Infinitely lived agents

- We will consider a model with H agents who live forever.
- There is no production, only one commodity. Agents have endowments in the commodity which are time-invariant functions of the shock
- Agents maximize time-separable expected utility
- For now, we assume that $H = 1$! This makes the analysis much easier but already motivates some of the necessary assumptions

Assets

- Lucas trees: Infinitely lived assets that pay a dividend at all nodes $\sigma_t \in \sigma$. The dividend is a function of the shock alone. Each agent faces the following budget constraint

$$c^h(s^t) = \bar{e}^h(s_t) + \sum_{j \in \mathcal{J}} \theta_j^h(s^{t-1})(q(s^t) + d(s_t)) - \theta^h(s^t)q(s^t).$$

- One period assets: Contracts written contingent on next period's shock. Easiest example is a bond that pays one unit next period independently of the shock.

  Often, people consider an economy with one tree and one bond. Asset one is the tree, asset 2 the bond, budget constraint becomes

$$c^h(s^t) = \bar{e}^h(s_t) + \theta_1^h(s^{t-1})(q_1(s^t) + d(s_t)) + \theta_2^h(s^{t-1}) - \theta^h(s^t)q(s^t).$$

- For now, only Lucas trees !

The simple Lucas model

- One infinitely lived agents and a single commodity in a pure exchange economy.
- Exogenous shocks s_t follow Markov chain with finite support $\mathcal{S}$ and transition π
- Endowments are $e(\sigma) > 0$ with $e(s^t) = \mathbf{e}^h(s_t)$
- h has von Neumann-Morgenstern utility over infinite consumption streams

$$U^h(c) = E_0 \sum_{t=0}^{\infty} \beta^t u_h(c_t)$$

The expectation is taken under the probabilities derived from the Markov chain !

The simple Lucas model (cont.)

- J infinitely lived assets in unit net supply. Each j pays shock dependent dividends $d_j(s)$, we denote its price at node s^t by $q_j(s^t)$.
- Agent trades these assets, taking prices as given. Interpretation: Continuum of identical agents
- Portfolios are $\theta \in \mathbb{R}^J$.

Utility maximization

At each node the faces the budget constraint

$$c(s^t) - e(s^t) \leq (q(s^t) + d(s_t))\theta(s^{t-1}) - q(s^t)\theta(s^t).$$

We collect the set of all non-negative consumption processes and portfolio processes which satisfy these constraints at all nodes in a budget set $\mathcal{B}(q)$.

The agent chooses consumption and portfolios at all nodes, (c, θ) to solve

$$U(c) \text{ subject to } (c, \theta) \in \mathcal{B}(q) \tag{1}$$

Mr. Ponzi

Given any candidate solution to (1) which yields finite utility, one can always improve by consuming one unit more today, borrowing that unit and rolling over the debt until infinity. Of course, this might involve infinite debt at infinity. So in order to rule out Ponzi-schemes, we need to **impose additional restrictions** on the agent's choices. There are several possible ways to do this, see Levine and Zame (1996) or Magill and Quinzii (1996) for a discussion. For now I want to impose as an additional constraint, the so-called implicit debt constraint

$$\inf_{\sigma \in \Sigma} q(\sigma)\theta(\sigma) > -\infty \tag{2}$$

This constraint implies that along all paths of the event tree, the agent can never get so much into debt that he cannot pay it off in finite time (maintaining non-negative consumption).

Euler Equation

The following condition, called Euler-equation in macroeconomics, is a necessary condition for an interior (finite) optimum

$$-q(s^t)u'(c(s^t)) + \sum_s \pi(s|s_t)(q(s^t) + d(s_t))u'(c(s^{t+1})) = 0 \tag{3}$$

Observe that this can only have a solution if there is no arbitrage.

Condition necessary and sufficient ?

- Given sequence of prices $q(s^t)$, need to show that first order conditions are necessary and sufficient for optimality
- Necessity of first order conditions is standard, e.g.:
  suppose (c^*, θ^*) is a solution to (1), (2) with strictly positive consumption. For any portfolio θ, because c^* is positive, there must exist an $\alpha > 0$ such that $c^*(s^t) - \alpha q \cdot \theta > 0$ and $c^*(s^{t+1}) + \alpha(q(s^{t+1}) + d(s_{t+1}) \cdot \theta > 0$ for all $s^{t+1} \succ s^t$. Define

$$g(\alpha) = u(c^*(s^t) - \alpha q \cdot \theta) + \beta \sum_s \pi(s|s_t) u(c^*(s^{t+1}) + \alpha(q(s^{t+1}) + d(s_{t+1})) \cdot \theta)$$

  By optimality of c^*, $g'(\alpha) = 0$ at $\alpha = 0$.

Sufficiency

- For sufficiency we need to make additional assumptions. Assume that Bernoulli utility $u(.)$ is bounded above. Suppose asset prices are bounded, i.e. $\sup_\sigma q(\sigma) < \infty$. A process $(\bar{c}(\sigma), \bar{\theta}(\sigma))$, with $\sup_\sigma q \cdot \bar{\theta}(\sigma) < \infty$ and with $\sup_\sigma u'(c(\sigma)) < \infty$ solves an agent h's optimization problem (1),(2) if for all s^t the Euler equation holds, i.e.

$$-q(s^t)u'(\bar{c}(s^t)) + \sum_s \pi(s|s_t)(q(s^{t+1}) + d(s_{t+1}))u'(\bar{c}(s^{t+1})) = 0$$

- Let $(c(\sigma), \theta(\sigma))$ be an arbitrary budget feasible process, satisfying (2). Note that concavity of u implies that for all σ,

$$u(\bar{c}(\sigma)) - u(c(\sigma)) \geq u'(\bar{c}(\sigma))(\bar{c}(\sigma) - c(\sigma)).$$

Sufficiency (cont)

Since consumption in 0 only differs by the value of the new portfolio, we have that $u(\bar{c}_0) \geq u(c_0) + u'(\bar{c}_0)q(s_0)(\theta(s_0) - \bar{\theta}(s_0))$.

For any T, need to show that

$$E\sum_{t=0}^{\infty} \beta^t u_h(\bar{c}(s^t)) \geq \; E\sum_{t=0}^{T} \beta^t u_h(c(s^t)) + E\sum_{t=T+1}^{\infty} \beta^t u_h(\bar{c}(s^t)) +$$
$$\beta^T E\left(u'(\bar{c}(s^t))q(s^t)(\theta(s^t) - \bar{\theta}(s^t))\right)$$

Since u is bounded above the second term on the right hand side will converge to zero as $T \to \infty$. The third term will converge to zero (or something positive), because consumption is bounded below and asset prices are bounded at equilibria we look at. $\square$

Equilibrium

- competitive equilibrium in this economy consists of processes for asset prices and individual choices $(q(\sigma), c(\sigma), \theta(\sigma))_{\sigma \in \Sigma}$ such that the agent solves (1) subject to (2) and markets clear, i.e. at all $\sigma \in \Sigma$,

$$\theta(\sigma) = \theta(s^{-1}), \quad c(\sigma) = e(\sigma) + \theta(s^{-1}) \cdot d(\sigma)$$

- Existence follows if we can find prices that satisfy the first order conditions, since these are sufficient

Equilibrium Prices

- Introduce notation:

$$x \circ y = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_S y_S \end{pmatrix} \in \mathbb{R}^S$$

  We denote the identity matrix of size S by I_S.

- The solution for asset prices is

$$q_j \circ p = [I_S - \beta\Pi]^{-1} \beta\Pi (p \circ d_j).$$

Is it a good model ?

- Historical equity premium in the US about 6-7 percent p.a.
- No way to get this in the model, given observed, smooth consumption
- The implied stochastic discount factor no good...

Is it a good model (cont) ?

- Several agents might help ?
- Expectations and beliefs ?
- Trading constraints