Spectral Measure of Certain Gram Random Matrices
Applications in Wireless Communications

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Gram matrices in this presentation

\[ H_n = Y_n + A_n \]

- \( Y_n \) is a \( N \times n \) random matrix with independent centered elements having possibly different variances.
- \( A_n \) is a deterministic matrix.

**Eigenvalue distribution of** \( H_n H_n^H \) **when** \( n \to \infty \) **and** \( \frac{N}{n} \to c > 0? \)
OUTLINE

1) Problem statement

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4) The general case: main steps of the proof

5) Towards a Central Limit Theorem
Problem Statement

Figure 1: Multiple Input Multiple Output (MIMO) wireless communication
\textbf{Problem Statement}

\begin{center}
\textbf{SHANNON'S MUTUAL INFORMATION}
\end{center}

Shannon’s mutual information per receive antenna of the $N \times n$ random MIMO channel $\mathbf{H}_n$:

$$C_n (\varsigma^2) = \frac{1}{N} \mathbb{E} \log \det \left( \mathbf{I}_N + \frac{1}{\varsigma^2} \mathbf{H}_n \mathbf{H}_n^H \right)$$

where $\varsigma^2$ is a known parameter (noise variance).

Information theory: $NC_n (\varsigma^2)$ is the maximum data rate attainable by the transmission system.

\textbf{Behaviour of $C_n (\varsigma^2)$ as $n \to \infty$ and $\frac{N}{n} \to c > 0$ ?}
Spectral measure and Stieltjes Transform

- $C_n (\zeta^2) = \mathbb{E} \frac{1}{N} \sum_{i=1}^{N} \log \left( 1 + \frac{\lambda_{i,n}}{\zeta^2} \right) = \mathbb{E} \int \log \left( 1 + \frac{t}{\zeta^2} \right) \mu_n(dt)$ where $\mu_n$ is the spectral measure (empirical distribution of eigenvalues $\{\lambda_{1,n}, \ldots, \lambda_{N,n}\}$) of $H_n H_n^\dagger$.

- Given a certain statistical model for $H_n$, one hopes that the spectral measure $\mu_n$ converges weakly to a deterministic Limit Spectral probability Measure (LSM) $\mu$, in order to have

$$C_n (\zeta^2) \xrightarrow{n \to \infty} C^* (\zeta^2) = \int \log \left( 1 + \frac{t}{\zeta^2} \right) \mu(dt).$$

- We study $\mu_n$ in the asymptotic regime, or equivalently, its Stieltjes Transform (ST)

$$f_{\mu_n}(z) = \int \frac{1}{t - z} \mu_n(dt).$$

- Weak convergence of $\mu_n$ towards $\mu$ is equivalent to convergence of $f_{\mu_n}(z)$ towards the ST $f_\mu(z)$ of the LSM $\mu$. 

Problem Statement
"Ricean" Channel Model

\[ \mathbf{H}_n = \mathbf{Z}_n + \mathbf{B}_n \]

- \( \mathbf{Z}_n = \begin{bmatrix} Z_{i,j}^{(n)} \end{bmatrix} \), elements of a Gaussian stationary two dimensional process with covariance function \( \kappa \):

\[ \mathbb{E} \left[ Z_{i_1,j_1}^{(n)} Z_{i_2,j_2}^{(n)*} \right] = \frac{1}{n} \kappa \left( i_1 - i_2, j_1 - j_2 \right) \]

- \( \mathbf{B}_n \) is a deterministic matrix (Rice component).
**Problem Statement**

**Channel statistical model 2**

Channel matrix is $F_N H_n F_n^H$ where $F_l$ is the $l \times l$ Fourier matrix and

$$H_n = Y_n + A_n$$

- Elements of $Y_n = \begin{bmatrix} Y_{i,j}^{(n)} \end{bmatrix}$ written $Y_{i,j}^{(n)} = \frac{\sigma_{ij}(n)}{\sqrt{n}} X_{ij}$ with $X_{ij}$ standard Gaussian independent random variables.

- $A_n$ is a deterministic matrix.

Sometimes we shall assume:

**(A)** Variance profile is $\sigma_{ij}(n)^2 = \sigma^2 \left( \frac{i}{N}, \frac{j}{n} \right)$ where $\sigma^2(x, y)$ is a continuous function on $[0, 1]^2$ called a limit variance profile.
Problem Statement

For asymptotic study, model 1 can be replaced with model 2 with

- Assumption (A) with \( \sigma^2(x, y) = \Gamma(x, y) \) where
  \[
  \Gamma(x, y) = \sum_{i,j} \kappa(i, j)e^{-2\pi i (ix-jy)}
  \]

  is the Spectral Density of the process \( Z_{i,j} \).

- \( A_n \) is the two-dimensional Fourier Transform of \( B_n \).

and some assumptions.
Argument formalized in Hachem, Loubaton and Najim’05.
Problem Statement

Model 2: $H_n = Y_n + A_n$ with size $N \times n$.

- $Y_n = \begin{bmatrix} Y_{i,j}^{(n)} \end{bmatrix}$ with $Y_{i,j}^{(n)} = \frac{\sigma_{i,j}^{(n)}}{\sqrt{n}} X_{i,j}$, random variables $X_{i,j}$ are centered unit variance iid.
  
  We release Gaussianity assumption on $X_{i,j}$.

- $A_n$ is a deterministic matrix.

With appropriate additional assumptions,

- Characterize the asymptotic behaviour of the spectral measure $\mu_n$ of $H_n H_n^H$ as $n \to \infty$ and $N/n \to c > 0$, or equivalently, its ST $f_{\mu_n}(z)$.

- Deduce the asymptotic behaviour of Shannon’s mutual information $C_n(\varsigma^2)$. 

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Some particular cases

THE CENTERED CASE \( (A_n = 0) \)

Assume \((A)\), i.e., \(\exists\) a limit variance profile.

- Girko’90: \( \mu_n \) converges weakly to a deterministic probability measure \( \mu \) which ST
  \[ f_{\mu}(z) = \int_{0}^{1} p(u, z) du. \]
  Function \( p(u, z) \) continuous in \( u \) for every \( z \), ST of a probability measure in \( z \) for
  every \( u \), defined as the unique solution of an implicit equation.

- Same result can be deduced from the work of Boutet de Monvel, Khorunzhyi and
  Vasilchuck (96).

- And also from Shlyakhtenko’s (96) result stated for Wigner Gaussian matrices. His
  approach based on the concept of freeness with amalgamation.
Some particular cases

**Remark on the general non centered case**

Even if we have a limit variance profile $\sigma^2(x, y)$ for the elements of $Y_n$ and if $A_n A_n^H$ has a limit spectral measure, the spectral measure $\mu_n$ of $H_n H_n^H$ does not converge except in some very specific cases.
Some particular cases

Specific Case 1: $\sigma(x, y)$ constant and $AA^H$ has a LSM

Case

- $\sigma(x, y) = \sigma$ is a constant, i.e., $Y_n$ has iid elements,

- The spectral measure $\nu_n$ of $A_nA_n^H$ converges weakly

$$\nu_n \rightharpoonup \nu$$

treated by Brent Dozier and Silverstein (04): $\mu_n$ converges to a deterministic probability measure which ST $f(z)$ is the unique solution to

$$f(z) = \int \frac{\nu(dt)}{-z(1 + c\sigma^2 f(z)) + (1 - c)\sigma^2 + \frac{t}{1 + c\sigma^2 f(z)}}$$

in the class of ST of probability measures over $\mathbb{R}_+$. 

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Some particular cases

Specific case 2: $\sigma^2(x, y)$ non trivial and $A$ diagonal

Hachem, Loubaton, Najim'05:

- Existence of a limit variance profile $(A)$.
- Moment assumption: $\exists \varepsilon > 0$ where $\mathbb{E} |X_{ij}|^{4+\varepsilon} < \infty$.
  
  Can be lightened by a truncation argument (Bai and Silverstein).

- $A_n$ diagonal, i.e., when $n \geq N$ (which we shall assume), has the form

$$A_n = \begin{bmatrix}
A_{11} & \cdots & \cdots & 0 \\
\vdots & & & \\
0 & A_{NN} & \cdots & 0
\end{bmatrix}$$

- $\frac{1}{N} \sum_{i=1}^{N} \delta_{(i/N, |A_{ii}|^2)} \Rightarrow H(dt, d\lambda)$, compactly supported pr. measure in $[0, 1] \times \mathbb{R}_+$.

"Stonger" than convergence of the empirical distribution $\frac{1}{N} \sum_{i=1}^{N} \delta_{|A_{ii}|^2}$. 
Some particular cases

**Specific Case 2: Technique**

- Resolvent is $Q_n(z) = (H_n H_n^H - zI_N)^{-1}$. ST associated with the spectral measure $\mu_n$ of $H_n H_n^H$:

  $$f_{\mu_n}(z) = \int \frac{1}{t-z} \mu_n(dt) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_{i,n} - z} = \frac{1}{N} \text{tr} Q_n(z)$$

- Let $\tilde{\mu}_N$ be the spectral measure of $H_n^H H_n$. Associated ST is $f_{\tilde{\mu}_n}(z) = \frac{1}{N} \text{tr} \tilde{Q}_n(z)$ with $\tilde{Q}_n(z) = (H_n^H H_n - zI_n)^{-1}$.

- We study jointly the convergence of $f_{\mu_n}$ and $f_{\tilde{\mu}_n}$ by considering the diagonal terms $Q_{ii}(z)$ and $\tilde{Q}_{jj}(z)$ of $Q_n(z)$ and $\tilde{Q}_n(z)$.
Some particular cases

**Specific Case 2: Technique**

- We establish convergence of measures

\[
L_n(z, du, d\lambda) = \frac{1}{N} \sum_{i=1}^{N} Q_{ii}(z) \delta\left(\frac{i}{N}, |A_{ii}|^2\right)(du, d\lambda)
\]

\[
\tilde{L}_n(z, du, d\lambda) = \frac{1}{n} \sum_{j=1}^{N} \tilde{Q}_{jj}(z) \delta\left(\frac{j}{n}, |A_{jj}|^2\right)(du, d\lambda)
\]

\[
+ \frac{1}{n} \sum_{j=N+1}^{n} \tilde{Q}_{jj}(z) \delta\left(\frac{j}{n}\right)(du) \otimes \delta_0(d\lambda)
\]
Some particular cases

**Specific case 2: Limit spectral measure**

- Consider the following system: for every bounded continuous $g$,

\[
\int g \, d\pi(z, du, d\lambda) = \int \frac{g(u, \lambda)}{-z - z \int \sigma^2(u, t) d\tilde{\pi}(z, dt, d\zeta) + \frac{\lambda}{1+c \int \sigma^2(t, cu) d\pi(z, dt, d\zeta)}} H(du, d\lambda)
\]

\[
\int g \, d\tilde{\pi}(z, du, d\lambda) = c \int \frac{g(cu, \lambda)}{-z - cz \int \sigma^2(t, cu) d\pi(z, dt, d\zeta) + \frac{\lambda}{1+c \int \sigma^2(u, t) d\tilde{\pi}(z, dt, d\zeta)}} H(du, d\lambda)
\]

\[
+ (1-c) \int_c^1 \frac{g(u, 0)}{-z - cz \int \sigma^2(t, u) d\pi(z, dt, d\zeta)} \, du
\]

System has a unique solution $(\pi, \tilde{\pi})$ in a certain class of complex measures (the Stieltjes kernels).

- $\pi$ and $\tilde{\pi}$ are the limits of $L_n$ and $\tilde{L}_n$ in the weak convergence of complex measures.

- The limit ST $f_\mu$ and $f_{\tilde{\mu}}$ are then

\[
f_\mu(z) = \int \pi(z, dt, d\lambda) \quad \text{and} \quad f_{\tilde{\mu}}(z) = \int \tilde{\pi}(z, dt, d\lambda)
\]
The general case

- We assume $\sigma^2(x, y)$ non trivial and $A_n$ has no particular structure.

- Difficult to devise simple conditions for the existence of a limit spectral measure, 
  i.e., an "extension" of assumption $\frac{1}{N} \sum_{i=1}^{N} \delta(i/N, |A_{ii}|^2) \Rightarrow H(dt, d\lambda)$ that we used for the case $A_n$ is diagonal.

- An alternative approach: look for a deterministic approximation of the empirical ST: there exists a a $N \times N$ deterministic matrix function $T_n(z)$ such that

$$f_{\mu_n}(z) - \frac{1}{N} \text{tr}T_n(z) \xrightarrow{n \to \infty} 0 \text{ almost surely}$$

This "deterministic approximation" dates back to Girko.
The general case

**Deterministic Approximation: Assumptions**

Hachem, Loubaton, Najim’05 (preprint): Extension of Girko’s result and simplification of his proof, approximation of Shannon’s mutual information.

Problem: approximate the spectral measure of $H_n = Y_n + A_n$ with

- $Y_{i,j}^{(n)} = \frac{\sigma_{ij}(n)}{\sqrt{n}} X_{i,j}$ with $X_{i,j}$ centered unit variance iid and $\mathbb{E}|X_{11}|^{4+\varepsilon} < \infty$ for some $\varepsilon > 0$. Last assumption can be lightened.

- $\sup_{i,j,n} \sigma_{ij}^2(n) < \infty$.

- Euclidean norms of rows and columns of $A_n$ uniformly bounded.

Girko assumed boundedness of $\ell_1$ norms of rows and columns.

In wireless communications, columns of $A_n$ have typically the form

$$\frac{C}{\sqrt{N}} [1, \exp(i\omega), \ldots, \exp(i(N-1)\omega)]^T$$

$\ell_1$ norm increases in $\sqrt{N}$ while Euclidean ($\ell_2$) norm is bounded.
The general case

Deterministic approximation: result

Let \( D^{(j)} = \text{diag} \left( [\sigma_{1j}^2, \ldots, \sigma_{Nj}^2] \right) \) and \( \tilde{D}^{(i)} = \text{diag} \left( [\sigma_{i1}^2, \ldots, \sigma_{in}^2] \right) \).

- The deterministic system of \( N + n \) equations:

\[
\psi^{(i)}(z) = \frac{-1}{z \left( 1 + \frac{1}{n} \text{tr} \left( \tilde{D}^{(i)} \tilde{T}(z) \right) \right)} \quad \text{for } 1 \leq i \leq N,
\]

\[
\tilde{\psi}^{(j)}(z) = \frac{-1}{z \left( 1 + \frac{1}{n} \text{tr} \left( D^{(j)} T(z) \right) \right)} \quad \text{for } 1 \leq j \leq n,
\]

where

\[
\Psi(z) = \text{diag} \left( [\psi^{(1)}(z), \ldots, \psi^{(N)}(z)] \right), \quad \tilde{\Psi}(z) = \text{diag} \left( [\tilde{\psi}^{(1)}(z), \ldots, \tilde{\psi}^{(n)}(z)] \right)
\]

\[
T(z) = \left( \Psi^{-1}(z) - z A \tilde{\Psi}(z) A^H \right)^{-1}, \quad \tilde{T}(z) = \left( \tilde{\Psi}^{-1}(z) - z A^H \Psi(z) A \right)^{-1}
\]

admits a unique solution \( (\psi^{(1)}, \ldots, \psi^{(N)}, \tilde{\psi}^{(1)}, \ldots, \tilde{\psi}^{(n)}) \) in the class of Stieltjes Transforms of probability measures over \( \mathbb{R}_+ \).
The general case

**Deterministic approximation: result**

- Almost surely,

\[
\left( \frac{1}{N} \text{tr} Q_n(z) - \frac{1}{N} \text{tr} T_n(z) \right) \xrightarrow{n \to \infty} 0 \quad \forall z \in \mathbb{C} - \mathbb{R}_+, \\
\left( \frac{1}{n} \text{tr} \tilde{Q}_n(z) - \frac{1}{n} \text{tr} \tilde{T}_n(z) \right) \xrightarrow{n \to \infty} 0 \quad \forall z \in \mathbb{C} - \mathbb{R}_+
\]
The general case

**Back to Mutual Information**

Mutual information can be written

\[
C_n(\varsigma^2) = \int_{\varsigma^2}^{\infty} \left( \frac{1}{\omega} - \mathbb{E} \frac{1}{N} \text{tr} Q_n(-\omega) \right) d\omega
\]

Combining this expression with the last result, we can establish:

Let

\[
\overline{C}_n(\varsigma^2) = \frac{1}{N} \log \det \left[ \frac{\Psi (-\varsigma^2)^{-1}}{\varsigma^2} + A \tilde{\Psi} (-\varsigma^2) A^H \right]
\]

\[
+ \frac{1}{N} \log \det \frac{\tilde{\Psi} (-\varsigma^2)^{-1}}{\varsigma^2} - \frac{\varsigma^2}{Nn} \sum_{i=1:N}^{i \neq j=1:n} \sigma_{ij}^2 T_{ii}(-\varsigma^2) \tilde{T}_{jj}(-\varsigma^2)
\]

where \(T_{ii}\) and \(\tilde{T}_{jj}\) are the diagonal elements of \(T_n(z)\) and \(\tilde{T}_n(z)\). Then

\[
C_n(\varsigma^2) - \overline{C}_n(\varsigma^2) \xrightarrow{n \to \infty} 0.
\]
General case: main steps of the proof

**Step 1: Existence and Unicity of \( T(z) \)**

Existence and unicity of \( T_n(z) \) and \( \tilde{T}_n(z) \) as solutions of the system of \( N + n \) equations above.

- Existence by an iterative scheme.
- Unicity in a certain region of \( \mathbb{C} \) in \( \mathbb{C} - \mathbb{R}_+ \) by analytic continuation.
- Use an extension of complex analysis results about Stieltjes transforms of probability measures over \( \mathbb{R}_+ \): let \( T(z) \) be an analytical matrix function on \( \mathbb{C}_+ = \{z : \Re z > 0\} \) such that \( \Re T(z) \geq 0 \) on \( \mathbb{C}_+ \) and \( \Re z T(z) \geq 0 \) on \( \mathbb{C}_+ \), (as nonnegative matrices). Then there exists a matrix \( C \geq 0 \) and a matrix valued measure \( \mu \) carried by \( \mathbb{R}_+ \) such as \( \mu(A) \geq 0 \) for every Borel set \( A \) of \( \mathbb{R}_+ \), and

\[
T(z) = C + \int \frac{1}{t-z} \mu(dt) \quad \text{with} \quad \text{tr} \int \frac{1}{1+t} \mu(dt) < \infty
\]
General case: main steps of the proof

**Step 2: Introducing new functions $R(z)$ and $\tilde{R}(z)$**

We introduce intermediate matrices $R_n(z)$ and $\tilde{R}_n(z)$ defined as:

\[
b^{(i)}(z) = \frac{-1}{z \left( 1 + \frac{1}{n} \text{tr} \left( \tilde{D}^{(i)} Q(z) \right) \right)}, \quad B(z) = \text{diag} \left( [b^{(1)}(z), \ldots, b^{(N)}(z)] \right),
\]

\[
\tilde{b}^{(j)}(z) = \frac{-1}{z \left( 1 + \frac{1}{n} \text{tr} \left( D^{(j)} Q(z) \right) \right)}, \quad \tilde{B}(z) = \text{diag} \left( [\tilde{b}^{(1)}(z), \ldots, \tilde{b}^{(n)}(z)] \right),
\]

\[
R(z) = \left( B^{-1}(z) - zA\tilde{B}(z)A^H \right)^{-1}, \quad \tilde{R}(z) = \left( \tilde{B}^{-1}(z) - zA^HB(z)A \right)^{-1}.
\]
**General case: main steps of the proof**

**Step 2: Introducing new functions $R(z)$ and $\tilde{R}(z)$**

We show that for any diagonal matrices $U_n$ and $\tilde{U}_n$ such that $\sup_n \|U_n\| < \infty$ and $\sup_n \|\tilde{U}_n\| < \infty$, we have on $\mathbb{C}_+$,

\[
\mathbb{E} \left| \frac{1}{n} \text{tr} \left( (Q_n(z) - R_n(z)) U_n \right) \right|^{2+\varepsilon/2} < \text{Cst} \times n^{-(1+\varepsilon/4)} \quad \text{and}
\]

\[
\mathbb{E} \left| \frac{1}{n} \text{tr} \left( (\tilde{Q}_n(z) - \tilde{R}_n(z)) \tilde{U}_n \right) \right|^{2+\varepsilon/2} < \text{Cst} \times n^{-(1+\varepsilon/4)}
\]

Derivations along the lines of those of Brent Dozier and Silverstein (04).
Bai and Silverstein’s (98) lemma is of prime importance: in our context, for any $p \geq 2$,

\[
\mathbb{E} \left| \frac{1}{N} x_N^H Z_N x_N - \frac{1}{N} \text{tr} Z_N \right|^p < \frac{\text{Cst}}{N^{p/2}}
\]

for $x_N = [X_1, \ldots, X_N]^T$ with $X_i$ iid centered unit variance random variables with $\mathbb{E}|X_{11}|^{2p} < \infty$, and $Z_N$ is a $N \times N$ random matrix independent of $x_N$ such that $\sup_N \|Z_N\| < \infty$. 

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General case: main steps of the proof

**Step 3:** $\frac{1}{n} \text{tr} \mathbf{R}$ is close to $\frac{1}{n} \text{tr} \mathbf{T}$

We show that in a certain region $\mathcal{D}$ of $\mathbb{C}_+$,

\[
    \mathbb{E} \left| \frac{1}{n} \text{tr} \left( \mathbf{R}(z) - \mathbf{T}(z) \right) \right|^{2+\varepsilon/2} < \frac{\text{Cst}}{n^{1+\varepsilon/4}}
\]

and

\[
    \mathbb{E} \left| \frac{1}{n} \text{tr} \left( \tilde{\mathbf{R}}(z) - \tilde{\mathbf{T}}(z) \right) \right|^{2+\varepsilon/2} < \frac{\text{Cst}}{n^{1+\varepsilon/4}}
\]

**Idea:**

Recall that $b^{(i)}(z) = \frac{-1}{z \left( 1 + \frac{1}{n} \text{tr} \left( \tilde{\mathbf{D}}^{(i)} \tilde{\mathbf{Q}}(z) \right) \right)}$.

From step 2 with $\tilde{\mathbf{U}} = \tilde{\mathbf{D}}^{(i)}$ we have $\frac{1}{n} \text{tr} \left( \tilde{\mathbf{D}}^{(i)} \tilde{\mathbf{Q}}(z) \right) = \frac{1}{n} \text{tr} \left( \tilde{\mathbf{D}}^{(i)} \tilde{\mathbf{R}}(z) \right) + \varepsilon^{(i)}$.

It results that $b^{(i)}(z) = \frac{-1}{z \left( 1 + \frac{1}{n} \text{tr} \left( \tilde{\mathbf{D}}^{(i)} \tilde{\mathbf{R}}(z) \right) \right)} + \varepsilon^{(i)}$ with

\[
    \mathbb{E} \left| \varepsilon^{(i)} \right|^{2+\varepsilon/2} < \text{Cst} \times n^{-(1+\varepsilon/4)}.
\]
**General case: main steps of the proof**

**Step 3: \( \frac{1}{n} \text{tr} \mathbf{R} \) is close to \( \frac{1}{n} \text{tr} \mathbf{T} \)**

Similarly, \( \tilde{b}^{(j)}(z) = \frac{-1}{z \left(1 + \frac{1}{n} \text{tr} \left( \mathbf{D}^{(j)} \mathbf{R}(z) \right) \right)} + \tilde{c}^{(j)} \).

Recall that

\[
\mathbf{R}(z) = \left( \mathbf{B}^{-1}(z) - z \mathbf{A} \tilde{\mathbf{B}}(z) \mathbf{A}^H \right)^{-1} \quad \text{and} \quad \tilde{\mathbf{R}}(z) = \left( \tilde{\mathbf{B}}^{-1}(z) - z \mathbf{A}^H \mathbf{B}(z) \mathbf{A} \right)^{-1}.
\]

So, up to the \( \epsilon^{(i)} \) and \( \tilde{\epsilon}^{(j)} \), matrices \( \mathbf{B} \) and \( \tilde{\mathbf{B}} \) satisfy the same system as \( \Psi \) and \( \tilde{\Psi} \).

With this idea, \( \left( \mathbf{R}, \tilde{\mathbf{R}} \right) \) can be approached by \( \left( \mathbf{T}, \tilde{\mathbf{T}} \right) \) for \( z \) carefully chosen (in the region \( \mathcal{D} \)).
General case: main steps of the proof

**Putting pieces together**

**Step 1:** $T_n(z)$ and $\tilde{T}_n(z)$ exist and are unique as solutions of a system of equations.

**Step 2:** $\mathbb{E}\left|\frac{1}{n}\text{tr} ((Q_n(z) - R_n(z)))\right|^{2+\varepsilon/2} < \text{Cst} \times n^{-(1+\varepsilon/4)}$ by taking $U_n = I_N$.

**Step 3:** $\mathbb{E}\left|\frac{1}{n}\text{tr} (R_n(z) - T_n(z))\right|^{2+\varepsilon/2} < \text{Cst} \times n^{-(1+\varepsilon/4)}$ in a region $D$.

**Consequence:** $\frac{1}{N}\text{tr} (Q_n(z) - T_n(z)) \xrightarrow[n \to \infty]{\text{a.s.}} 0$ almost surely on $\mathbb{C} - \mathbb{R}_+$ by Borel-Cantelli’s lemma and by analytic continuation.
Towards a Central Limit Theorem

Let \( I_n (\varsigma^2) = \frac{1}{N} \log \det \left( I_N + \frac{1}{\varsigma^2} H_n H_n^H \right) \) so that \( C_n (\varsigma^2) = \mathbb{E} I_n (\varsigma^2) \).

- CLT over \( I_n \) as \( n \to \infty \) and \( N/n \to c > 0 \), at least in some particular cases such as \( A_n = 0 \) in the model \( H_n = Y_n + A_n \). We shall assume this case.
- By means of the "Gaussian approximation", we have an idea of the "outage probability" \( \mathbb{P}(I_n < \text{a given threshold } R) \).
  In certain situations, this gives the probability that the channel cannot provide data rate \( R \).

- Two terms:
  - CLT over \( \chi_{1,n} = N (I_n - C_n) \) and variance derivation.
  - Bias \( \chi_{2,n} = N (C_n - \overline{C}_n) \) between mutual information \( NC_n \) and the deterministic approximation \( N\overline{C}_n \).
Towards a Central Limit Theorem

The term $\chi_{1,n}$

Approach: CLT for martingales as in Girko and in Bai and Silverstein’04.

Notations:

$Y^{(j)}$ is the $N \times (n - 1)$ matrix that remains after extracting column $j$ denoted as $y^{(j)}$ from $Y$.

$Q^{(j)}(z)$ is the resolvent $Q^{(j)}(z) = \left( Y^{(j)} Y^{(j)\text{H}} - zI_n \right)^{-1}$.

$\mathcal{F}^{(j)}$ is the $\sigma$-field $\mathcal{F}^{(j)} = \sigma(y^{(j)}, \ldots, y^{(n)})$.

$E^{(j)}$ is the conditional expectation $E \left[ . \mid \mathcal{F}^{(j)} \right]$.

$I^{(j)}_n (s^2) = \frac{1}{N} \log \det \left( I_N + \frac{1}{s^2} Y^{(j)}_n Y^{(j)\text{H}}_n \right)$. 
Towards a Central Limit Theorem

**The term $\chi_{1,n}$**

- We have

$$N \left( I_n - \mathbb{E}I_n \right) = N \sum_{j=1}^{n} \left( \mathbb{E}(j) - \mathbb{E}(j+1) \right) I_n$$

$$= N \sum_{j=1}^{n} \left( \mathbb{E}(j) - \mathbb{E}(j+1) \right) \left( I_n - I_n^{(j)} \right) \quad \text{due to } \mathbb{E}(j) I_n^{(j)} = \mathbb{E}(j+1) I_n^{(j)}.$$

- By standard matrix manipulations, we have

$$N \left( I_n - I_n^{(j)} \right) = \log (\varsigma^2) + \log \left( 1 + y^{(j)^H} Q^{(j)} (-\varsigma^2) y^{(j)} \right)$$

- Sequence $\gamma^{(j)} = \left( \mathbb{E}(j) - \mathbb{E}(j+1) \right) \log \left( 1 + y^{(j)^H} Q^{(j)} (-\varsigma^2) y^{(j)} \right)$ is a martingale difference sequence with respect to the increasing filtration $\mathcal{F}^{(n)}, \ldots, \mathcal{F}^{(1)}$. Apply the CLT for martingales to $\sum_{j=1}^{n} \gamma^{(j)}$.

- Variance of $\chi_{1,n}$ is $O(1)$.
Towards a Central Limit Theorem

**The Bias Term $\chi_{2,n}$**

$$\chi_{2,n} = N \left( C_n - \overline{C}_n \right)$$

We get back to $ST$ by taking the derivative with respect to $\zeta^2$:

$$\frac{d\chi_{2,n}}{d\zeta^2} = -\text{tr} \left( \mathbb{E} Q_n \left( -\zeta^2 \right) - T_n \left( -\zeta^2 \right) \right)$$

We obtain

$$\frac{d\chi_{2,n}}{d\zeta^2} \xrightarrow{n \to \infty} \left( \mathbb{E} |X_{11}|^4 - 2 \right) \times \text{Cst}$$

$\chi_{2,n} \to 0$ in the case elements of $Y_n$ are Gaussian.