template (with structure matrix $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$), then the knot type of the attractor or the repeller is either a unknot, a trefoil or a $(p, 3)$-torus knot.*