Eigenvalues and Biorthogonal Eigensystems of Scaling Operators

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Organization of Talk

1. Introduction

2. Appell Systems of Random Variables

3. Biorthogonal Systems of Eigenfunctions

4. Gaussian, B-splines, Hermite and Bernoulli Polynomials

1 Introduction

1.1 Appell Sequences

- Take a compactly supported distribution ϕ, normalized so that

$$\widehat{\phi}(0) = 1.$$

- Generate an Appell sequence

$$\frac{e^{xz}}{\widehat{\phi}(iz)} = \sum_{m=0}^{\infty} Q_m(x) z^m \ . \tag{1}$$

- Action of $\phi^{(k)}$ on (1)

$$\frac{\langle \phi^{(k)}(x), e^{xz} \rangle}{\widehat{\phi}(iz)} = \sum_{m=0}^{\infty} \langle \phi^{(k)}, Q_m \rangle z^m$$

$$\Downarrow$$

$$(-1)^k z^k = \sum_{m=0}^{\infty} \langle \phi^{(k)}, Q_m \rangle z^m$$

$$\Downarrow$$

$$\langle (-1)^k \phi^{(k)}, Q_m \rangle = \delta_{k,m} \ .$$

1.2 Scaling Functions

- A scaling function satisfies a *scaling equation,*

$$\phi(x) = 2\sum_j h(j)\phi(2x-j), \quad x \in \mathbb{R}, \qquad (2)$$

  where $h \in \ell_0(\mathbb{Z})$ with sum 2.

- In the Fourier domain

$$\widehat{\phi}(u) = \widehat{h}(u/2)\widehat{\phi}(u/2), \quad u \in \mathbb{R}, \qquad (3)$$

  where $\widehat{h}(u) = \sum_j h(j)e^{-iju}$.

- **Fact 1.** (2) has a unique solution ϕ, which is a compactly supported distribution whose FT is

$$\widehat{\phi}(u) = \prod_{j=1}^{\infty} \widehat{h}(u/2^j), \quad u \in \mathbb{R}$$

- **Fact 2.** The cascade sequence $\prod_{j=1}^{n} \widehat{h}(u/2^j)$ converges locally uniformly to $\widehat{\phi}(u)$.

- **Fact 3.**

$$\phi^{(k)}(x) = 2^{k+1}\sum_j h(j)\phi^{(k)}(2x-j), \quad x \in \mathbb{R},$$

  showing that $\phi^{(k)}$ is an eigenfunction of the *scaling operator*

$$T_h f = 2\sum_j h(j)f(2\cdot -j), \quad f \in \ell(\mathbb{Z}),$$

  with eigenvalues 2^{-k}, $k = 1, 2, \ldots$.

1.3 Appell Sequences of Scaling Functions

Take a scaling function that satisfies (2) and generate an Appell sequence by (1).

- **Example 1.** Take $\phi = \chi_{(0,1]}$. Then $\widehat{\phi}(u) = \left(\frac{1-e^{iu}}{iu}\right)$, and (1) becomes

$$\frac{ze^{xz}}{e^z - 1} = \sum_{m=0}^{\infty} B_m(x) z^m,$$

which generates the classical Bernoulli polynomials.

- **Example 2.** Take $\phi = \chi_{(0,1]} * \cdots * \chi_{(0,1]}$, N times. Then ϕ is the B-spline of order N, $\widehat{\phi}(u) = \left(\frac{1-e^{iu}}{iu}\right)^N$, and (1) becomes

$$\frac{z^N e^{xz}}{(e^z - 1)^N} = \sum_{m=0}^{\infty} B_{N,m}(x) z^m,$$

which generates the classical Bernoulli polynomials, $B_{N,m}(x)$, of order N.

1.4 Aim of Talk

- To explore some connections between scaling functions and the corresponding Appell sequences.

- Formulate the problem using simple probability.

2 Appell Systems of Random Variables

2.1 Stochastic Multiscale Operators

Consider the conditional expectation,

$$E\{f(Y+Z)|Y\}$$

of functions of two independent random variables Y and Z.

Examples

$$E\{\alpha f(\alpha X - T)|X\} = \alpha \int_{\mathbb{R}} f(\alpha X - t) d\psi_T(t) =: W_{T,\alpha} f(X) \tag{4}$$

is a *scaling operator*, and

$$E\left\{f\left(\frac{X+T}{\alpha}\right)|X\right\} = \int_{\mathbb{R}} f\left(\frac{X+t}{\alpha}\right) d\psi_T(t) =: W^*_{T,\alpha} f(X) \tag{5}$$

is its adjoint, where $\alpha > 1$ is the dilation.

2.2 Appell Systems

- Let Y_n, $n = 1, 2, 3, \ldots$, be i.i.d. random variables with c.f. $\widehat{\psi}$, analytic at 0,

$$S_n := \frac{Y_1}{\alpha} + \frac{Y_2}{\alpha^2} + \cdots + \frac{Y_n}{\alpha^n} \ ,$$

where $\alpha > 1$.

- The c.f. of S_n (*cascade sequence*),

$$\widehat{\phi}_n(u) = \prod_{j=1}^{n} \widehat{\psi}(u/\alpha^j),$$

generates two families of Appell sequences

$$\widehat{\phi}_n(iz)e^{-xz} = \sum_{m=0}^{\infty} P_m(x;n)z^m$$

and

$$\frac{e^{xz}}{\widehat{\phi}_n(iz)} = \sum_{m=0}^{\infty} Q_m(x;n)z^m \ ,$$

which we shall call the *Appell system of* S_n.

- $\widehat{\phi}_n(iz) \to \widehat{\phi}(iz)$ locally uniformly as $n \to \infty$, where

$$\widehat{\phi}(iz) = \prod_{j=1}^{\infty} \widehat{\psi}(iz/\alpha^j).$$

- $P_m(x;n) \to P_m(x)$ and $Q_m(x;n) \to Q_m(x)$, $x \in \mathbb{R}$, as $n \to \infty$, where (P_m) and (Q_m) are generated by

$$\widehat{\phi}(iz)e^{-xz} = \sum_{m=0}^{\infty} P_m(x) z^m$$

and

$$\frac{e^{xz}}{\widehat{\phi}(iz)} = \sum_{m=0}^{\infty} Q_m(x) z^m \ .$$

- (P_m) and (Q_m) are an Appell system of

$$S = \sum_{j=0}^{\infty} Y_j/\alpha^j \ .$$

Proposition 3.1

Let $(P_m(\cdot\,;n))_m$, $(Q(\cdot\,;n)_m$, $n = 1, 2, \ldots,$ be an Appell system of S_n. For any two independent random variables X and T, where T has finite moments of all order, and for any $n \geq 1$,

$$\sum_{m=0}^{\infty} E\left\{P_m(X+T;n)|X\right\}\frac{z^m}{m!} = \prod_{j=1}^{n} \widehat{\psi}(iz/\alpha^j)e^{-Xz}E\{e^{-Tz}\}$$

and

$$\sum_{m=0}^{\infty} E\left\{Q_m(X+T;n)|X\right\}\frac{z^m}{m!} = \frac{e^{Xz}E\{e^{Tz}\}}{\prod_{j=1}^{n} \widehat{\psi}(iz/\alpha^j)}$$

in a neighborhood of the origin.

Proof of Proposition 3.1

We shall prove only the first relation.

$$\begin{aligned}
\sum_{m=0}^{\infty} E\left\{P_m(X+T;n)|X\right\}\frac{z^m}{m!} &= E\left\{\sum_{m=0}^{\infty} P_m(X+T;n)\frac{z^m}{m!}|X\right\} \\
&= E\left\{\prod_{j=1}^{n}\psi(iz/\alpha^j)e^{-(X+T)z}|X\right\} \\
&= \prod_{j=1}^{n}\psi(iz/\alpha^j)e^{-Xz}E\left\{e^{-Tz}|X\right\},
\end{aligned}$$

which leads to the first relation, since T and X are independent.

QED

Theorem 3.2

Let $(P_m(x;n))_m$, $(Q(x;n))_m$, $n = 1, 2, \dots$, be an Appell system of S_n. If X is independent of Y_j, then for any $m \geq 0$ and $n \geq 1$, we have the following relations,

$$E\left\{P_m(\alpha X - Y_j; n)|X\right\} = \alpha^m P_m(X; n+1),$$

$$E\left\{P_m(X - Y_j/\alpha^{n+1}; n)|X\right\} = P_m(X; n+1),$$

$$E\left\{Q_m\left(\frac{X+Y_j}{\alpha}; n\right)|X\right\} = \alpha^{-m} Q_m(X; n-1),$$

and

$$E\left\{Q_m\left(X + Y_j/\alpha^n; n\right)|X\right\} = Q_m(X; n-1).$$

Corollary 3.3

Let $(P_m(x;n))_m$, $(Q_m(x;n))_m$, $n = 1, 2, \ldots$, be an Appell system of $S_n = \sum_{j=1}^{n} Y_j/\alpha^j$. Then for any $m \geq 0$ and $n \geq 1$,

$$E\left\{Q_m\left(\frac{S_{n-1}+Y_n}{\alpha};n\right)|S_{n-1}\right\} = \alpha^{-m}Q_m(S_{n-1};n-1),$$

and

$$E\left\{Q_m\left(S_n;n\right)|S_{n-1}\right\} = Q_m(S_{n-1};n-1).$$

Remark

- Since

  $$E\left\{E\left\{Q_m\left(S_n;n\right)|S_{n-1}\right\}|S_{n-2}\right\} = E\left\{Q_m\left(S_n;n\right)|S_{n-2}\right\},$$

  a repeated application of the last relation in Corollary 3.3 gives the martingale equality

  $$E\left\{Q_m\left(S_n;n\right)|S_k\right\} = Q_m(S_k;k), \quad k = 1,2,\ldots,n-1,$$

  which shows that for each $m \geq 0$, $(Q_m(S_n;n))_{n\geq 1}$ is a martingale sequence.

- This martingale relation has been observed by Wim Schoutens (1998) in a different context.

Corollary 3.4

Let (P_m) and (Q_m) be an Appell system of $S = \sum_{j=1}^{\infty} Y_j/\alpha^j$. If X is independent of Y_j, then for any $m \geq 0$, we have the following equalities,

$$E\left\{P_m(\alpha X - Y_j)|X\right\} = \alpha^m P_m(X),$$

$$E\left\{Q_m\left(\frac{X+Y_j}{\alpha}\right)|X\right\} = \alpha^{-m} Q_m(X).$$

3.5 Biorthogonal Systems of Eigenfunctions

Let ψ be a probability measure whose c.f. is analytic in a neighborhood of the origin.

- The first relation in Corollary 3.4 implies that the Appell polynomials P_m are eigenfunctions with eigenvalues α^{m+1}, $m = 0, 1, \ldots,$ of the scaling operator

$$W_{\psi,\alpha} f(x) := \alpha \int_{\mathbb{R}} f(\alpha x - y) d\psi(y). \qquad (6)$$

- The second implies that Q_m are eigenfunctions of its adjoint $W^*_{\psi,\alpha}$ with eigenvalues α^{-m}, $m = 0, 1, \ldots$.

- There is a unique probability measure ϕ, which is the invariant distribution of $W_{\psi,\alpha}$, and

$$\widehat{\phi}(u) = \prod_{j=1}^{\infty} \widehat{\psi}(u/\alpha^j).$$

- (6) shows that the distributional derivatives $\phi^{(m)}$ are eigenfunctions of $W_{\psi,\alpha}$ with eigenvalues α^{-m}, $m = 0, 1, \ldots,$.

Theorem 3.5

Let ψ be a probability measure whose F.T. is analytic in a neighbourhood of the origin and $\widehat{\phi}(u) = \prod_{j=1}^{\infty} \widehat{\psi}(u/\alpha^j)$. Then $(\phi^{(m)})_{m \geq 0}$ and $(Q_m)_{m \geq 0}$ form a biorthogonal eigensystem, i.e.,

$$\langle (-1)^m \phi^{(m)}, Q_n \rangle = \delta_{m,n}, \quad m, n = 0, 1, \ldots.$$

where $\langle \cdot, \cdot \rangle$ denotes the action of a distribution on a test function.

4 Gaussian, B-splines, Hermite and Bernoulli Polynomials

4.1 Gaussian Function

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

- Optimal in time-frequency localization, i.e. it attains the bound in the Heissenberg uncertainty product.

- Invariant density of $W_{\psi,\alpha}$, i.e.

$$W_{\psi,\alpha} G = G,$$

  where, in this case, ψ is the Gaussian density function

$$\psi(x) = \frac{1}{\sqrt{2\pi(\alpha^2 - 1)}} e^{-x^2/2(\alpha^2-1)}.$$

4.2 Gaussian and Hermite Polynomials

- Take i.i.d. Gaussian random variables Y_j with density

$$\psi(x) = \frac{1}{\sqrt{2\pi(\alpha^2-1)}} e^{-x^2/2(\alpha^2-1)} \ .$$

- Then $S = \sum_{j=1}^{\infty} Y_j/\alpha^j$ has density

$$\phi(x) = G(x),$$

which is the invariant density of $W_{\psi,\alpha}$, and

$$\widehat{\phi}(u) = e^{-u^2/2} \ .$$

- The same method applies giving the generating function

$$e^{xz-z^2/2} = \sum_{m=0}^{\infty} H_m(x) z^m \ ,$$

which generates the Hermite polynomials.

- It is known that

$$\phi^{(n)}(x) = (-1)^n H_n(x) G(x) \ .$$

- Hence the biorthogonal relation becomes

$$\delta_{m,n} = \langle (-1)^n \phi^{(n)}, H_m \rangle = \int_{\mathbb{R}} H_n(x) H_m(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \ ,$$

  which is the familar orthogonal relations of Hermite polynomials.

4.3 B-splines

$$B_N = \chi_{[0,1)} * \cdots * \chi_{[0,1)}, \quad (N \text{ times}),$$

is a scaling function that satisfies the scaling equation

$$\phi(x) = 2 \sum_{j=0}^{N} \frac{1}{2^N} \binom{N}{j} \phi(2x - j).$$

Properties

- Piecewise polynomials with knots at integers.
- $\mathrm{Supp}(B_N) = [0, N]$.
- $\widehat{B}_N(u) = \left(\frac{1-e^{-iu}}{iu}\right)^N$.
- B_N approximates the Gaussian function:

  $$\widetilde{B}_N \to G \;\text{ in }\; L^p(\mathbb{R}), \quad 1 \le p \le \infty,$$

  where

  $$\widetilde{B}_N(x) = \frac{1}{2}\sqrt{\frac{N}{3}} B_N \left(\frac{1}{2}\sqrt{\frac{N}{3}} x + \frac{N}{2} \right).$$

- provides efficient algorithm for computation.

4.4 B-splines and Bernoulli Polynomials

- Take $\alpha = 2$, and i.i.d. random variables Y_j whose distribution, ψ, is the measure concentrated on the integers with binomial weights

  $$\frac{1}{2^N}\binom{N}{m}, \; m = 0, 1, \ldots, N, \text{ and } 0, \text{ otherwise.}$$

- The density of $S = \sum_{j=1}^{\infty} Y_j/\alpha^j$, $\phi = B_N$, and

  $$\widehat{\phi}(u) = \left(\frac{1 - e^{-iu}}{iu}\right)^N .$$

- The corresponding generating function is

  $$\frac{z^N e^{xz}}{(e^z - 1)^N} = \sum_{m=0}^{\infty} B_{N,m}(x) z^m ,$$

  which generates the Bernoulli polynomials $B_{N,m}$ of order N.

- Further,

  $$T_\psi^* B_{N,m} = 2^{-m} B_{N,m},$$

  and

  $$\langle (-1)^n B_N^{(n)}, B_{N,m} \rangle = \delta_{n,m}, \quad n, m = 0, 1, \ldots .$$

4.5 Normalized B-splines and Bernoulli Polynomials

- Take

$$\phi = \widetilde{B}_N := \sigma_N B_N(\sigma_N \cdot + \mu_N),$$

the normalized B-splines.

- Then

$$\widehat{\phi}(u) = e^{iu\mu_N/\sigma_N} \left(\frac{1 - e^{iu/\sigma_N}}{iu/\sigma_N} \right)^N \ .$$

- The corresponding generating function is

$$\frac{(z/\sigma_N)^N e^{(\sigma_N x + \mu_N)(z/\sigma_N)}}{(e^{z/\sigma_N} - 1)^N} = \sum_{m=0}^{\infty} \widetilde{B}_{N,m}(x) z^m \ ,$$

which generates the normalized Bernoulli polynomials of order N,

$$\widetilde{B}_{N,m}(x) = \sigma_N^{-m} B_{N,m}(\sigma_N x + \mu_N) \ .$$

- The biorthogonal relation is

$$\langle (-1)^n \widetilde{B}_N^{(n)}, \widetilde{B}_{N,m} \rangle = \delta_{n,m}, \quad n, m = 0, 1, \dots \ .$$

$$
\begin{array}{ccc}
\widetilde{B}_N & \stackrel{L^p}{\longrightarrow} & G \\
\Updownarrow & & \Updownarrow \\
\widetilde{B}_{N,m} & \stackrel{?}{\longrightarrow} & H_m
\end{array}
$$

$$
\begin{aligned}
\lim_{N\to\infty}\sum_{m=0}^{\infty}\widetilde{B}_{N,m}(x)z^m &= \lim_{N\to\infty}\frac{(z/\sigma_N)^N e^{(\sigma_N x+\mu_N)(z/\sigma_N)}}{(e^{z/\sigma_N}-1)^N} \\
&= \cdots = \\
&= e^{xz-z^2/2} \\
&= \sum_{m=0}^{\infty} H_m(x)z^m \ ,
\end{aligned}
$$

which shows that for $m = 0, 1, \ldots,$

$$
\lim_{N\to\infty}\widetilde{B}_{N,m}(x) = H_m(x), \quad x \in \mathbb{R} \ .
$$