Caplets: wavelets without wavelets

Amos Ron

Wavelet decompositions are implemented and inverted by fast algorithms, the so-called *fast wavelet transform* (FWT). The FWT is the primary reason for the popularity of wavelet-based methods in so many different scientific and engineering disciplines. The second most important reason for the popularity of wavelets is their mathematical theory: that theory shows that the wavelet coefficients record faithfully the precise smoothness class of the underlying dataset/function. These characterizations are instrumental for the mathematical analysis of wavelet-based algorithms in the areas of image and signal analysis. The third most important reason for the popularity of wavelets (which is closely related to the first one) is the vehicle of MultiResolution Analysis (MRA) which allows for the construction of a wide variety of wavelet systems. This approach is epitomized in the univariate Mallat's algorithm. The effective construction of wavelet systems is more cumbersome in higher dimensions. For example, in 4D (and dyadic downsampling) one employs (at least) 15 different highpass filters in any MRA-based wavelet system. And the struggle in higher dimensions to balance optimally between time localization (short filters) and frequency localization is hampered by the need to adhere to the MRA-based construction principles.

Some relief to the above is offered by inserting redundancy. The resulting theory is known as *framelet theory*. While framelets offer more flexibility (for example, one can use spline filters for both the decomposition and the reconstruction), the redundancy entails that we use an even larger number of highpass filters. From the theoretical point of view, framelets suffer from the so-called vanishing moment phenomenon, which in down-to-earth terms says that the ability to characterize function spaces via framelet coefficients is limited in some unexpected artificial way (by the number of vanishing moments in the framelets), and not by the (usually twice larger) "performance grade" of the system (viz. its approximation order).

For these and other reasons, practitioners sometimes abandon wavelets altogether. Instead, they use some method for (linear) coarsening of their data, and a complementary method for (linear) prediction of the original data from the coarsened one. The "wavelet coefficients" are then trivially defined to be the difference between the predicted coefficients and the actual ones. The approach is intimately related to the notion of *hierarchical bases* and is sometimes referred to as *pyramidal representation*. It is a simpler approach, and, indeed, was introduced and used a few years before wavelets made their debut.

Caplets, which are introduced and analysed in the talk, is the melting pot of the above three ideas. From the algorithmic point of view, they employ the same coarsening-prediction methodology alluded to above, but add the (optional) intermediate step of *alignment*. They are therefore based merely on three filters: the lowpass decomposition filter, the lowpass prediction filter, and the third (fullpass) alignment filter. They are implemented by simple, fast, wavelet-like decomposition, and by trivial (not wavelet-like) reconstruction. They do not require the construction of any wavelet, framelet, zilch. They work for any dilation process (e.g., binary, ternary, quincunx), and for any triplet of C(oasification)-A(lignment)-P(rediction) filters.

As far as redundancy is concerned, the caplet representation is redundant, with the redundancy rate decreasing with the spatial dimension. For example, in 4D the caplet representation has a redundancy rate of 1.066. We prove that the caplet representation coincides with the representation offered by a carefully designed (redundant) framelet system. That framelet system does not exhibit the aforementioned performance-degradation.

From the theoretical point of view, we prove that the caplet coefficients provide characterizations of function spaces that are analogous to the characterizations provided by wavelet coefficients. This is, actually, our main finding.

There are other attractions in caplats, in addition to the simplicity and the universality of their construction. For example, in image processing, the caplat representation produces a *single* image at each scale. Another advantage is improved time-frequency localization. That latter point is illustrated by the simplest 2D caplat system: the filters involved in that system employ, on average, four coefficients (same as the 2D Haar system), but the performance is equivalent to the 3/5 biorthogonal system (whose filters involve, on average, 16 coefficients).

This is a joint project with Youngmi Hur, a graduate student at the University of Wisconsin.