Pseudo-Chaotic Orbits of Kicked Oscillators

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Outline

1. One-dimensional oscillators driven by impulsive kicks, periodic in $x$ and $t$
   - Resonant $\sin(x)$ kick amplitude; chaos and pseudochaos in stochastic webs
2. Resonant sawtooth oscillator with quadratic irrational parameter
   - Hamiltonian and equations of motion
   - Poincaré section
   - Local and global (piecewise) rotations
   - Dynamical self-similarity: return-map scaling, recursive tiling
   - Scale factors, exponents, fractal dimensions
   - Symbolic representation and dynamics
   - Lifting to the plane: appearance of a new scale factor
   - Long-$t$ behavior of aperiodic orbits: diffusive, super-diffusive, sub-diffusive, ballistic
3. Resonant sawtooth oscillator with cubic irrational parameter
   - Infinitely many invariant components; multi-fractal structure; long-$t$ behavior
4. Nonresonant sawtooth oscillator
   - Invariant measure of aperiodic orbits; numerical explorations
Hamiltonian and Equations of Motion

\[ H(x,p) = \frac{1}{2}(p^2 + x^2) + F(x) \sum_{n} (t - 2\pi n) \]

\[ \bar{r} = \text{rotation number} = \# \text{kicks per natural period} \quad \bar{r} = 2\pi \]

(Resonant case: rational rotation number)

\[ F(x) = F(x + 2\pi) \]

\[ \dot{x} = \frac{\partial H}{\partial p} = p \]

\[ \dot{p} = -\frac{\partial H}{\partial x} = -x - F'(x) \sum_{n} (t - 2\pi n) \]

Free oscillation for fraction \( \bar{r} \) of a natural period, followed by momentum shift \( p \rightarrow p + \Delta p \), \( \Delta p = -F'(x) \) = "kick amplitude"

Sawtooth kick amplitude: piecewise linear function of \( x \)
Sawtooth Kick Function ($\bar{a} = \sqrt[3]{2}$)

$$f(y) = \bar{a} (y \mod \bar{a})$$

$$y \mod \bar{a} \begin{cases} [0, \bar{a}) & \bar{a} > 0 \\ (\bar{a}0] & \bar{a} < 0 \end{cases}$$

Kicked oscillator map on the plane:

$$W \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{a} x \\ \bar{a} (y \mod \bar{a}) \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Kicked oscillator map folded onto the fundamental cell:

$$K \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{a} x \\ \bar{a} (y \mod \bar{a}) \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}$$
Map $K$ in the Dynamical Systems Literature


2. Sawtooth standard map (Ashwin, Dana)

3. Rounded-off oscillator (JHL, Vivaldi, et al)

4. Pseudochaos (Zaslavsky et al)

5. Piecewise isometry (Adler, Kitchens, Tresser (AKT), Goetz, Kahng, Poggiaspalla, et al)

Direct antecedents of this work:

G. Poggiaspalla, J.H.L. unpublished

Scaling and recursive tiling for quadratic irrational parameter.
Symbolic dynamics

Lifting $K$ to obtain $W$. Some questions:

Do periodic orbits get promoted to accelerator modes?
Can aperiodic orbits escape to infinity? If so, is there an asymptotic power law? (sub-diffusive? diffusive? super-diffusive?) Do orbits on the discontinuity set play a special role?
Scaling Domain \((\frac{a}{b} = -\sqrt{2})\)

Limit point of scaling sequence

**Diagram:**
- **Scaling Domain:** A square is shown with the scaling domain marked. The domain is labeled as \(D(0)\) and \(D(1)\).
- **Return Map:** An arrow points from \(D(0)\) to \(D(1)\) indicating a return map action.
- **Limit Point:** A point labeled as the limit point of the scaling sequence is indicated.

**Mathematical Notation:**
- \(W_D(0)\)
- \(0, 1, 2, 0\)
- \(D(0), D(1)\)
- \(D_0(0), D_1(0)\)
Recursive Tiling

Tiling of $\tilde{\omega}$ by sub-domains of $D(0)$

Tiling of $D(0)$ by sub-domains of $D(1)$

Tiling of $\tilde{\omega}$ by sub-domains of $D(1)$
Action of $K$ on $\square$

$$K u = C u - d(u)\square$$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$d(u) = \begin{cases} 1 & u \square 0 \\ 0 & u \square 1 \\ -1 & u \square 2 \end{cases}$$
Action of map $W$

Compare action of $K$:

\[ K u = C u - d(u) \]

\[ W [u, z] = [K u, F z + d(u)] \]

\[ F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ d(u) = d_j , \quad u \in \mathbb{Z}_2^j , \quad z \in \mathbb{Z}^2 \]
Tiles

$D_0(0) = D_0^0(0), D_0^1(0), \ldots, D_0^{18}(0)$

$K$ - orbit of $D_0(0)$ in $\mathbb{R}^2$ = $\mathbb{R} \times \mathbb{Z}^2$

$W$ - orbit of $D_0(0)$ in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{Z}^2$

Tiles $[D_0^0(0), (0,0)], [D_0^1(0), (0,1)], [D_0^2(0), (1,-1)], \ldots, [D_0^{18}(0), (0,0)]$
Combining orbits initiated in cells \([D(0), (0,0)]\), \([D(0), (0,1)]\), \([D(0), (1,1)]\), and \([D(0), (1,0)]\) with the lowest-level periodic orbits (yellow octagons) produces a **supertile**. These invariant non-convex polygonal regions tile the plane with periodicity \(2t\) in both \(x\) and \(y\) directions. Here \(t = -\sqrt{2}/2\).
Local Scaling

Level \( L = 0, 1, 2, \ldots \)

\[
D_0(L+1) = K D_0(L)
\]

\[
\square(L+1) = K \circ \square(L+1) \circ K^{-1}
\]

\( K \) = local geometric scale factor

If \( \square_j(L) \) is the restriction of \( \square(L) \) to sub-domain \( D_j(L) \),

\[
\square_j(L+1) = \square_{p(j, j^{-1})}(L) \circ \ldots \circ \square_{p(j, 1)}(L) \circ \square_{p(j, 0)}(L)
\]

\( p(j, t) = \) path function \( (j=0, \ldots, J-1, \ t=0, 1, \ldots, j^{-1}) \)
Let $T_j(L)$ be the **first-return time** (number of $K$ iterations) for the orbit of $D_j(L)$.

Let $A$ be the **incidence matrix** defined by

$$A_{ij} = \# \{ t : p(j,t) = i \} , \ i,j = 0, \ldots, J-1$$

where $p(j,t)$ is the path function. Thus

$A_{ij}$ is the number of times the first-return orbit of $D_j(L+1)$ visits $D_i(L)$.

Recursion relation for return times:

$$T_j(L+1) = \sum_{k=0}^{J-1} T_{p(j,k)}(L+1) = \sum_{i=0}^{J-1} T_i(L) A_{ij}$$

$\Box_T = \text{largest eigenvalue of transpose of } A$

$\Box_T = \text{temporal scale factor}$

Fractal dimension (Hausdorff or box-counting) for aperiodic orbits of $K$:

$$\frac{\log \Box_T}{\log \Box_K}$$
Symbolic Representation and Dynamics

For aperiodic \( Z \),

\[
j_0 \searrow D^{t_1}_{j_1}(1) \searrow D^{t_1t_2}_{j_2}(2) \searrow D^{t_1t_2t_3}_{j_3}(3) \searrow \cdots \searrow \{ Z \}
\]

we write

\[
z \iff (j_1, j_2, j_3, \ldots)
\]

\[
j_k = (j_k, t_k)
\]

Path constraint: \( j_k = p(j_{k+1}, t_{k+1}) \)

**Action of local map or global map:**

\[
Kz \iff (\bar{j}_1', j_2, j_3, \ldots) \iff Wz
\]

where

if \( t_1 < j_1 = 1 \), \( j_1' = (j_1, t_1+1) \), \( j_2' = j_2 \), \( j_3' = j_3 \), ...

if \( t_1 = j_1 = 1 \), \( j_1' = (p(j_2, t_2+1), 0) \),
\( t_2 < j_2 = 1 \), \( j_2' = (j_2, t_2+1) \), \( j_3' = j_3 \), \( j_4' = j_4 \), ...

etc.

**Vershik map** (example: Gregorian calendar)
Scaling for the lifted map: a third scale factor

Represent $\mathbb{R}^2$ as

$$\mathbb{R}^2 \times \mathbb{Z}^2 = \{ [(x,y),(m,n)] : (x,y) \in \mathbb{R}^2, (m,n) \in \mathbb{Z}^2 \}$$

Then, for $u \in \mathbb{R}^j$, $z \in \mathbb{Z}^2$,

$$W[u,z] = [Ku, Fz + dj] = [Cu - \square dj, Fz + dj]$$

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For the return maps, $u \in \mathbb{R}D_j(L+1)$, $z \in \mathbb{Z}^2$,

$$W^{(L+1)}[u,z] = [\square_{K(L+1)}u, F^{(L+1)}z + d_j(L+1)]$$

where

$$d_j(L+1) = F^{(j,0)}d_{p(j,0)}(L) + F^{(j,1)}d_{p(j,1)}(L) + \cdots + d_{p(j,J-1)}(L)$$

$$\square(j,t) = \begin{pmatrix} T_{p(j,k)} \\ k = t+1 \end{pmatrix}$$

$$d_j(L+1) = \begin{pmatrix} M_{j} \end{pmatrix} d_k(L)$$

$M_{jk} = 2 \times 2$ matrix of integers
In place of \( \mathbb{R} \times \mathbb{Z}^2 \), we may represent \( \mathbb{R}^2 \) as \( \mathbb{R} \times \mathbb{C} \), \( \mathbb{C} = \) complex plane. The formulas of the previous slide remain valid with the replacements

\[
d_j(L) = (m,n) \mathbb{Z}^2 \xrightarrow{\longrightarrow} n + i m \quad (i = \sqrt{-1})
\]

\[
F^p (m,n) \xrightarrow{\longrightarrow} i^p (n + i m)
\]

Now the recursion matrix \( \mathbf{M} \) has entries which are Gaussian integers (Re and Im parts are integers). The asymptotic scaling properties, for \( L \xrightarrow{\longrightarrow} \infty \), of the translation vectors \( d_j(L) \) are revealed by examining the Jordan canonical form \( \mathcal{J}(\mathbf{M}) \) of \( \mathbf{M} \). The **global geometric scale factor** \( \mathbb{W}_W \) is defined as the absolute value of the largest-magnitude diagonal element of \( \mathcal{J}(\mathbf{M}) \).

Fractal dimension associated with aperiodic orbits (or asymptotically long periodic orbits)

\[
D_W = \frac{\log \mathbb{W}_T}{\log \mathbb{W}_W}
\]

Asymptotic distance from initial point \( \sim t \mathbb{W}^{-1} \)

\[
\mathbb{W} = D_W^{-1}
\]
## Classification of Quadratic-Kicked Oscillator Models

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$K_\alpha$</th>
<th>$T$</th>
<th>$W$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1 + \sqrt{5})/2$</td>
<td>$(3 \sqrt{5})/2$</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$(1 - \sqrt{5})/2$</td>
<td>$(3 - \sqrt{5})/2$</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td>$3 - 2\sqrt{2}$</td>
<td>9</td>
<td>1</td>
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<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\sqrt{2}$ (AKT-R)</td>
<td>$3 - 2\sqrt{2}$</td>
<td>9</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\sqrt{3}$</td>
<td>$7 - 4\sqrt{3}$</td>
<td>25</td>
<td>4</td>
<td>.43067... sub-diffusive</td>
</tr>
<tr>
<td>$\sqrt{3}$ (A)</td>
<td>$2 - \sqrt{3}$</td>
<td>4</td>
<td>2</td>
<td>.5 diffusive</td>
</tr>
<tr>
<td>$\sqrt{3}$ (B)</td>
<td>$2 - \sqrt{3}$</td>
<td>5</td>
<td>2</td>
<td>.43067... sub-diffusive</td>
</tr>
<tr>
<td>$\sqrt{2}$ (AKT-D)</td>
<td>$3 - 2\sqrt{2}$</td>
<td>9</td>
<td>5</td>
<td>.732487... super-diffusive</td>
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<td>$(1 + \sqrt{5})/2$</td>
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</table>
\[ \Box = 3 \quad \Box = \sqrt{3} \]

Generating domain \[ \Box \]

Scaling domains

A: limit point (0,0)

B: limit point \((41+72\Box, 41+72\Box)\)
\[ \frac{p}{q} = \frac{5}{12} \]
p/q=5/12

Tiling of the Fundamental Domain near the Limit Points at $(\frac{41}{1}, \frac{72}{1})$ and $(41+72, 41+72)$
Accelerator Mode ($\square = -\sqrt{3}$, sequence B)

Level 2, 1000 periods, 5,707,236,000 itns. of $W$
Accelerator Mode \( \sqrt{3} \)

Level=6, 9 local periods, 10,643,503,428 itns. of \( W \)
Aperiodic Orbit ((3,2),(3,1),(3,1),(3,1),...)
1438699 iterations of $r(0)$
$\ln (m^2 + n^2), (m,n) \in \mathbb{Z}^2$

slope = $\frac{\ln 4}{\ln 5}$

Total iterations: 1,438,699
Histogram of Recurrences ($\nabla = -\sqrt{3}$, B)

bin size = 0.1

$N(t)$

$\ln N(t)$

$\ln t$

$\ln t$

slope $\sim \sqrt{34}$
Aperiodic Orbit(A) : \(((3,4),(7,1),(7,1),(7,1),\ldots)\)

55344 iterations of $r(0)$
\( \ln(m^2+n^2) \)

\[ \text{slope} = 1 \]
\( \Box = \Box \sqrt{2} \)  
Origin at Center of the Square

(Adler, Kitchens & Tresser, Kouptsov, JHL & Vivaldi)
Level 8 Periodic Orbit, First 10000 Iterations

K period: 129,140,163
W period: 516,560,652

Max. excursion from (0,0) approx. = 35
Aperiodic Orbit Starting at $\frac{1}{17} (19 + 29 \cdot 4 + 7)$

30,000 iterations
Same aperiodic orbit uniformly sampled on a logarithmic time scale. Total time interval:

2778986358613839774206999581799344345700854998386139914133990537
1952551860548740895787499137478593

*Lattice coordinates*
Aperiodic orbit beginning at corner point \((-t, -t)\)
(8000 iterations of \(W\))
Orbit of \((0, \square)\) on the discontinuity set.

8000 iterations of \(W\).
Orbit of $W^4$, starting at $(0,\frac{\pi}{2})$ on the discontinuity set.

Log($x$) versus Log($t$) ($10^8$ iterations)
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