The Ramanujan Entire Function

Mourad E.H. Ismail, University of Central Florida

Ramanujan was a self-educated college drop out who did some of the best mathematics of the twentieth century. He extensively worked on the

$$F(z) = 1 + \sum_{n=1}^{\infty} \frac{(-z)^n q^{n^2}}{(1 - q)(1 - q^2)\ldots(1 - q^n)},$$

which we refer to as the Ramanujan entire function. We demonstrate the significance of this function in number theory and analysis and give a new interpretation of the statement

$$1 + \sum_{n=1}^{\infty} \frac{z^n q^{n^2}}{(1 - q)(1 - q^2)\ldots(1 - q^n)} = \prod_{n=1}^{\infty} \left( 1 + \frac{z q^{2n-1}}{1 - c_1 q^n - c_2 q^{2n} - \ldots} \right)$$

in Ramanujan’s lost note book.

The coefficients $c_1, c_2, \ldots$ turned out to have very interesting patterns and many open problems with be mentioned.