THE HOMOTOPY GROUPS RELATED TO $L_2T(m)/(v_1)$

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- Part I Background
- Part II Toward the homotopy group of $L_2T(m)/(v_1)$
- Part III Construction of the cocycle for $m > 1$

1 Background

We will work in the stable homotopy category, all the spectra(spaces) will be localized at an odd prime p.

• The generalized homologies:

In the stable homotopy category, the generalized homology is referred as a spectrum.

(1) BP-spectrum :

$$BP_* = \pi_*(BP) = Z_{(p)}[v_1, v_2, \cdots, v_n, \cdots]$$
$$Z_{(p)} = \{\frac{m}{n} | p \nmid n\}$$

$$BP_*(BP) = \pi_*(BP \wedge BP) = BP_*[t_1, t_2, \cdots, t_n, \cdots]$$

(2) E(n):

Consider the BP_*-modules

$$E(n)_* = v_n^{-1} BP_*/(v_{n+i}|i > 0) = Z_{(p)}[v_1, \cdots, v_{n-1}, v_n, v_n^{-1}]$$

$E(0)_*$ is the rational Q. By the exact functor theorem of Landweber, that

$$E(n)_*(X) = E(n)_* \otimes_{BP_*} BP_*(X)$$

is a homology theory. Then using the representation theorem, we have a spectrum $E(n)$ such that

$$E(n)_*(X) = \pi_*(E(n) \wedge X)$$

(3) $\mathbf{E_m(2)}$:

Consider the $E(2)_*$-modules,

$$E_m(2)_* = E(2)_*[v_3, \cdots v_{m+2}] = Z_{(p)}[v_1, v_2^{\pm 1}, v_3, \cdots, v_{m+2}]$$

$E_m(2)$ is also a spectrum.

• Localization with respect to generalized homology

If E is a spectrum, a spectrum X is **E-acyclic** if $E \wedge X$ is null. A spectrum is **E-local** if every map from an E-acyclic spectrum to it is null. A map $X \to Y$ is an **E-equivalence** if it induces an isomorphism on E_*-homologies. Bousfield shows that there is a functor called E-localization $L_E X$, and a natural transformation

$$X \longrightarrow L_E X$$

which is an E-equivalence.

Examples:

A spectrum X localized with respect to the Eilenberg-MacLane spectrum KZ/p is the spectrum itself, i.e., $L_{KZ/p}X = X$. We also have $L_{BP}X = X$.

The localization with respect to the Johnson-Wilson spectrum $E(n)$ is referred to L_n, i.e., $L_nX = L_{E(n)}X$.

By the chromatic convergence theorem, there is a tower

$$\cdots \longrightarrow L_nX \longrightarrow L_{n-1}X \longrightarrow \cdots \longrightarrow L_0X \tag{1.1}$$

for any finite spectrum X. And the inverse limit of L_nX is X. So from L_EX, we could study the part of $\pi_*(X)$ which E sees.

• The Ravenel spectrum $\mathrm{T(m)}$ and homotopy groups of localized spectra:

The Ravenel spectrum $T(m)$ is characterized by BP_*-homology

$$BP_*(T(m)) = BP_*[t_1, \cdots, t_m] \subset BP_*(BP) = BP_*[t_1, t_2, \cdots, t_n, \cdots].$$

There is a tower of spectra

$$S^0 = T(0) \hookrightarrow T(1) \hookrightarrow T(2) \hookrightarrow \cdots \hookrightarrow T(n) \hookrightarrow \cdots \hookrightarrow BP$$

$\pi_*(\mathbf{L_2 S^0})$

- $\pi_*(L_2S^0)$ for $p > 3$ is detected by Shimomura and Yabe. (Topology, 1995)

- Shimomura and Wang determined $\pi_*(L_2S^0)$ at the prime 3. (Topology, 2002)
- Shimomura and Wang determined the Adams-Novikov E_2-term for $\pi_*(L_2S^0)$ at the prime 2 also. But because of the Adams-Novikov differentials, it is far from the full understanding of the homotopy groups.

$\pi_*(L_2T(m))$:

Let $T(m)/(v_1)$ denote the cofiber of the self map $v_1 : \Sigma^{2(p-1)}T(m) \longrightarrow T(m)$. Then by the cofiber sequence:

$$\Sigma^{2(p-1)}T(m) \xrightarrow{v_1} T(m) \longrightarrow T(m)/(v_1)$$

We should consider $L_2T(m)/(v_1)$. In this lecture,we will consider the homotopy groups of $L_2T(m)/(v_1)$.

2 Toward the homotopy group of $L_2T(m)/(v_1)$

For a spectrum X, a basic method to compute its hopmotopy groups is the Adams-Novikov spectral sequence, the E_2-term of the spectral sequence is $Ext^{s,t,}_{BP_*(BP)}(BP_*, BP_*(X))$ and it converges to $\pi_*(X)$.

To compute the E_2-term $Ext^{s,t,}_{BP_*(BP)}(BP_*, BP_*(X))$, we have the chromatic spectral sequence and the Bockstein spectral sequence.

• The chromatic spectral sequence:

Let $T(m)/(p^\infty, v_1)$ be the cofiber of the localization map $T(m)/(v_1) \to L_0T(m)/(v_1)$

$$T(m) \longrightarrow L_0T(m) \longrightarrow T(m)/(p^\infty)$$

Apply the localization functor L_2, we have the following cofiber sequence:

$$L_2T(m)/(v_1) \longrightarrow \underline{L_0T(m)/(v_1)} \longrightarrow L_2T(m)/(p^\infty, v_1) \qquad (2.1)$$

These cofiber sequences induce long exact sequences of homotopy groups. Thus it suffices to compute the homotopy groups

$$\pi_*(L_0T(m)/(v_1)),\ \ \pi_*(L_2T(m)/(p^\infty, v_1))$$

The homotopy groups of $\pi_*(L_0T(m)/(v_1))$ is known to be rational homotopy.

- **The Bockstein spectral sequence:**

To compute $\pi_*(L_2T(m)/(p^\infty, v_1))$, denote the cofiber of $T(m) \xrightarrow{p} T(m)$ by $T(m)/(p)$

$$T(m) \xrightarrow{p} T(m) \longrightarrow T(m)/(p).$$

There is the cofiber sequence

$$L_2T(m)/(p, v_1) \longrightarrow L_2T(m)/(p^\infty, v_1) \xrightarrow{p} L_2(T(m)/(p^\infty, v_1) \qquad (2.2)$$

which induces a long exact sequence of homotopy groups.

Let

$$\Gamma(2,m) = E_m(2)_*[t_{m+1}, t_{m+2}, \cdots] \otimes_{BP_*} E_m(2)_*$$

By change of rings theorem, we could get

$$Ext^{s,t,}_{BP_*BP}(BP_*, BP_*(L_2(T(m) \wedge X)) \cong Ext^{s,t}_{\Gamma(2,m)}(E_m(2)_*, E_m(2)_*(X)).$$

By the equations above, we simplify all the items from (BP_*, BP_*BP) to $(E_m(2)_*, \Gamma(2,m))$.

Let $M_2^0 = E_m(2)_*/(p, v_1)$,$L_1^1 = E_m(2)_*/(p^\infty, v_1)$,$L(k,1) = E_m(2)_*/(p^k, v_1)$, $Ext^{s,t}_{\Gamma(2,m)}(E_m(2)_*, M) = H^{s,t}(M)$ for short. Then we will have the short exact sequence of $E_m(2)_*E_m(2)$-comodules

$$0 \longrightarrow M_2^0 \longrightarrow L_1^1 \xrightarrow{p} L_1^1 \longrightarrow 0.$$

This short exact sequence induces the following long exact sequence

$$\cdots \to H^{s,*}M_2^0 \longrightarrow H^{s,*}L_1^1 \xrightarrow{p} H^{s,*}L_1^1 \xrightarrow{\delta} H^{s+1,*}M_2^0 \to \cdots .$$

$H^{*,*}M_2^0 = \mathbb{Z}/p[v_2^{\pm 1}, v_3, \cdots, v_{m+2}] \otimes E[h_{m+1}^0, h_{m+1}^1, h_{m+2}^0, h_{m+2}^1]$ where h_i^j denotes the cohomology class represented by $t_i^{p^j}$.

The connecting homomorphism δ:

A simple description of δ is necessary. By the long exact sequence above and the structure of $H^{*,*}M_2^0$, the essential part is the differential of t^p_{m+1} and t^p_{m+2}.

A cocycle of t^p_{m+1} or t^p_{m+2} is necessary, if not, the structure of δ will be too complex to compute.

When $m = 1$, we could find a cocycle in $H^{*,*}L_1^1$, and finally we get $\pi_*(L_2T(1)/(v_1))$. while $m > 1$,unluckily, we did not find a cocycle in $H^{*,*}L_1^1$.

$\mathbf{L_2T(m)/(p^{[\frac{m+1}{2}]+2}, v_1)}$:

We find cocycle just in $H^{s,*}L(k,1)$ while $k = [\frac{m+1}{2}] + 1$. Finally, we get the homotopy group of $L_2T(m)/(p^{[\frac{m+1}{2}]+2}, v_1)$.

When $m > 1$,the short exact sequence will be

$$0 \to M_2^0 \xrightarrow{p^k} L(k+1,1) \longrightarrow L(k,1) \to 0.$$

This short exact sequence induces the following long exact sequence

$$\cdots \to H^{s,*}M_2^0 \xrightarrow{p^k} H^{s,*}L(k+1,1) \longrightarrow H^{s,*}L(k,1) \xrightarrow{\delta} H^{s+1,*}M_2^0 \to \cdots.$$

$\pi_*(\mathbf{L_2T(1)/(v_1)})$:

$$
\begin{aligned}
\widetilde{C}_0 =& \mathbb{Z}_{(p)}\{v_2^{sp^n} v_3^{tp^m}/p^{min\{m,n\}+1} | 0 \leqslant m,n \leqslant \infty\},\\
\widetilde{C}_1 =& \widetilde{C}_1^0 \oplus \widetilde{C}_1^1,\\
& \widetilde{C}_1^0 = \mathbb{Z}_{(p)}\{v_2^{sp^n} v_3^{tp^m-1} h_3^0/p^{n+1} | 0 \leqslant n < m < \infty\},\\
& \widetilde{C}_1^1 = \mathbb{Z}_{(p)}\{v_2^{sp^n-1} v_3^{tp^m} h_2^0/p^{m+1} | 0 \leqslant m \leqslant n \leqslant \infty\},\\
h_3^1\widetilde{C_0} =& \mathbb{Z}_{(p)}\{v_2^{sp^n} v_3^{tp^m} h_3^1/p\}.
\end{aligned}
$$

Theorem 2.3 $H^{*,*}L_1^1$ and then the homotopy groups of $L_2T(1)/(p^\infty, v_1)$ are isomorphic to the direct sum of

$$(\widetilde{C_0} \oplus \widetilde{C_1}) \otimes E[h_2^1] \quad \text{and} \quad h_3^1\widetilde{C_0} \otimes E[h_2^0, h_3^0].$$

Theorem 2.4 The homotopy groups of $L_2T(1)/(v_1)$ are isomorphic to

$$\pi_*(L_2T(1)/(v_1)) = \mathbb{Z}_{(p)} \oplus (\pi_*(L_2T(1)/(p^\infty, v_1)) - \mathbf{Q}/\mathbb{Z}_{(p)}).$$

3 Construction of the cocycle for $m > 1$

We do it by the structure map Δ of $(E_m(2)_*, \Gamma(2,m))$:

In (BP_*, BP_*BP),we have

$$\Delta(t_n) = \sum_{i+j=n} t_i \otimes t_j^{p^i} + v_{n-1}b_1^{n-2} + v_{n-2}b_2^{n-3} + \cdots + v_1 b_{n-1}^0$$

$$pb_n^k = \sum_{i=1}^{n-1} v_i^{p^{k+1}} b_{n-i}^{k+i} + \sum_{i+j=n} t_i^{p^{k+1}} \otimes t_j^{p^{i+k+1}} - \Delta(t_n)^{p^{k+1}}.$$

Then, in $(E_m(2)_*, \Gamma(2,m))$, we compute Δ until $n = 2m+3$. After that, we construct elements $c_{i,j}(m+1 \leqslant i \leqslant 2m+1)$ such that

$$d(c_{i,j}) = p^{[\frac{i+1}{2}]} b_{m+i}^{j-1}$$

With these elements, we can eliminate the items b_i^j. Finally we will get a cocycle.

THANK YOU VERY MUCH!!!