Arbitrage Models of Commodity Prices

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What is a Commodity?

- **Property-based definition**: A commodity is any physical good or service traded “in bulk” (i.e., with no differentiation) within predetermined qualitative specifications and across an organized market.

- **Listing-based definition**: A commodity as any entry in a list of goods and services labeled as “commodities”:
  - *Precious metals*: gold, silver, platinum, palladium;
  - *Base metals*: aluminium, copper, iron;
  - *Traditional energy sources*: crude oil, refined products (gas oil, kerosene), natural gas, coal, electricity;
  - *Renewables*: pellets, biofuels (ethanol);
  - *Plastics*;
  - *Agricultural products*: sugar, wheat, corn;
  - *Standardized services*: CO2 permits, shipping freight.
Spot Price Assumptions

- We **assume** that the **asset**:
  1. has a spot price $S$,  
     Example: Oil price cannot be bought spot $\rightarrow$ proxy $=$ forward with the shortest maturity;
  2. is storable,  
     Example: Electricity is (claimed to be) not storable $\rightarrow$ production power is a form of storage.
  3. holding on $[t, T]$ provides cumulated collateral **benefits** $B$ and requires bearing cumulated **costs** $C$ (besides funding costs for buying the asset, *i.e.*, interest).
    Assumptions: a) benefits and costs are known in advance ($=$no randomness) and b) are expressed in EUR at time $T$.
- Additionally, we consider a deterministic **time value of money**.
Convenience Revenue and the Cost of Carry

- **Net Convenience Revenue** = *Net side* benefit from asset holding:
  - **Additive revenue**
    
    \[ +B - C \text{ time } T \text{ EUR}; \]
  - **Yield multiplicative:**
    \[
    P&L_{[t, t+dt]} = dS(t) + (\beta - \chi) \times S(t) \, dt 
    \]
    continuously reinvested in \( S \):
    
    \[ +S(T) e^{(\beta - \chi)(T-t)} - S(T) \text{ time } T \text{ EUR}. \]

- \( c := \beta - \chi = \text{Inst. Avg. Convenience Yield} \) on \([t, T]\).
- **Cost of Carry** = Cost for buying & carrying the asset to delivery:
  
  \[
  CC = (\text{Funding costs} + \text{Side costs } C) - \text{Side benefits } B 
  
  = \text{Funding costs} - (\text{Side benefits } B - \text{Side costs } C) 
  
  = \text{Funding costs} - \text{Net Convenience Revenue} \]
### Examples

<table>
<thead>
<tr>
<th>Asset</th>
<th>benefit</th>
<th>cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dividend paying stock</td>
<td>dividend</td>
<td>-</td>
</tr>
<tr>
<td>Consumption commodity</td>
<td>consumption</td>
<td>inventory</td>
</tr>
<tr>
<td>Perishable good</td>
<td>consumption</td>
<td>replacing</td>
</tr>
<tr>
<td>Investment commodity</td>
<td>-</td>
<td>inventory</td>
</tr>
<tr>
<td>Foreign currency</td>
<td>foreign int.rate</td>
<td>-</td>
</tr>
<tr>
<td>Stock index</td>
<td>index yield</td>
<td>-</td>
</tr>
<tr>
<td>Futures contract</td>
<td>interest savings</td>
<td>-</td>
</tr>
</tbody>
</table>
A **Forward** is a contract whereby the holder (the long party) receives in the future one unit of an asset $S$ (the underlying) for a fixed amount $x$ (the delivery price).

The transaction occurs at a future time $T$ (the **delivery date**), but the **delivery price** $x$ is fixed at the contract issuing time $t < T$.

**Pay-off profile:**

<table>
<thead>
<tr>
<th>$t$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fix $x$</td>
<td>$\exists V = ?$</td>
</tr>
<tr>
<td>no intermediate cash flow</td>
<td>$\backslash$ Pay-off $V = S(T) - x$</td>
</tr>
</tbody>
</table>

Forwards are issued for a delivery price $x = f_T^S(t)$ (**Forward price**) making the initial value of the contract equal to zero:

$$\text{Value of the fwd} = V_{fwd}^{t, T, x, S}(t) = 0 \text{ if } x = f_T^S(t) = \text{fwd price.}$$

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We consider two alternative, yet equivalent definitions of static arbitrage. They are used for the purpose of pricing simple derivatives (e.g., fwd’s, swaps).

**Assumption**: Let portfolio $\pi$ be self-financed on $[t, T]$ (e.g., static).

**Def.1 (Standard Arbitrage)**: A portfolio $\pi$ whose value satisfies:

$$V_\pi(t) = 0,$$
$$V_\pi(T) \text{ is deterministic and } > 0.$$

(i.e., we invest nothing at $t$ and obtain a positive amount for sure at a later time.)

**Def.2 (Replication-based)**: A portfolio $\pi$ whose value satisfies:

$$V_\pi(t) > 0, V_\pi(T) = 0.$$

(i.e., we gain from shorting the portfolio and our position is perfectly hedged.)
**Strategy:** Consider a fwd maturing at $T$ on a flow asset $S$. Let $B - C$ be the time $T$ value of the net benefit (=benefit − costs) accrued to the holder of the underlying asset over the fwd lifetime. A replication argument leads to:

<table>
<thead>
<tr>
<th></th>
<th>Value at $t$</th>
<th>Value at $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 short fwd</td>
<td>$-V_f(t)$</td>
<td>$-S(T) + x(t)$</td>
</tr>
<tr>
<td>Borrowing $$</td>
<td>$- [x(t) + B] e^{-r(T-t)}$</td>
<td>$-x(t) - B$</td>
</tr>
<tr>
<td>Lending $$</td>
<td>$Ce^{-r(T-t)}$</td>
<td>+$C$</td>
</tr>
<tr>
<td>1 long asset</td>
<td>$+S(t)$</td>
<td>$+S(T) + B - C$</td>
</tr>
<tr>
<td>Portfolio $\pi =$</td>
<td>$-V_f(t) - [x(t) + B - C] \times e^{-r(T-t)} + S(t)$</td>
<td>$V_\pi(T) = 0$</td>
</tr>
</tbody>
</table>

**Conclusion:** $V_\pi(T) = 0$ (hedged) $\Rightarrow$ $V_\pi(t)$ must equal 0:

$$V_\pi(t) = 0 \iff V_f(t) = S(t) - [x(t) + B - C] e^{-r(T-t)}.$$
Spot-Forward Parities

- **Additive spot-forward parity:** setting $V_f(t)$ and solving for $x(t) =: f_T(t)$ leads to:

$$f_T(t) = S(t) e^{r(T-t)} - (B - C)$$

- **Multiplicative formulations:**

  $$f_T(t) = S(t) e^{(r-c)\times[T-t]}$$  
  Yield-based Multiplicative

  $$f_T(t) = S(t) \left( 1 + (R - C) [T - t] \right)$$  
  Simple rate-based Multiplic.

where $r - c$ (resp., $R - C$) represents the **yield (resp., simple rate)-based Cost of Carry**.

( (*) See Appendix 1 for a derivation.)
Example: Gold Market (Lease Rate)

- **Two rates** are important in the gold markets:
  1. Gold lease rate $LR_T(t)$,
  2. Gold forward rate $GOFO_T(t)$.

- **Large inventories $\Rightarrow$ Lease market** (typical of gold):
  - Lender (Central bank, fin.institution) earns income from an asset that has not intrinsic return;
  - Borrower (producer, manufacturer, speculator) may use or sell metal and invest in US deposits.

- **Lease Rate** $LR_T(t) =$ simple rate *per annum* at which lenders offer gold at $t$ for lease on $[t, T]$:
  - **Lease cost** $= S(t) \times LR_T(t) \times (T - t)$ EUR;
  - Quotation is *per annum* under **day-count convention** $=$ Actual/360
  - $LR$ may be negative! On April 2005, a short-term lease rate was $-0.07\%$ (bid, by market maker as a borrower of gold) and $0.00\%$ (offer, by lender of gold). The borrower was charging the lender for the side costs of the operation (storage).
Example: Gold Market (GOFO Rate)

- **GOFO** = “gold forward offered rate” (or “gold swap rate”) is the **Cost of Carry** for gold measured as a *simple rate per annum*:

\[
GOFO_T (t) := L_T (t) - C_T (t),
\]

- \( L_T (t) \) = spot LIBOR on \([t, T]\),
- \( C_T (t) \) = gold spot conv.rev.on \([t, T]\) as a *simple rate per annum*.

**Gold Spot-Forward Parity:**

\[
f_T (t) = S (t) \times \left[ 1 + GOFO_T (t) \times (T - t) \right]
\]

\( \Leftrightarrow \quad GOFO_T (t) = \frac{f_T (t) / S (t) - 1}{T - t} \)

- **Remarks:** Cost of Carry \( C = S (t) \times GOFO \times (T - t) \).
**Example: Gold Market (Lease-GOFO Rate Parity)**

- **Proposition:** Gold lease rate measures the Gold Convenience Revenue stemming from holding gold (= gross benefit − storage costs) as a simple rate *per annum*.

  - **Proof:** Lending gold is financially equivalent to the strategy of spot selling gold, investing the proceeds at LIBOR, and forward repurchasing gold at the GOFO rate. While lending gold earns the Lease rate, the equivalent strategy earns LIBOR - GOFO. So, Lease rate = LIBOR - GOFO. Since GOFO = Cost of carry = LIBOR - Convenience Revenue (as a simple rate *per annum*). Hence Lease rate = Convenience Revenue.

- **Gold Rate Parity:**

  \[
  GOFO_T(t) = \text{Cost of money} - \text{Revenue - storage} = \text{LIBOR} - \text{Lease rate} = L_T(t) - LR_T(t).
  \]

- **Example:**

<table>
<thead>
<tr>
<th>Maturity</th>
<th>GOFO%</th>
<th>LIBOR%</th>
<th>Lease rate%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-month</td>
<td>2.95714%</td>
<td>3.05000%</td>
<td>0.09286%</td>
</tr>
</tbody>
</table>
Principle → Focus on:

1. **Primitives** = Input state variables
   → should be quantities with:
   - Reliable observations;
   - Economic significance.

2. **Structural elements** = Form of drift, volatility, jump, if any
   → should be identified using statistical analysis of historical data and then fitted to observed prices.

3. **Driving noise terms** = Number (&nature) of noise terms
   → should be assessed based on historical price analysis (e.g., exam of the trajectorial properties of price paths, Principal Components Analysis, jump filtering).
Model Frameworks

**Reference:** Roncoroni: *Commodity Price Models, in: Cont et al., Encyclopedia of Quantitative Finance, Wiley (forthcoming)*, we identify four classes of arbitrage free models for commodity prices according to selection of primitives:

1. **[SC]** Spot Price-Convenience Yield Models (Gibson-Schwartz (1990))
   → primitives = spot price + instantaneous spot convenience yield;

   → primitive = forward price curve;

3. **[FC]** Forward Convenience Yield Models (Cortazar-Schwartz (1994))
   → primitives = spot price + instantaneous fwd convenience yield;

4. **[SP]** Spot Price Models (Black (1976))
   → primitive = spot price (deterministic convenience yield)

Instantaneous Convenience Yield

- Let $dD(t)$ be the **additive convenience revenue** accrued to the commodity holder on $[t, t + dt]$.

- **Definition**: The **Instantaneous Convenience Yield** $c$ is the net benefit stemming from holding commodity $S$ as computed as a *per annum* dividend rate per unit of value of the commodity:

$$c(t) : dD(t) = c(t) S(t) \, dt.$$ 

- Holding one unit of commodity on $[t, t + dt]$ yields a **Gain**:

$$\frac{dG(t)}{G(t)} = \frac{dS(t)}{S(t)} + \frac{dD(t)}{S(t)},$$

$$\Rightarrow G(t) = S(t) e^{\int_0^t c(s) \, ds}.$$
Spot Dynamics

- $B(t) := e^{\int_0^t r(s)ds} = \text{Locally risk-free asset.}$
- **Spot price $\mathbb{P}$-dynamics:** \[ \frac{dS(t)}{S(t)} = \mu(t) dt + \sigma(t) d\bar{W}(t). \]
- By the **first theorem of APT**, there is a measure $\mathbb{P}^* \sim \mathbb{P}$ such that the **Gain Loc.risk-free asset** is a martingale: \[ \frac{G(t)}{B(t)} = \mathbb{P}^*\text{-martingale} \iff \mathbb{P}^\text{-drift} \left( \frac{S(t) e^{\int_0^t c(s)ds}}{e^{\int_0^t r(s)ds}} \right) = 0 \]
  \[ \iff \mu^*(t) := \% \mathbb{P}^\text{-drift of } S = r(t) - c(t). \]

- **Spot price $\mathbb{P}^*$-dynamics:**
  \[ \frac{dS(t)}{S(t)} = (r(t) - c(t)) dt + \sigma(t) dW(t), \]
  \[ S(t) = \mathbb{E}^\mathbb{P} \left( e^{-\int_t^T [r(s) - c(s)]ds} S(T) \right). \]  \[ (1) \]
Forward Prices and Parity

- Consider a **forward contract** \( f = \text{fwd} (t, T, x, S) \) issued at \( t \) for delivery of \( S \) at \( T > t \) at a price \( x \).

- **Contract value** at \( s \in [t, T] \):

\[
V_f(s) = \mathbb{E}_s^{\text{P}^*} \left( e^{-\int_s^T r(u)du} \left( S(T) - f_T(t) \right) \right)
\]

- Fwd is issued for free: \( V_f(t) = 0 \iff \text{Forward price:} \)

\[
f_T(t) = \frac{\mathbb{E}_t^{\text{P}^*} \left( e^{-\int_t^T r(u)du} S(T) \right)}{P_T(t)}
\]  

(2)

where \( P_T(t) \) denotes the time \( t \) default-free zero price for maturity \( T \).

- **Remark**: Dependence on the convenience yield goes through \( S(T) \); we now examine cases of explicit dependence.
Explicit Spot - Forward Parity (1/2)

\[
\text{Futures price } F_T(t) = \mathbb{E}_t^{P^*}(S(T)) = \mathbb{E}_t^{P^*}(f_T(T)):
\]

- **Case** \( r \) det. \( \Rightarrow f_T(t) = \mathbb{E}_t^{P^*}(S(T)) = \mathbb{E}_t^{P^*}(f_T(T))): f_T \) is a \( P^* \)-martingale.

- **Case** \( c \) det. \( \Rightarrow S(t) e^{-\int_t^T c(s)ds} (1) = \mathbb{E}_t^{P^*} \left( e^{-\int_t^T r(s)ds} S(T) \right) \) plugged into (2) gives:

\[
f_T(t) \overset{(2)}{=} S(t) e^{-\int_t^T c(s)ds} \frac{e^{\int_t^T [r(s)-c(s)]ds}}{P_T(t)} \overset{r=\text{det.}}{=} S(t) e^{\int_t^T [r(s)-c(s)]ds}.
\]
Case $r, c$ stoch + $\langle \lg S, \lg Q \rangle_t = \text{deterministic}$:

$$Q_T(t) := \mathbb{E}^{P^*} \left( e^{-\int_t^T c(s)ds} \right);$$

Jamshidian (1993) proves that:

$$f_T(t) = S(t) \frac{Q_T(t)}{P_T(t)} e^{\int_t^T \sigma_Q \sigma_S du}$$

Spot Price - Convenience Yield Modelling

- **Input:**
  1. Spot price dynamics: \( \frac{dS(t)}{S(t)} = [r(t) - c(t)] \, dt + \sigma(t) \, dW(t) \), \( S(0) = S_0 \);
  2. Instantaneous spot convenience yield dynamics:
     \( dc(t) = \mu_c(t) \, dt + \sigma_c(t) \, dW_c(t) \), \( c(0) = c_0 \);

- **Output:**
  - Fwd/Futures price: \( f_T(t) \)
  - Fwd/Futures price dynamics: \( df_T(t) = f_T(t) \sigma_T(t) \, dW(t) \), where \( \sigma_T(t) \) is the forward volatility structure.
  - Initial fwd curve: \( (f_T(0), T \geq 0) \).
## Popular Spot-Convenience Yield Models

<table>
<thead>
<tr>
<th>Name</th>
<th>Dynamics</th>
<th>Features</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gibson-Schwartz</td>
<td>(\frac{dS_t}{S_t} = (r - c_t) , dt + \sigma_S , dW_1)</td>
<td>(\rho &gt; 0) \rightarrow\text{mean-rev.}</td>
</tr>
<tr>
<td></td>
<td>(dc_t = [\kappa (\alpha - c_t) + \lambda \sigma_c] , dt + \sigma_c , dW_2)</td>
<td></td>
</tr>
<tr>
<td>Nielsen-Schwartz</td>
<td>(\frac{dS_t}{S_t} = (r - c_t) , dt + \sqrt{\alpha_t , c_t + \beta_t} , dW_1)</td>
<td>Spot vol. (=f(\text{curve } \Delta))</td>
</tr>
<tr>
<td></td>
<td>(dc_t = (\gamma - \delta c_t) , dt + \sigma \sqrt{\alpha_t , c_t + \beta_t} , dW_2)</td>
<td></td>
</tr>
<tr>
<td>Korn</td>
<td>(d \log S_t = \alpha (\xi_t - \log S_t) , dt + \sigma , dW_1)</td>
<td>Stationary spot \rightarrow\text{long term hedg.}</td>
</tr>
<tr>
<td></td>
<td>(d\xi_t = \beta (\gamma - \xi_t) , dt + \nu , dW_2)</td>
<td></td>
</tr>
<tr>
<td>Ritcher-Sorensen</td>
<td>(\frac{dS_t}{S_t} = (r - c_t) , dt + \alpha_t \sqrt{v_t} , dW_1)</td>
<td>Seasonality (+\text{ stoch.vol.})</td>
</tr>
<tr>
<td></td>
<td>(dc_t = (\gamma_1 (t) - \delta_1 c_t) , dt + \sigma \alpha_t \sqrt{v_t} , dW_2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(dv_t = (\gamma_2 - \delta_2 v_t) , dt + \tau \alpha_t \sqrt{v_t} , dW_3)</td>
<td></td>
</tr>
</tbody>
</table>

References:  
- Richter, M.C, Sorensen, C., Stochastic Volatility and Seasonality in Commodity Futures and Options: The Case of Soybeans, WP, Copenhagen Business School.
Problems Arising with Convenience Yield Models

- **Issue 1**: The current futures price curve and term structure of volatilities must be calibrated → need closed-from formulae for forward/futures prices.

- **Issue 2**: The spot convenience yield is unobservable → filtering techniques → unstable calibration + need linear linear fwd/fut. prices in the state vector.

- **An alternative approach**: directly modelling the forward curve (like in HJM for int.rates) (Jamshidian (1991)).

Forward Curve Modelling [FD]

- **Input:**
  1. Quoted forward/futures price curve $F_T(0)$ for $T > 0$;
  2. **Forward/Futures Price Dynamics:**

  \[
  \frac{dF_T(t)}{F_T(t)} = \underbrace{\mu(t, T)}_{\text{vol.str.}} \, dt + \underbrace{\sigma(t, T)dW(t)}_{\text{vol.str.}}
  \]

- **Output:**
  - **Spot Price Dynamics** $S(t) = F_t(t)$:

  \[
  \frac{dS(t)}{S(t)} = [\mu(t, t) + \partial_T F_T(t)|_{T=t}] \, dt + \sigma(t, t) \, dW(t).
  \]

  - **Convenience yield** $c(t)$: via instantaneous forward convenience yield $c(t, T)$ (see below).
Forward Convenience Yield Modelling [FC]

**Idea:** If $F_T$ is $T$-differentiable, there is $c(t, T)$:

$$F_T(t) = \frac{S(t)}{P_T(t)} e^{-\int_t^T c(t,u) du}.$$ 

**Input:**

1. **Quoted forward/futures price curve** $F(0) \rightarrow c(0, \cdot)$;
2. **Forward Convenience Yield Dynamics**:
   $$dc(t, T) = \mu_c(t, T) dt + \sigma_c(t, T) dW_c(t);$$
3. **Spot price dynamics**:
   $$\frac{dS(t)}{S(t)} = \mu_S(t) dt + \sigma_S(t) dW_S(t).$$

1. The **initial forward/futures price curve** is observable via combined:
   - Quotes stripping (=energy forwards are for continuous delivery), and
   - Functional interpolation (=for unobserved maturities).

2. **Principal Components Analysis** of time-series of forward prices can be used to assess:
   - a number of driving noise terms and
   - the volatility structure \( (\sigma_i(t, T), T > t) \) for each noise term.

3. **Drift:**
   - For pricing purposes, no drift assessment is required.
   - For RM purposes (e.g., scenario simulation), one should estimate a cross-sectional trend function \( \mu_T(t) \).

Gibson-Schwartz (1990): Spot-convenience yield formulation under $\mathbb{P}$:

$$
\frac{dS(t)}{S(t)} = (\mu - \delta(t))dt + \sigma_1 dW_1(t),
$$

$$
d\delta(t) = \kappa[\hat{a} - \delta(t)]dt + \sigma_2 dW_2(t),
$$

with:

- $\hat{a} = a - \lambda \sigma_2 / \kappa$,
- $\lambda =$ market price of convenience risk,
- $\rho =$ Corr $(W_1, W_2)$.

Estimation method: Kalman-filter (James, Webber (1999): Interest Rate Modelling, Wiley.)
Issue: Does Commodity Model Framework Matter?

- **Gibson-Schwartz (1990):** Forward price formulation under $\mathbb{P}$:

$$\frac{dF_T}{F_T} = \left[ \mu - r + \lambda \frac{e^{-\kappa(T-t)} - 1}{\kappa} \right] dt + \sigma_1 dW_1$$

$$-\sigma_2 \left[ \frac{1 - e^{-\kappa(T-t)}}{\kappa} \right] dW_2,$$

$$F_T(0) = S(0) e^{(r-\hat{\alpha} + \frac{\sigma_2^2}{2\kappa} - \rho \frac{\sigma_1 \sigma_2}{\kappa}) T + \frac{\sigma_2^2}{4} \frac{1-e^{-2\kappa T}}{\kappa^3} + \left( \alpha \kappa + \rho \frac{\sigma_1 \sigma_2}{\kappa} - \frac{\sigma_2^2}{\kappa} \right) \frac{1-e^{-\kappa T}}{\kappa} }.$$

- **Estimation method:** Maximum likelihood (Bhar, Chiarella, To (2003): A Maximum Likelihood Approach to Estimation of the Heath-Jarrow-Morton Models, WP, UTS.
Data and Estimation Results

- **Market**: NYMEX WTI crude oil futures.

**Results:**

<table>
<thead>
<tr>
<th>Setting</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\rho$</th>
<th>$\kappa$</th>
<th>$\mu$</th>
<th>$\lambda$</th>
<th>$\alpha$</th>
<th>$S(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fwd</td>
<td>0.3310</td>
<td>0.3387</td>
<td>0.7717</td>
<td>2.0905</td>
<td>0.6450</td>
<td>0.0523</td>
<td>0.0934</td>
<td>42.0129</td>
</tr>
<tr>
<td>Sc</td>
<td>0.3074</td>
<td>0.1936</td>
<td>0.7803</td>
<td>0.9102</td>
<td>0.5599</td>
<td>0.0785</td>
<td>0.1884</td>
<td></td>
</tr>
</tbody>
</table>
Test 1: Trajectorial Properties

Simulation

Futures Contracts

Simulation

Forward

Historical

Spot - Convenience Yield
Test 2: Re-estimation Stability: Forward Model

**Description:**

1. Sample 50 paths (1 path = 4 ttm’s (1, 3, 9, 18m) × 150d);
2. Estimation on simulated paths;
3. Descriptive statistics of discrepancy re-estimated/initial par.

<table>
<thead>
<tr>
<th>Fwd-2005</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\rho$</th>
<th>$\kappa$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$\alpha$</th>
<th>S(0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>-0.0037</td>
<td>-0.0142</td>
<td>0.0215</td>
<td>0.0467</td>
<td>0.0482</td>
<td>0.0855</td>
<td>-0.0001</td>
<td>-0.7887</td>
</tr>
<tr>
<td>Std.Dev.</td>
<td>0.0163</td>
<td>0.0343</td>
<td>0.0476</td>
<td>0.3371</td>
<td>0.1520</td>
<td>0.2508</td>
<td>0.0122</td>
<td>5.9490</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.2790</td>
<td>-1.6778</td>
<td>0.4512</td>
<td>-4.3216</td>
<td>0.2820</td>
<td>-0.6367</td>
<td>5.8256</td>
<td>-6.8571</td>
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<tr>
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<td>$\sigma_1$</td>
<td>$\sigma_2$</td>
<td>$\rho$</td>
<td>$\kappa$</td>
<td>$\lambda$</td>
<td>$\mu$</td>
<td>$\alpha$</td>
<td></td>
</tr>
<tr>
<td>--------</td>
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<td>-----------</td>
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<td>----------</td>
<td>-----------</td>
<td>--------</td>
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<tr>
<td>Mean</td>
<td>-0.0214</td>
<td>-0.0777</td>
<td>0.0789</td>
<td>-0.4364</td>
<td>0.0189</td>
<td>0.0117</td>
<td>0.0208</td>
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<tr>
<td>Std.Dev.</td>
<td>0.0321</td>
<td>0.0290</td>
<td>0.0281</td>
<td>0.2361</td>
<td>0.0143</td>
<td>0.0209</td>
<td>0.0338</td>
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<tr>
<td>Skewness</td>
<td>0.6484</td>
<td>-1.5947</td>
<td>-0.5614</td>
<td>-1.9651</td>
<td>1.8062</td>
<td>-1.0884</td>
<td>2.2702</td>
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</table>
**Description:** Sample 50 paths perturbed at 10 rand pts by $\mathcal{N}(0, 1)$.

<table>
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<tr>
<th></th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\rho$</th>
<th>$\kappa$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$\alpha$</th>
<th>$s(0)$</th>
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<tbody>
<tr>
<td><strong>FD Model</strong></td>
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<td></td>
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<tr>
<td>Mean</td>
<td>0.0037</td>
<td>0.0281</td>
<td>-0.0024</td>
<td>0.0493</td>
<td>-0.0034</td>
<td>0.0069</td>
<td>0.0051</td>
<td>0.0085</td>
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<td>SE</td>
<td>0.0083</td>
<td>0.1190</td>
<td>0.0460</td>
<td>0.6813</td>
<td>0.0092</td>
<td>0.0208</td>
<td>0.0244</td>
<td>0.1476</td>
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<tr>
<td>Skewness</td>
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<td>0.2581</td>
<td>1.6474</td>
<td>1.1414</td>
<td>-2.9557</td>
<td>2.9021</td>
<td>2.9021</td>
<td>1.3868</td>
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<tr>
<td><strong>SC Model</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.0320</td>
<td>-0.1065</td>
<td>-0.0761</td>
<td>-0.3460</td>
<td>0.0155</td>
<td>0.0537</td>
<td>0.0181</td>
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<tr>
<td>Std.Dev.</td>
<td>0.0331</td>
<td>0.0480</td>
<td>0.0689</td>
<td>0.1952</td>
<td>0.0137</td>
<td>0.0890</td>
<td>0.0138</td>
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<tr>
<td>Skewness</td>
<td>0.1520</td>
<td>-0.4963</td>
<td>-1.1401</td>
<td>1.1888</td>
<td>0.9438</td>
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<td>2.0937</td>
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<tr>
<td>Kurtosis</td>
<td>3.6762</td>
<td>2.4367</td>
<td>3.6116</td>
<td>5.6231</td>
<td>5.4545</td>
<td>12.2756</td>
<td>9.3147</td>
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</table>
Test 4: Convergence with Increasing Information

- **Descriptions**: Estimation across thicker&thicker term structures.

<table>
<thead>
<tr>
<th>Tenor (Fwd model)</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\rho$</th>
<th>$\kappa$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$\alpha$</th>
<th>$S(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,3,9,18</td>
<td>0.3310</td>
<td>0.3387</td>
<td>0.7717</td>
<td>2.0908</td>
<td>0.0523</td>
<td>0.6450</td>
<td>0.0934</td>
<td>42.0129</td>
</tr>
<tr>
<td>1,3,6,9,12,18</td>
<td>0.3305</td>
<td>0.3369</td>
<td>0.7640</td>
<td>2.0663</td>
<td>0.0522</td>
<td>0.6419</td>
<td>0.0954</td>
<td>41.9480</td>
</tr>
<tr>
<td>1,2,3,6,9,12,15,18</td>
<td>0.3296</td>
<td>0.3350</td>
<td>0.7620</td>
<td>2.0730</td>
<td>0.0525</td>
<td>0.6465</td>
<td>0.0988</td>
<td>42.0119</td>
</tr>
<tr>
<td>All</td>
<td>0.3273</td>
<td>0.3193</td>
<td>0.7569</td>
<td>2.0012</td>
<td>0.0531</td>
<td>0.6586</td>
<td>0.1009</td>
<td>41.9354</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Tenor (SC model)</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\rho$</th>
<th>$\kappa$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,3,9,18</td>
<td>0.3074</td>
<td>0.1936</td>
<td>0.7803</td>
<td>0.9102</td>
<td>0.0785</td>
<td>0.5599</td>
<td>0.1884</td>
</tr>
<tr>
<td>1,3,6,9,12,18</td>
<td>0.3610</td>
<td>0.1942</td>
<td>0.7856</td>
<td>0.9587</td>
<td>0.0595</td>
<td>0.5673</td>
<td>0.1488</td>
</tr>
<tr>
<td>1,2,3,6,9,12,15,18</td>
<td>0.3877</td>
<td>0.1882</td>
<td>0.7960</td>
<td>1.0050</td>
<td>0.0542</td>
<td>0.5673</td>
<td>0.1592</td>
</tr>
<tr>
<td>All</td>
<td>0.3653</td>
<td>0.1837</td>
<td>0.7869</td>
<td>1.0485</td>
<td>0.0568</td>
<td>0.5620</td>
<td>0.1884</td>
</tr>
</tbody>
</table>
Tests 5-6: Computational Time and Volatility Structure

- **Computations time** over increasing tenors:

<table>
<thead>
<tr>
<th>Tenor</th>
<th>1,3,9,18</th>
<th>1,3,6,9,12,18</th>
<th>1,2,3,6,9,12,15,18</th>
<th>all time to maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td>SC</td>
<td>50 sec,</td>
<td>100 sec,</td>
<td>120 sec,</td>
<td>380 sec,</td>
</tr>
<tr>
<td>Fwd</td>
<td>1 sec,</td>
<td>2-3 sec,</td>
<td>2-4 sec,</td>
<td>5-6 sec,</td>
</tr>
</tbody>
</table>

- **Recovery of volatility structure:**

\[\text{Volatility Structure}\]

\[\text{Time-to-Maturity (month)}\]

\[\text{FD estimation} \quad \text{SC estimation} \quad \text{Empirical}\]
Conclusions

- **Kalman filter on spot-convenience yield model estimation:**
  - Several parameters to estimate,
  - Weak statistical stability and convergence,
  - The optimizing function displays several or even no local maxima,
  - Time intensive computation.

- **GMM/Exact Likelihood on forward model estimation:**
  - Rather statistically stable ...
  - ... and quick to compute.
Andrea Roncoroni is Associate Professor of Finance at ESSEC Business School (Paris - Singapore) and regular Lecturer at Bocconi University (Milan). He holds a BS in Economics from Bocconi University (Italy), an MS in Mathematics from the Courant Institute of Mathematical Sciences (New York) and PhD’s in Applied Mathematics and Finance from the University of Trieste (Italy) and University Paris Dauphine (France), respectively. His research interests cover Energy Finance, Financial Econometrics and Derivative Structuring. He consulted for private companies (e.g., Gaz de France, Fideuram Asset Management, Italian Stock Exchange) and lectured for public institutions (e.g., International Energy Agency, Central Bank of France, Italian Authority for Electricity and Gas) and regularly publishes on academic journals and book series. Handbook "Implementing Models in Quantitative Finance: Methods and Cases" (with G.Fusai), edited by Springer-Verlag in 2008, has been the 2nd best seller of the Series in the year.

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Web page: http://www45.essec.edu/faculty/andrea-roncoroni