Mean Value Representation and Curvatures of Compact Convex Hypersurfaces

S. L. Lee

Department of Mathematics
National University of Singapore

Abstract

Representation of points and functionals on a set by its extreme points or boundary is an important problem in mathematics and its applications. Barycentric coordinates, the Krein-Millman Theorem and Choquet's Theorem are examples of such a representation. Recently, in conjunction with the construction of one-one transformation and parametrization of meshes in $\mathbb{R}^3$, Floater has constructed new coordinates, called the *mean value coordinates,* for the representation of points in the kernel of star-shaped polygons in $\mathbb{R}^2$ and polyhedrons in $\mathbb{R}^3$ in terms of extreme points of the star-shaped regions. Floater's construction was motivated by the mean value property of harmonic functions, but it was shown recently that the mean value coordinates can be derived from a *mean value representation,* which is more naturally associated with conservative vector fields in $\mathbb{R}^2$ and divergence free vector fields in $\mathbb{R}^3$.

Interestingly, in this divergence free framework, the mean value representation is intimately connected with Minkowski problem that relates positive functions on the unit $(n-1)$-dimensional sphere in $\mathbb{R}^n$ that are orthogonal to the first harmonics, to the curvature of compact strictly convex hypersurfaces through its Gauss map. The vector fields are also intimately related to homogeneous functions, which provide a general method for the construction of a large class of coordinates based on the curvature of compact strictly convex hypersurfaces. In the divergence free framework, the vector fields, $\mathbf{F}(\mathbf{r}) = \mathbf{r}/||\mathbf{r}||^{n+1}$, $\mathbf{r} \in \mathbb{R}^n$, that produce Floater's mean value coordinates for $n = 2, 3$ are associated the curvature of the unit $(n-1)$-dimensional sphere. This shows that the mean value coordinates are indeed a simple case of mean value representation based on the geometry of compact strictly convex hypersurfaces.

In this talk we give the relationship between homogeneous functions and the corresponding vector fields for mean value representation and use them, in conjunction with Minkowski problem, to construct mean value representation based on the curvatures of compact strictly convex hypersurfaces in $\mathbb{R}^n$.