Korteweg-de Vries Equation

Roger Grimshaw
Loughborough University, UK

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1. Historical Introduction

The Korteweg-de Vries (KdV) equation is

$$A_t + 6AA_x + A_{xxx} = 0\,, \tag{1}$$

written here in canonical form. In (1) $A(x,t)$ is an appropriate field variable, t is time, and x is a space coordinate in the direction of propagation. The KdV equation is widely recognised as a paradigm for the description of weakly nonlinear long waves in many branches of physics and engineering. It describes how waves evolve under the competing but comparable effects of weak nonlinearity and weak dispersion. Indeed, if it is supposed that x-derivatives scale as ϵ where ϵ is the small parameter characterising long waves (i.e. typically the ratio of a relevant background length scale to a wavelength scale), then the amplitude scales as ϵ^2 and the time evolution takes place on a scale of ϵ^{-3}.

2. Solitary Wave

The KdV equation is characterised by its solitary wave solutions,

$$A = a\,\mathrm{sech}^2(\gamma(x - Vt))\,, \tag{2}$$

$$\text{where} \qquad V = 2a = 4\gamma^2\,. \tag{3}$$

This solution describes a family of steady isolated wave pulses of positive polarity, characterized by the wavenumber γ; note that the speed V is proportional to the wave amplitude a, and to the square of the wavenumber γ^2. Thus the larger waves are thinner and travel faster.

3. History

The KdV equation (1) owes its name to the famous paper of Korteweg and de Vries, published in 1895, in which they showed that small-amplitude long waves on the free surface of water could be described by the equation

$$\zeta_t + c\zeta_x + \frac{3c}{2h}\zeta\zeta_x + \frac{ch^2}{6}(1 - \frac{Bo}{3})\zeta_{xxx} = 0\,. \tag{4}$$

Here $\zeta(x,t)$ is the elevation of the free surface relative to the undisturbed depth h, $c = (gh)^{1/2}$ is the linear long wave phase speed, and $Bo = T/gh^2$ is the Bond number measuring the effects of surface tension (ρT is the coefficient of surface tension and ρ is the water density). Transformation to a reference frame moving with the speed c (i.e. (x,t) is replaced by $(x-ct,t)$, and subsequent rescaling readily establishes the equivalence of (1) and (4). Although equation (1) now bears the name KdV, it was apparently first obtained by Boussinesq in 1877.

4. Cnoidal Waves

Korteweg and de Vries found the solitary wave solutions (2) and, importantly, they showed that they are the limiting members of a two-parameter family of periodic travelling wave solutions, described by elliptic functions and commonly called cnoidal waves,

$$A = b + a\,\mathrm{cn}^2(\gamma(x - Vt); m)\,, \tag{5}$$

$$\text{where} \quad V = 6b + 4(2m-1)\gamma^2\,,\; a = 2m\gamma^2\,. \tag{6}$$

Here $\mathrm{cn}(x; m)$ is the Jacobian elliptic function of modulus m $(0 < m < 1)$. As $m \to 1$, $\mathrm{cn}(x; m) \to \mathrm{sech}(x)$ and then the cnoidal wave (5) becomes the solitary wave (2), now riding on a background level b. On the other hand, as $m \to 0$, $\mathrm{cn}(x; m)$ $\to \cos 2x$ and so the cnoidal wave (5) collapses to a linear sinusoidal wave (note that in this limit $a \to 0$).

5. Russell's Observations

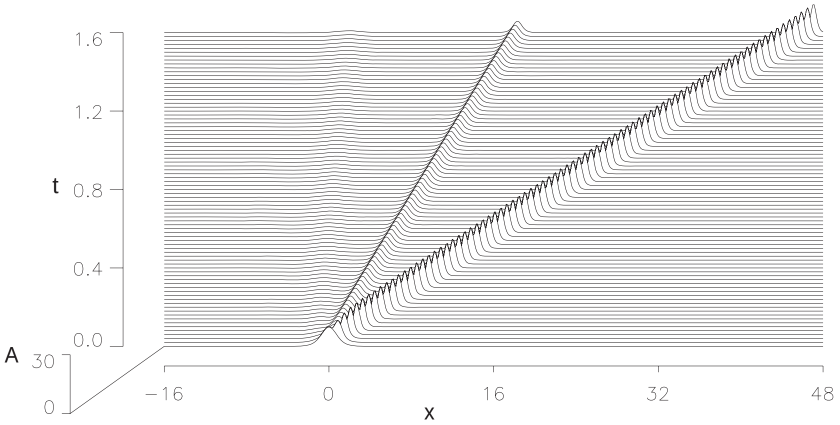
This solitary wave solution found by Korteweg and de Vries had earlier been obtained directly from the governing equations (in the absence of surface tension) independently by Boussinesq in 1871,and Rayleigh in 1876, who were motivated to explain the now very well-known observations by John Scott Russell in the Union Canal in 1834, and his subsequent experiments.

6. Solitons

After this ground-breaking work of Korteweg and de Vries, interest in solitary water waves and the KdV equation declined until the dramatic discovery of the **soliton** by Zabusky and Kruskal in 1965. Through numerical integrations of the KdV equation they demonstrated that the solitary wave (2) could be generated from quite general initial conditions, and could survive intact collisions with other solitary waves, leading them to coin the term soliton. Their remarkable discovery, followed almost immediately by the theoretical work of Gardner, Greene, Kruskal and Miura showing that the KdV equation was **integrable** through an **inverse scattering transform**, led to many other startling discoveries and marked the birth of soliton theory as we know it today. The implication is that the solitary wave is the key component needed to describe the behaviour of long, weakly nonlinear waves. In particular, a general localized initial condition will lead as $t \to \infty$ the generation of a finite number of solitons and some dispersing radiation.

7. Soliton Generation

The generation of three solitons from a localized initial condition for the KdV equation (1).



8. Kadomtsev-Petviashvili Equation

The KdV equation is uni-directional. A two-dimensional version of the KdV equation is the KP equation, obtained by Kadomtsev and Petviashvili in 1970,

$$(A_t + 6AA_x + A_{xxx})_x \pm A_{yy} = 0\,. \tag{7}$$

This equation includes the effects of weak diffraction in the y-direction, in that y-derivatives scale as ϵ^2 whereas x-derivatives scale as ϵ. Like the KdV equation it is an integrable equation. When the "$+$"-sign holds in (7), this is the KPII equation, and it can be shown that then the solitary wave (2) is stable to transverse disturbances. On the other hand if the "$-$"-sign holds, this is the KPI equation for which the solitary wave is unstable; instead this equation supports "lump" solitons. Both KPI and KPII are integrable equations. Water waves belong to the KPII case.

9. KdV as a Canonical Model Equation

Although the KdV equation (1) is historically associated with water waves, it occurs in many other physical contexts, where it can be derived by an asymptotic multi-scale reduction from the relevant governing equations. Typically the outcome is

$$A_t + cA_x + \mu AA_x + \lambda A_{xxx} = 0\,. \tag{8}$$

Here c is the relevant linear long wave speed for the mode whose amplitude is $A(x,t)$, while μ and λ, the coefficients of the quadratic nonlinear and linear dispersive terms respectively, are determined from the properties of this same linear long wave mode and, like c depend on the particular physical system being considered. Note that the linearization of (8) has the linear dispersion relation $\omega = c\,k - \lambda k^3$ for linear sinusoidal waves of frequency ω and wavenumber k; this expression is just the truncation of the full dispersion relation for the wave mode being considered, and immediately identifies the origin of the coefficient λ. Similarly, the coefficient μ can be identified with the an amplitude-dependent correction to the linear wave speed.

10. KdV as a Canonical Model Equation

Transformation to a reference frame moving with a speed c and subsequent rescaling shows that (8) can be transformed to the canonical form (1). Specifically, let

$$\mu A = 6U\tilde{A}\,, \quad x - ct = \left(\frac{\lambda}{U}\right)^{1/2}\tilde{x}\,, \quad t = \left(\frac{\lambda}{U^3}\right)^{1/2}\tilde{t}\,. \qquad (9)$$

Here U is a constant velocity scaling factor inserted to make the transformed variables dimensionless; a convenient choice is often $U = |c|$ provided $c \neq 0$. Then, after removing the superscript, equation (8) collapses to the canonical form (1). Equations of the form (8) arise in the study of internal solitary waves in the atmosphere and ocean, mid-latitude and equatorial planetary waves, plasma waves, ion-acoustic waves, lattice waves, waves in elastic rods and in many other physical contexts. Later we shall give a brief outline of the derivation of (8) for surface and internal waves.

11. Generalized Solitary Waves

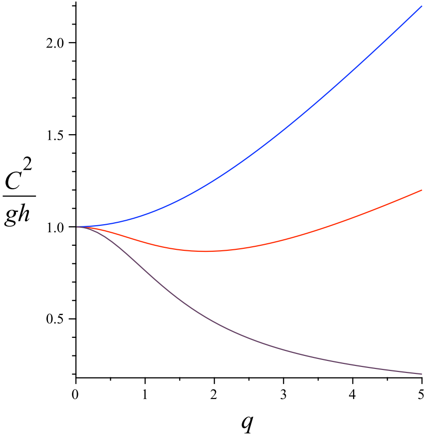
The strict validity of the asymptotic expansion leading to the KdV equation (8) is that there should be no resonance between the linear long wave mode with speed c, and any other part of the linear spectrum of the system being considered. There is an implicit assumption in deriving (8) that the solitary wave is spatially localized. In the far-field of any solution linearized dynamics apply, we can use the dispersion relation $\omega = \omega(k)$ of the full linearized system for sinusoidal waves of wavenumber k and frequency ω to test this.. Since all solutions of (8) travel with a speed close to the linear long wave speed c, spatial localization requires that there be no resonance between c and any linear phase speed $C(k) = \omega(k)/k$; that is, there are no real solutions for any finite real-valued non-zero k of the resonance condition $c = C(k)$. Pure solitary waves exist in the gaps in the linear spectrum. Otherwise, any attempt to construct a solitary wave may lead to co-propagating small-amplitude oscillations at infinity, the so-called generalized solitary wave.

12. Effect of Surface Tension

The dispersion relation for water waves with surface tension is

$$\frac{C^2(k)}{gh} = \frac{(1 + Boq^2)\tanh q}{q} \qquad q = kh\,,$$

where the plots are for $Bo = 0.0, 0.2, 0.4$.



13. Effect of Surface Tension

For water waves without surface tension, $Bo = 0$, pure solitary waves can exist for speeds greater than c. However, if the Bond number is such that $0 < Bo < 1/3$ then a resonance occurs between a long gravity wave and a short capillary wave. In this case, the full system cannot support a spatially localized solitary wave and instead there exist generalized solitary waves. They have a central core, described by the KdV solitary wave (2) for small amplitudes, but in the far-field have co-propagating non-decaying oscillations with a wavenumber approximately given by the resonance condition. The amplitudes of these oscillations are exponentially small relative to the amplitude of the central core, and this is why any multi-scale asymptotic expansion leading to the KdV equation (8) cannot find them. But note that when $Bo > 1/3$ the graph is again monotonic and there again exist genuine solitary waves, now for speeds less than c.

14. Extended Korteweg-de Vries Equation

In some physical situations, it is necessary to extend the KdV equation (8) with a higher-order cubic nonlinear term of the form $\sigma A^2 A_x$. After transformation and rescaling, the amended equation (8) can be transformed to the so-called extended KdV (or Gardner) equation

$$A_t + 6AA_x + 6\beta A^2 A_x + A_{xxx} = 0\,. \qquad (10)$$

Like the KdV equation, the Gardner equation is integrable by the inverse scattering transform. Here the coefficient β can be either positive or negative, and the structure of the solutions depends crucially on which sign is appropriate.

The solitary wave solutions are given by

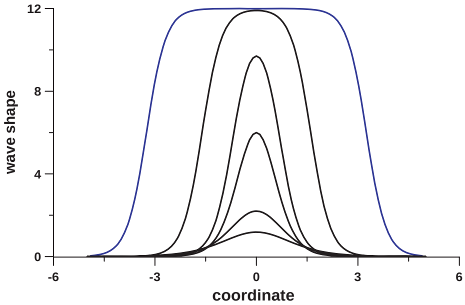
$$A = \frac{a}{b + (1-b)\cosh^2 \gamma(\theta - V\tau)}\,, \tag{11}$$

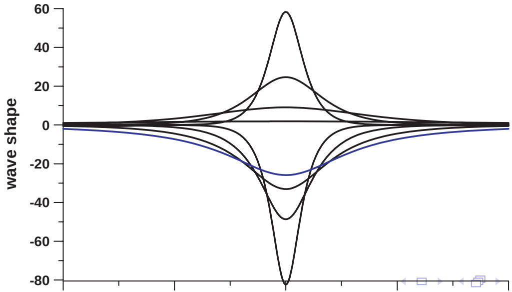
$$\text{where} \quad V = a(2+\beta a) = 4\gamma^2\,, \quad b = \frac{-\beta a}{(2+\beta a)}\,. \tag{12}$$

There are two cases to consider. If $\beta < 0$, then there is a single family of solutions such that $0 < b < 1$ and $a > 0$. As b increases from 0 to 1, the amplitude a increases from 0 to a maximum of $-1/\beta$ while the speed V also increases from 0 to a maximum of $-1/\beta$. In the limiting case when $b \to 1$ the solution (11) describes the so-called "thick" solitary wave, which has a flat crest of amplitude $a_m = -1/\beta$.

The figure shows solitary wave solutions (11) of the extended KdV equation; upper panel for $\beta < 0$; lower panel for $\beta > 0$.

16. Solitary Waves of the Extended KdV Equation





17. Solitary Waves of the Extended KdV Equation

For the case when $\beta > 0$, $b < 0$ and there are two families of solitary waves. One is defined by $-1 < b < 0$, has $a > 0$, and as b decreases from 0 to -1, the amplitude a increases from 0 to ∞, while the speed V also increases from 0 to ∞. The other is defined by $-\infty < b < -1$, has $a < 0$ and, as b increases from $-\infty$ to -1, the amplitude a decreases from $-2/\beta$ to ∞. In the limit $b \to -1$

$$A = a \,\mathrm{sech}\, 2\gamma(\theta - V\tau) \,, \quad V = \beta a^2 = 4\gamma^2 \,,$$

where here a can take either sign. On the other hand, as $b \to -\infty$, $\gamma \to 0$ and the solitary wave (11) reduces to the algebraic form

$$A = \frac{a_0}{1 + \beta a_0^2 \theta^2/4}, \quad a_0 = -\frac{2}{\beta} \,.$$

The parameter space between this lower amplitude limit of $-2/\beta$ and 0 is occupied by breathers .

18. Solitary Waves in a Variable Environment

In many physical situations it is necessary to take account of the fact that solitary waves propagate through a variable environment. This means that the coefficients c, μ and λ in (8) are functions of x, while an additional term $c(Q_x/2Q)A$ needs to be included, where $Q(x)$ is a magnification factor. Thus (8) is replaced by

$$A_t + cA_x + c\frac{Q_x}{2Q}A + \mu AA_x + [\sigma A^2 A_x] + \lambda A_{xxx} = 0\,. \qquad (13)$$

After transforming to new variables, $X = (\int^x dx/c) - t$, $B = Q^{1/2}A$, the variable-coefficient KdV equation (when we ignore the cubic nonlinear term, $\sigma = 0$) is obtained for $B(x, X)$,

$$B_x + \nu(x)BB_X + \delta(x)B_{XXX} = 0\,. \qquad (14)$$

$$\text{where} \qquad \nu = \mu/cQ^{1/2}, \delta = \lambda/c^3\,.$$

It is assumed here that $\frac{\partial}{\partial x} << \frac{\partial}{\partial X}$. In general, equation (14) is not an integrable equation and must be solved numerically, although we shall exhibit some asymptotic solutions. There are two distinct limiting situations to be considered.

19. Fission of Solitary Waves

First, let it be supposed that the coefficients $\nu(x), \delta(x)$ in (14) vary rapidly with respect to the wavelength of a solitary wave, and consider then the case when these coefficients make a rapid transition from the values ν_-, δ_- in $x < 0$ to the values ν_+, δ_+ in $x > 0$. Then a steady solitary wave can propagate in the region $x < 0$, given by

$$B = b\,\mathrm{sech}^2(\gamma(X - Wx)) \text{where} \quad W = \frac{\nu_- b}{3} = 4\delta_-\gamma^2 . \qquad (15)$$

It will pass through the transition zone $x \approx 0$ essentially without change, but on arrival into the region $x > 0$ it is no longer a permissible solution of (14), which now has constant coefficients ν_+, δ_+. Instead, with $x = 0$, the expression (15) forms an effective initial condition for the new constant-coefficient KdV equation. Using the inverse scattering transform, the solution in $x > 0$ can now be constructed; indeed in this case the spectral problem has an explicit solution. The outcome is that the initial solitary wave fissions into N solitons, and some radiation.

20. Fission of Solitary Waves

The number N of solitons produced is determined by the ratio of coefficients $R = \nu_+\delta_-/\nu_-\delta_+$. If $R > 0$ (i.e. there is no change in polarity for solitary waves), then $N = 1 + [((8R+1)^{1/2} - 1)/2]$ ($[\cdots]$ denotes the integral part); as R increases from 0, a new soliton (initially of zero amplitude) is produced as R successively passes through the values $m(m+1)/2)$ for $m = 1, 2, \cdots$. But if $R < 0$ (i.e. there is a change in polarity) no solitons are produced and the solitary wave decays into radiation.

For instance, for water waves,

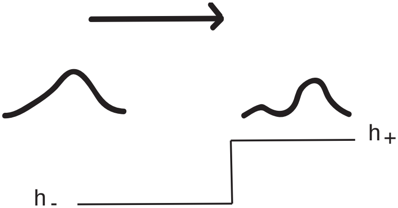
$$c = (gh)^{1/2}, \mu = 3c/2h, \lambda = ch^2/6, Q = c\,,$$

$$\text{and so} \quad \nu = 3/(2hc^{1/2})\,, \delta = h^2/(6c^2)\,,$$

where h is the water depth. It can then be shown that a solitary water wave propagating from a depth h_- to a depth h_+ will fission into N solitons where N is given as above with $R = (h_-/h_+)^{9/4}$; if $h_- > h_+$, $N \geq 2$, but if $h_- > h_+$ then $N = 1$ and no further solitons are produced.

21. Fission of a water wave at a step

Fissionong of a solitary water wave at a step change in the bottom topography.



22. Slowly-varying Solitary Wave

Next, consider the opposite situation when the coefficients $\nu(x), \delta(x)$ in (14) vary slowly with respect to with respect to the wavelength of a solitary wave. In this case a multi-scale perturbation technique can be used in which the leading term is

$$B \sim b\,\mathrm{sech}^2\gamma(X - \int_{x_0}^{x} W\,dx)\,, \tag{16}$$

$$\text{where} \quad W = \frac{\nu b}{3} = 4\delta\gamma^2\,. \tag{17}$$

Here the wave amplitude $b(x)$, and hence also $W(x), \gamma(x)$, are slowly-varying functions of x.

Their variation is most readily determined by noting that the variable-coefficient KdV equation (14) possesses a conservation law,

$$\int_{-\infty}^{\infty} B^2 dX = \text{constant}\,. \tag{18}$$

which expresses conservation of wave-action flux. Substitution of (16) into (18) gives

$$\frac{2b^2}{3\gamma} = \text{constant}\,, \qquad \text{so that} \quad b = \text{constant}\left(\frac{\delta}{\nu}\right)^{1/3}. \tag{19}$$

This is an explicit equation for the variation of the amplitude $b(x)$ in terms of $\nu(x), \delta(x)$.

24. Slowly-varying Solitary Wave

However, the variable-coefficient KdV equation (14) also has a conservation law for mass ,

$$\int_{-\infty}^{\infty} B dX = \text{constant} . \qquad (20)$$

Thus, although the slowly-varying solitary wave conserves wave-action flux it cannot simultaneously conserve mass. Instead, it is accompanied by a trailing shelf of small amplitude but long length scale given by B_s, so that the conservation of mass gives

$$\int_{-\infty}^{\phi} B_s \, dX + \frac{2b}{\gamma} = \text{constant} ,$$

where $\phi = \int_{x_0}^{x} W \, dx$ ($X = \phi$ gives the location of the solitary wave) and the second term is the mass of the solitary wave (16).

25. Slowly-varying Solitary Wave

The amplitude of the shelf at the rear of the solitary wave is then

$$B_{-} = B_s(X = \phi) = \frac{3\gamma_x}{\nu\gamma^2}\,. \tag{21}$$

This shows that if the wavenumber γ increases (decreases) as the solitary wave deforms, then the trailing shelf amplitude B_{-} has the same (opposite) polarity to the solitary wave.

For a solitary water wave propagating over a variable depth $h(x)$ these results show that the amplitude varies as h^{-1}, while the trailing shelf has positive (negative) polarity relative to the wave itself according as $h_x < (>) 0$.

26. Critical Point

A situation of particular interest occurs if the coefficient $\nu(x)$ changes sign at some particular location (note that in most physical systems the coefficient δ of the linear dispersive term in (14) does not vanish for any x). This commonly arises for internal solitary waves in the coastal ocean, where typically in the deeper water, $\nu < 0, \delta > 0$ so that internal solitary waves propagating shorewards are waves of depression. But in shallower water, $\nu > 0$ and so only internal solitary waves of elevation can be supported. The issue then arises as to whether an internal solitary wave of depression can be converted into one or more solitary waves of elevation as the critical point, where ν changes sign, is traversed. The solution depends on how rapidly the coefficient ν changes sign. If ν passes through zero rapidly compared to the local width of the solitary wave, then the solitary wave is destroyed, and converted into a radiating wavetrain.

27. Critical Point

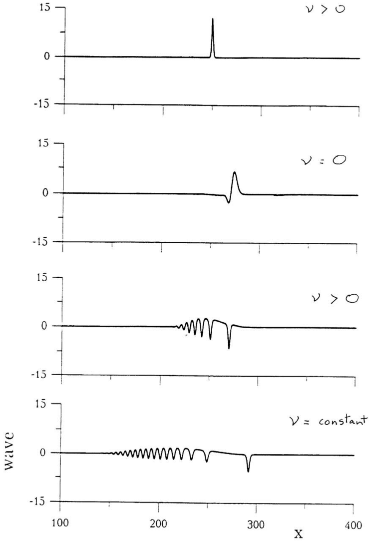
On the other hand, if ν changes sufficient slowly for the present theory to hold (i.e. (19) applies), we find that as

$$\nu \to 0 \quad \text{then} \quad A \sim |\nu|^{1/3} \to 0\,,$$

$$\text{while} \quad B_{-} \sim |\nu|^{-8/3} \to \infty\,.$$

Thus, as the solitary wave amplitude decreases, the amplitude of the trailing shelf, which has the opposite polarity, grows indefinitely until a point is reached just prior to the critical point where the slowly-varying solitary wave asymptotic theory fails. A combination of this trailing shelf and the distortion of the solitary wave itself then provide the appropriate "initial" condition for one or more solitary waves of the opposite polarity to emerge as the critical point is traversed.

28. Passage through a critical point

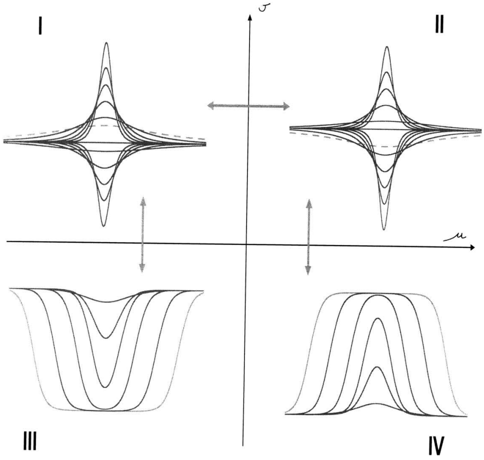


29: Passage through a Critical Point, Extended KdV Equation

However, it is clear that in situations, as here, where $\nu \approx 0$, it will be necessary to include a cubic nonlinear term in (13), thus converting it into a variable-coefficient Gardner equation. The outcome depends on the sign of the coefficient (σ) of the cubic nonlinear term at the critical point. If $\sigma > 0$ so that solitary waves of either polarity can exist when $\nu = 0$, then the solitary wave preserves its polarity (i.e. remains a wave of depression) as the critical point is traversed. On the other hand if $\sigma < 0$ so that no solitary wave can exist when $\sigma = 0$ then the solitary wave of depression may be converted into one or more solitary waves of elevation.

30. Passage through a Critical Point, Extended KdV Equation

Solitary wave transitions as μ, σ change sign.



31. The generation of solitary waves by flow over topography

To this point, we have considered only the situation when solitary waves arise as a solutions of the initial value problem for the KdV equation (1). This is appropriate in many cases. For instance, in the ocean or atmosphere, or in laboratory experiments, internal solitary waves can be generated when the pycnocline is given a localized displacement. This gives an initial condition for the KdV equation. When this initial displacement has the correct polarity, it will evolve into a finite number of solitary waves. This scenario often occurs in the coastal oceans when the barotropic ocean tide interacts with the continental shelf to generate an internal tide, which provides the necessary pycnocline displacement. This, in turn, evolves into propagating internal solitary waves.

32. The generation of solitary waves by flow over topography

However, KdV solitary waves can also arise by direct forcing mechanisms. A common such situation is when a fluid flow interacts with a localized topographic obstacle in a situation of near criticality. Here a critical flow is one which supports a linear long wave of zero speed. In this case, the waves generated are unable to escape the vicinity of the obstacle, and hence can be said to be directly forced. The inability of the waves to propagate away rapidly from the forcing region means that nonlinearity is needed from the outset. As we will show below the KdV equation (1) is then replaced by a forced KdV equation, whose principal solutions resemble "undular bores". Hence, we first provide a description of the "undular bore".

33. Undular Bore

The term "undular bore " is widely used in the literature in a variety of contexts and several different meanings. Here, we need to make it clear that we are concerned with non-dissipative flows, in which case an undular bore is intrinsically unsteady. In general, an undular bore is an oscillatory transition between two different basic states. A simple representation of an undular bore can be obtained from the solution of the KdV equation (1) with the initial condition that

$$A = A_0 H(-x)\,, \tag{22}$$

where we assume at first that $A_0 > 0$. Here $H(x)$ is the Heaviside function (i.e. $H(x) = 1$ if $x > 0$ and $H(x) = 0$ if $x < 0$). The solution can in principle be obtained through the inverse scattering transform.

34. Undular Bore

However, it is more instructive to use the asymptotic method developed by Gurevich & Pitaevskii and Whitham independently in 1974. In this approach, the solution of (1) with this initial condition is represented as the modulated periodic wave train (5) supplemented here with a mean term d, that is

$$A = a\{b(m) + \mathrm{cn}^2(\gamma(x - Vt); m)\} + d\,, \tag{23}$$

$$\text{where} \quad b(m) = \frac{1-m}{m} - \frac{E(m)}{mK(m)}\,, \quad a = 2m\gamma^2\,, \tag{24}$$

$$\text{and} \quad V = 6d + 2a\left\{\frac{2-m}{m} - \frac{3E(m)}{mK(m)}\right\}. \tag{25}$$

We recall that as the modulus $m \to 1$, this becomes a solitary wave, but as $m \to 0$ it reduces to sinusoidal waves of small amplitude.

The asymptotic method of Gurevich & Pitaevskii and Whitham is to let the expression (23) describe a modulated periodic wavetrain in which the amplitude a, the mean level d, the speed V and the wavenumber γ are all slowly varying functions of x and t. The cnoidal wave contains three free parameters, and so three equations are needed. The Whitham method obtains these by averaging three conservation laws of the KdV equation. The outcome is three modulation equations, whose form is nonlinear hyperbolic. But, because the KdV equation is an integrable equation, these equations can be put into diagonal Riemann form. The general solution can then be obtained. Here the relevant solution corresponding to the initial condition (22) is constructed in terms of the similarity variable x/t, and is given by

36. Similarity Solution for an Undular Bore

$$\frac{x}{t} = 2A_0\{1 + m - \frac{2m(1-m)(K(m)}{E(m) - (1-m)K(m)}\}, -6A_0 < \frac{x}{t} < 4A_0 , \tag{26}$$

$$a = 2A_0 m\,,\ d = A_0\{m - 1 + \frac{2E(m)}{K(m)}\}\,. \tag{27}$$

Ahead of the wavetrain where $x/t > 4A_0$, $A = 0$ and at this end, $m \to 1$, $a \to 2A_0$ and $d \to 0$; the leading wave is a solitary wave of amplitude $2A_0$ relative to a mean level of 0. Behind the wavetrain where $x/t < -6A_0$, $A = A_0$ and at this end $m \to 0$, $a \to 0$, and $d \to A_0$; the wavetrain is now sinusoidal with a wavenumber γ given by $6\gamma^2 \approx A_0$. Further, it can be shown that on any individual crest in the wavetrain, $m \to 1$ as $t \to \infty$. In this sense, the undular bore evolves into a train of solitary waves.

37. Rarefraction wave

If $A_0 < 0$ in the initial condition (22), then an "undular bore" solution analogous to that described by (26, 27) does not exist. Instead, the asymptotic solution is a rarefraction wave,

$$
\begin{aligned}
A &= 0 \quad \text{for} \quad x > 0\,, \\
A &= \frac{x}{6t} \quad \text{for} \quad A_0 < \frac{x}{6t} < 0\,, \\
A &= A_0\,, \quad \text{for} \quad \frac{x}{6t} < A_0(< 0)\,.
\end{aligned}
\tag{28}
$$

Small oscillatory wavetrains are needed to smooth out the discontinuities in A_x at $x = 0$ and $x = -6A_0$.

The generation of an undular bore requires an initial condition in which $A \to A_{\pm}$ with $A_- > A_+$ as $x \to \pm\infty$; note that (22) is the simplest such condition. A common situation where this type of initial condition may be generated occurs when a steady transcritical flow encounters a localized topographic obstacle, in the context of the flow of a density-stratified fluid.

38. Forced KdV Equation

A flow $U_0 > 0$ is said to be critical if it can support a wave mode whose speed $c \approx 0$, in the frame of reference of the topographic obstacle. Let us suppose that the bottom boundary of the stratified fluid is given by $z = -h + F(x)$, where $F(x)$ is spatially localized, and for a KdV balance F is order the wave amplitude squared. Let the speed $c = \Delta$ where $\Delta << 1$ is a detuning parameter of the same order as the wave amplitude. Then it can be shown that the KdV equation (**??**) is replaced by the forced KdV (fKdV) equation

$$A_T + \Delta A_X + \mu A A_X + \lambda A_{XXX} + \Gamma F_X(X) = 0\,, \qquad (29)$$

Here the coefficients μ, λ, Γ are all known, and we can assume that $\lambda < 0$. The $\Delta > 0 (< 0)$ defines supercritical (subcritical) flow respectively. Also it then follows that $\mu < 0 (> 0)$ for a solitary wave of elevation (depression). The fKdV equation (29) has been derived in many physical contexts, and is a canonical model equation to describe transcritical flow interaction with an obstacle.

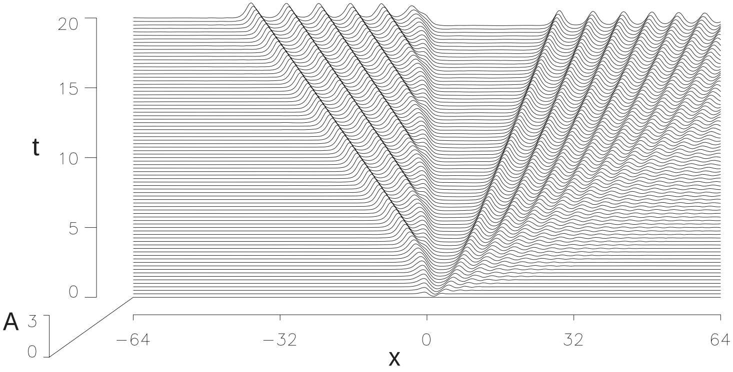
39. Forced KdV Equation

As for the KdV equation (8) we may now rescale the fKdV equation (29) into a canonical form

$$-A_t - \Delta A_x + 6AA_x + A_{xxx} + F_x(x) = 0\,. \qquad (30)$$

This is to be solved with the initial condition that $A(x,0) = 0$, which corresponds to a slow introduction of the topographic obstacle. An important issue here is the polarity of the forcing in (30), that is, whether it has positive (negative) polarity $F(x) \geq 0 (\leq 0)$. Taking account of the scaling positive polarity in the original dimensional coordinates leads to positive (negative) polarity in the dimensionless equation (30) according as $\mu\Gamma > 0 (< 0)$.

39. Solution of the Forced KdV Equation for Critical Flow

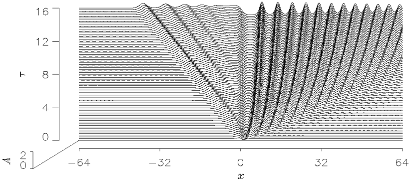


The solution of the fKdV equation (30) at exact criticality, $\Delta = 0$ an for isolated positive forcing, $F(x)$ is positive, and non-zero only in a vicinity of $x = 0$, with a maximum value of $F_M > 0$.

40. Solution of Forced KdV Equation for Subcritical/Supercritical Flow

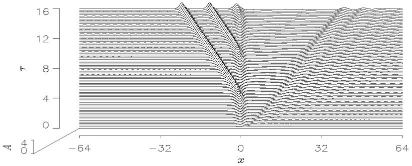
Δ=-1.5

subcritical



Δ=1.5

supercritical



41. Solution of the Forced KdV Equation

The solution is characterised by upstream and downstream wavetrains connected by a locally steady solution over the obstacle. For supercritical flow ($\Delta < 0$) the upstream wavetrain weakens, and for sufficiently large $|\Delta|$ detaches from the obstacle, while the downstream wavetrain intensifies and for sufficiently large $|\Delta|$ forms a stationary lee wave field. On the other hand, for supercritical flow ($\Delta > 0$) he upstream wavetrain develops into well-separated solitary waves while the downstream wavetrain weakens and moves further downstream The origin of the upstream and downstream wavetrains can be found in the structure of the locally steady solution over the obstacle.

42. Solution of the Forced KdV Equation: Hydraulic Approximation

In the transcritical regime this is characterised by a transition from a constant state A_- upstream of the obstacle to a constant state A_+ downstream of the obstacle, where $A_- > 0$ and $A_+ < 0$. It is readily shown that $\Delta = 3(A_+ + A_-)$ independently of the details of the forcing term $F(x)$. Explicit determination of A_+ and A_- requires some knowledge of the forcing term $F(x)$. However, in the "hydraulic" limit when the linear dispersive term in (30) can be neglected, it is readily shown that

$$6A_\pm = \Delta \mp (12F_M)^{1/2} \,. \tag{31}$$

This expression also serves to define the transcritical regime, which is

$$|\Delta| < (12F_M)^{1/2} \,. \tag{32}$$

Thus upstream of the obstacle there is a transition from the zero state to A_-, while downstream the transition is from A_+ to 0; each transition is effectively generated at $X = 0$.

43. Solution of the Forced KdV Equation: Undular Bores

Both transitions are resolved by "undular bore" solutions as described above. That in $x < 0$ is exactly described by (26, 27) with x replaced by $\Delta t - x$, and A_0 by A_-. It occupies the zone

$$\Delta - 4A_- < \frac{x}{t} < \max\{0,\ \Delta + 6A_-\}\,. \tag{33}$$

Note that this upstream wavetrain is constrained to lie in $x < 0$, and hence is only fully realised if $\Delta < -6A_-$. Combining this criterion with (31) and (32)) defines the regime

$$-(F_M)^{1/2} < \Delta < -\frac{1}{2}(F_M)^{1/2}\,, \tag{34}$$

where a fully developed undular bore solution can develop upstream.

44. Solution of the Forced KdV Equation: Undular Bores

On the other hand, the regime $\Delta < -6A_-$ or

$$-\frac{1}{2}(F_M)^{1/2} < \Delta < (F_M)^{1/2}\,, \tag{35}$$

is where the upstream undular bore is only partially formed, and is attached to the obstacle. In this case the modulus m of the Jacobian elliptic function varies from 1 at the leading edge (thus describing solitary waves) to a value m_- (< 1) at the obstacle, where m_- can be found from (26) by replacing x with Δ and A_0 with A_-.

45. Solution of the Forced KdV Equation: Undular Bores

The transition in $x > 0$ can also be described by (26, 27) where we now replace x with $(\Delta + 6A_+)t - x$, A_0 with $-A_+$, and d with $d - A_+$. This "undular bore" solution occupies the zone

$$\max\{0,\ \Delta - 2A_+\} < \frac{x}{t} < \Delta - 12A_+ \,. \tag{36}$$

Here, this downstream wavetrain is constrained to lie in $x > 0$, and hence is only fully realised if $\Delta > 2A_+$. Combining this criterion with (31) and (32) defines the regime (35), and so a fully detached downstream undular bore coincides with the case when the upstream undular bore is attached to the obstacle.

46. Solution of the Forced KdV Equation: Undular Bores

On the other hand, in the regime (34), when the upstream undular bore is detached from the obstacle, the downstream undular bore is attached to the obstacle, with a modulus $m_{+}(< 1)$ at the obstacle, where m_{+} can be founding from (26) by replacing x with $\Delta - 6A_{+}$ and A_0 with A_{+}. Indeed now a stationary lee wavetrain develops just behind the obstacle.

For the case when the obstacle has negative polarity (that is $F(x)$ is negative, and non-zero only in the vicinity of $x = 0$), the upstream and downstream solutions are qualitatively similar to those described above for positive forcing. However, the solution in the vicinity of the obstacle remains transient, and this causes a modulation of the "undular bore" solutions.

47. References

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THE END, THANK YOU