INTERNAL SOLITARY WAVES IN THE ATMOSPHERE AND OCEAN

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Outline:

1. Observations and effects --- ocean & atmosphere
2. Analytical modelling --- Korteweg-de Vries (KdV) theory
3. Generation mechanisms --- current/wind interaction with topography, gravity currents, undular bores
4. Propagation over topography --- wave shoaling, transformation and breakdown
Oceanic observations

• Direct observations --- in situ measurements, at a fixed spatial location, giving time-series of temperature, currents etc., as functions of depth.

• Indirect observations --- the internal waves have a surface signature, due to the distortion of the surface wind-wave field by the near-surface currents of the internal wave. This, in turn, can be detected visually, by radar, and by satellite.
Figure 2. (Upper) A color contour time series of temperature profiles from the surface to 35m depth measured by the LMP over a one-day period. The 10°C span color contour scale is shown the right of the time series panel. The low frequency, semidiurnal internal tide displacement can clearly be seen along the yellow isotherm. (Lower) A profile time series of the first 1.7 hours of the time series shown in Figure 1a. White areas indicate times with no data. [From Stanton and Ostermey, 1998]
Oregon Coast - Colombia River

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Figure 1. Radarsat-1 image of the Colombia River and Oregon coast showing internal waves. The river plume induces seaward propagating internal waves. The shoreward propagating waves were generated at the coastal shelf break.
STRAIT OF MESSINA
October 25, 1995

start time: 15:50 UTC
position: 38.305 N 15.752 E
end time: 16:14 UTC
position: 38.281 N 15.722 E

ship speed (m/s): 2.5
ship heading (degrees): 225

MAX. NORTHWARD TIDAL FLOW: 16:02 UTC
Figure 5. ERS-1 (C-band VV) SAR image of the Strait of Messina acquired on 11 July 1993 at 0941 UTC (orbit 10387, frame 2835). The image shows internal wave signatures radiating out of the strait in both the northern and southern directions. Northwards propagating internal waves are less frequently observed than southward propagating ones. Imaged area is 65 km x 65 km. ©ESA 1993. [From The Tropical and Subtropical Ocean Viewed by ERS SAR http://www.ifm.uni-hamburg.de/ers-sar/]

Figure 3. RADARSAT-1 (C-band HH) Standard Mode SAR image acquired over the North Rankin oil and gas field of Western Australia on 12 February 1997 (orbit 11881). The image shows a complex pattern of internal wave signatures. Imaged area is approximately 100 km x 100 km. [Image courtesy of George Cresswell, CSIRO Marine Research, Hobart, Tasmania, Australia. RADARSAT image acquired as part of ADRO Project #72 Cresswell and Tildesley]
Atmospheric observations:

- Direct observations --- in situ measurements, at a fixed spatial location, giving time-series of potential temperature, winds etc., as functions of height.

- Visual observations --- under certain conditions, roll clouds form, which can be seen from the ground and by satellite.

- Indirect observations --- over the sea, the internal waves have a surface signature, due to the distortion of the surface wind-wave field by the near-surface winds of the internal wave. This, can be detected (as for oceanic internal waves) visually, by radar, and by satellite.
Morning Glory Waves of the Gulf of Carpentaria
Satellite view
Morning Glory Clouds - Australia
Morning glory waves at Mornington Island
Atmospheric internal waves

MODIS TERRA image Mozambique channel (16 AUG 2002)
Analytical Modelling

- Euler equations for conservation of mass and momentum for an inviscid, incompressible fluid
- Boundary conditions at free surface (z=f) and at the bottom (z=-h)
- Basic state of background density $\rho(z)$ and horizontal current $u(z)$
Multi-scale asymptotic expansion

• Long-wave scaling: \( X=\varepsilon x, \ T=\varepsilon t, \ \varepsilon<<1 \)
• Small-amplitude waves: \( \eta \) scales as \( \alpha<<1 \)
• At leading order: Linear long waves

\[
\eta = \alpha A(X-cT) \phi(z)
\]

c is the linear long-wave speed

\( \phi(z) \) is the modal function

\[
\{\rho(c-u)^2\phi_z\}_z - g\rho_z\phi = 0 ,
\]

\( \phi=0 \) at \( z=-h \), \( c^2\phi_z=g\phi \) at \( z=0 \)
At the next order, impose KdV balance:

\[ \alpha = \varepsilon^2, \quad \tau = \alpha T, \quad \xi = X - cT. \]

Outcome is the Korteweg-de Vries (KdV) equation:

\[ A_{t} + \mu AA_{\xi} + \lambda A_{\xi \xi \xi} = 0 \]

The coefficients are integrals over the waveguide, \(-h < z < 0\):

\[ I_{\mu} = \int 3\rho(c-u)^2(\phi_z)^3 dz, \quad I_{\lambda} = \int \rho(c-u)^2\phi^2 dz, \]

\[ I = \int 2\rho(c-u)(\phi_z)^2 dz. \]
KdV equation is integrable:

**Solitons**, IST, conservation laws, etc.

- Solitary wave (soliton):
  \[ A = a \text{ sech}^2 \{ \gamma (\xi - V\tau) \} \; ; \; \mu a = 12\lambda \gamma^2 = 3V \ . \]

- Localised initial state develops into a finite number of solitons, and dispersing radiation.
Extended KdV (Gardner) equation to take account of stronger nonlinearity:

$$A_\tau + \mu A A_\xi + \nu A^2 A_\xi + \lambda A_{\xi\xi\xi} = 0$$

In canonical form:

$$A_\tau + 6 AA_\xi + 6 \beta A^2 A_\xi + A_{\xi\xi\xi} = 0$$

Both equations ( $\beta > 0$ and $\beta < 0$) integrable, with solitons and breathers ($\beta > 0$).

For $\beta < 0$, there is a single family of positive solitons with a limiting amplitude of $-1/\beta$.

For $\beta > 0$, there are two families of solitons, of both polarities.
Upper panel, $\beta<0$; lower panel, $\beta>0$. 
Generation Mechanisms

Interaction of large-scale barotropic tide with continental slope, which generates a meso-scale internal tide. As it propagates shoreward (off-shore) the internal tide steepens and a soliton wave-train forms, the undular bore: Modelled by (e)KdV equation

Australian NWS: Displacement of the 25°C isotherm observed on 2 April (left) and 25 March (right) 1992.
Critical flow (internal Froude number F -- ratio of current speed to linear long wave speed -- close to one) over a sill, or through a strait, generating upstream and downstream propagating undular bores.

Modelled by forced KdV equation

\[-A_t - \Delta A_x + 6AA_x + A_{xxx} + F_x(X) = 0\]

Here \( \Delta = F-1 \) measures the degree of criticality, and \( F(X) \) is the projection of the topography onto the relevant (critical) mode.
Simulation at exact criticality, $\Delta=0$, and for positive forcing of amplitude 1.
Δ = -1.5
subcritical

Δ = 1.5
supercritical
**Undular Bore**: This can be modelled as the outcome from a steepening wave-front in the framework of a nonlinear, dispersive wave equation, e.g. the KdV equation.
Korteweg-de Vries equation:

$$A_t + 6AA_x + A_{xxx} = 0$$

One-phase travelling periodic wave, **cnoidal wave**, with 3 free parameters:

$$A = a\{b(m) + cn^2 (k[x-ct];m)\} + d$$

Where the modulus m, $0 < m < 1$, and

- $b(m) = \{[1-m]K(m) - E(m)\}/\{mK(m)\}$
- $c-6d = 2a\{[2-m]K(m) - 3E(m)\}/\{mK(m)\}$
- $a = 2mk^2$

$m \rightarrow 1$ is the KdV **solitary wave**, "sech$^2"$

$m \rightarrow 0$ are small-amplitude **sinusoidal waves**
Use Whitham modulation theory in which one develops equations for the modulation of the parameters (amplitude, wavelength, speed, mean, modulus $m$ of the cnoidal function) of an exact one-phase periodic travelling wave solution. These modulation equations are obtained by averaging conservation laws, and form (3) nonlinear hyperbolic equations, which are diagonalizable, and have a similarity solution in which $m = m(x/t)$.

Leading edge, $m \to 1$: Soliton with amplitude twice that of the initial jump $H > 0$ located at $x=0$.

Trailing edge, $m \to 0$: Small-amplitude sinusoidal waves.

Unsteady: $-6H_t < x < 4H_t$
Propagation over topography

The topography and hydrology often vary along the propagation path. Hence we replace the eKdV equation with an generalised KdV equation:

\[ S = \alpha \int \frac{dX}{c}, \quad \varphi = \int \frac{dX}{c} - T \]

\[ A_S + \left(\frac{Q_S}{2Q}\right)A + \alpha AA_\varphi + \alpha_1 A^2 A_\varphi + \beta A_{\varphi\varphi} = 0 \]

\[ \alpha = \mu/c, \quad \alpha_1 = \nu/c, \quad \beta = \lambda/c^3, \quad Q = c^2I \]

The coefficients all vary with S along the propagation path.
This generalized KdV equation conserves mass and momentum:

\[ M = \int Q^{1/2} A \, d\varphi = \text{constant} \]
\[ P = \int Q A^2 \, d\varphi = \text{constant} \]

**Slowly-varying solitary wave**: Deforms adiabatically, conserving \( P \). But cannot then also conserve \( M \), and instead a trailing shelf of small amplitude but long wavelength develops.

For the KdV equation, this yields the simple expression for the amplitude \( A \):

\[ \beta Q A^3 / \alpha = \text{constant} \]

Breakdown as \( \alpha \to 0 \)?
For the generalized KdV equation, conservation of momentum yields:

$$ G(B) = \text{constant} \sqrt{\left| \frac{\alpha_1^3}{Q^2} \beta \alpha^2 \right|} $$

where

$$ G(B) = |B-1|^{3/2} \int \frac{du}{(1 + B \cosh u)^2}, $$

and the integral is from $-\infty$ to $+\infty$.

Breakdown can now occur for $\alpha \to 0$ ($G \to \infty$),

or as $\alpha_1 \to 0$ ($G \to 0$). The zeros of the coefficients $\alpha$ and $\alpha_1$ thus define critical locations along the path of the wave.
Simulations for the NWS of Australia

\[ S = \alpha \int^X dX/c, \quad \varphi = \int^X dX/c - T \]

\[ A_S + \left(\frac{Q_S}{2Q}\right)A + \alpha AA_\varphi + \alpha_1 A^2 A_\varphi + \beta A_{\varphi\varphi\varphi} = 0 \]

\[ \alpha = \mu/c, \quad \alpha_1 = \nu/c, \quad \beta = \lambda/c^3, \quad Q = c^2l \]

This is solved numerically for the actual coefficients obtained from the ocean topography and hydrology.
The graph shows the relationship between wave amplitude (m) and distance (km) for two different cases, labeled as 1 and 2. The wave amplitude decreases as the distance increases. The amplitude for case 1 remains relatively stable, while case 2 shows a more pronounced decrease in amplitude with increasing distance.
Plot of linear speed $c$

Plot of dispersion coefficient $\beta$
Plot of quadratic coefficient $\alpha$

Plot of cubic coefficient $\alpha_1$
References:


The end

Thank you