Pseudo-splines, Wavelets and Framelets

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Abstract

The first type of pseudo-splines were introduced by [Daubechies, Han, Ron and Shen, 2003] (DHRS) to construct tight framelets with desired approximation orders via the *unitary extension principle* of [Ron and Shen, 1997]. In the spirit of the first type of pseudo-splines, we introduce here a new type (the second type) of pseudo-splines to construct symmetric or antisymmetric tight framelets with desired approximation orders. Pseudo-splines provide a rich family of refinable functions. B-splines are one of special classes of pseudo-splines; orthogonal refinable functions (whose shifts form an orthonormal system given in [Daubechies, 1988]) are another class of pseudo-splines; and so are the interpolatory refinable functions (which is the Lagrange interpolatory function at $\mathbb{Z}$ and was first discussed in [Dubuc, 1986]). The other pseudo-splines with various orders fill in the gaps between the B-splines and orthogonal refinable functions for the first type, and between B-splines and interpolatory refinable functions for the second type. This gives a wide range of choices of refinable functions that meets various demands for balancing the approximation power, the length of the support and the regularity in applications. This paper is to give a regularity analysis of pseudo-splines of the both types and provide various constructions of wavelets and framelets. It is known that the regularity of the first type of pseudo-splines is between B-spline and orthogonal refinable function of the same order. However, there is no precise regularity estimate for pseudo-splines in general. In this paper, an optimal estimate of the decay of the Fourier transform of the pseudo-splines is given. This deduces the regularities of pseudo-splines, hence, the regularities of the corresponding wavelets and framelets. The asymptotical regularity analysis, as the order of the pseudo-splines goes to the infinity, is also provided. From a given pseudo-spline, a short support Riesz wavelet (that has the same support as that of the pseudo-spline) is constructed. The construction is rather simple and natural, however, the proof of the Riesz property of the corresponding wavelet system is highly nontrivial. Furthermore, this short support wavelet is one of the tight framelets constructed from the same pseudo-spline derived from the methods provided in both [DHRS] and our currant paper. This reveals that in almost all pseudo-spline tight frame systems constructed so far, there is one framelet whose dilations and shifts already form a Riesz basis for $L_2(\mathbb{R})$.

References

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