General Equilibrium on an event tree

- Given a tree Σ with M nodes $\sigma \in \Sigma$
- Suppose there is one (perishable) commodity per node
- There are H agents with individual endowments $e^h \in \mathbb{R}_+^M$ and utility

$$u^h : \mathbb{R}_+^M \longrightarrow \mathbb{R}$$

Debreu's solution

- Debreu in Chapter 7 of his 'Theory of Value':
  'A contract for the transfer of commodities now specifies, in addition to its physical properties, its location and its date, an event on the occurrence of which the transfer is conditional. This new definition of a commodity allows one to obtain a theory of uncertainty free from any probability concepts and formally identical with the theory of certainty'.
- With this redefinition of a commodity, our results on existence, efficiency uniqueness etc. must hold true for an economy with time and uncertainty.

Debreu's solution

- Commodity space becomes $\mathbb{R}_+^M$
- Agents face (Debreu) prices ρ and the budget constraint is

$$\rho \cdot x \leq \rho \cdot e^h.$$

- We do not have to prove any more theorems !
- But where can you buy a future contract for an orange in Singapore if it rains on Dec 23rd 10 am ?

General equilibrium and finance

- Instead of assuming spot markets, assume that there are J assets and agents trade in these assets to maximize utility
- A Stochastic Finance Economy is a collection of agents, a stochastic structure, preferences, endowments and asset payoffs
- Assume without loss of generality that assets are in zero net supply

GEI equilibrium

A GEI equilibrium for a 'Stochastic Finance Economy' is defined as a collection of consumption processes $(\bar{c}^h)_{h=1,\ldots,H}$, portfolio holdings $(\bar{\theta}^h)_{h=1,\ldots,H}$ and an asset price process $\bar{q}$ that satisfy the following conditions:

(1) For all agents $h = 1, \ldots, H$:

$$(\bar{c}^h, \bar{\theta}^h) \in \arg\max_{c,\theta} \; u^h(c) \; s.t. \; (c, \theta) \in \mathcal{B}^h(q)$$

(2) $\sum_{h \in \mathcal{H}} \bar{\theta}^h(\sigma) = 0$ at all nodes $\sigma \in \Sigma$.

Dynamic completeness

- What does complete markets mean ?
- An economy is dynamically complete if the markets can be completed through trading over time, i.e. for each consumption process $(c(\sigma)_{\sigma \in \Sigma}) \in \mathbb{R}^M$ there exists a trading strategy θ such that $c(\sigma) = D^\theta(\sigma)$ for all $\sigma \neq \sigma_0$.
- If this is true, the agent faces a single budget constraint because we can price uniquely any consumption $c \in \mathbb{R}_+^M$
- But if some assets pay in other assets, this is endogenous !

Condition for dynamic completeness

Markets are dynamically complete, if and only if for all non-terminal nodes $\sigma \in \Sigma$ there are S assets with linearly independent payoffs (in numéraire commodity terms) at direct successor nodes. I.e. if a given node σ has S direct successors which we denote by $(\sigma 1), (\sigma 2), \ldots, (\sigma S)$, it must hold that

$$\text{rank} \begin{pmatrix} (q(\sigma 1) + d(\sigma 1) \\ \vdots \\ (q(\sigma S) + d(\sigma S)) \end{pmatrix} = S$$

Condition for dynamic completeness (cont).

- Markets are complete if and only if for each node (except the root node) σ there exists a trading strategy θ^σ such that $D^{\theta^\sigma}(\sigma) = 1$ and $D^{\theta^\sigma}(\sigma') = 0$ for all $\sigma' \neq \sigma$, $\sigma' \neq \sigma_0$
- Construct the strategy as follows

$$\theta^\sigma(\sigma') = 0 \text{ unless } \sigma \succ \sigma'$$

Suppose a given σ' has S direct successors. If $(\sigma' s)$ is a predecessor of σ or if it is σ itself (i.e. if $\sigma' = \sigma_-$) then let

$$\theta^\sigma(\sigma') = \begin{pmatrix} (q(\sigma' 1) + d(\sigma' 1)) \\ \vdots \\ (q(\sigma' S) + d(\sigma' S)) \end{pmatrix}^{-1} \iota_s$$

GEI equilibria and Walras

If in a given GEI equilibrium markets are dynamically complete then there exists a Walrasian equilibrium for this economy with the same equilibrium allocation. The Walrasian equilibrium prices ρ are a state price vector. $\rho(\sigma)$ is the period 0 price of trading strategy θ^σ.

Unfortunately, there is no easy way to get the converse...

Possible non-existence - Intuition I

- The equilibrium allocation for complete markets will generally by (very) different to an allocation where some assets are missing
- We can construct asset payoffs such that the unique supporting state prices for the Walrasian equilibrium allocation lead to incomplete asset markets
- When markets are incomplete, the Walrasian equilibrium allocation will generally not be a GEI equilibrium allocation, state prices will be different

Possible non-existence - Intuition II

- Standard existence proofs do not work because aggregate excess demand function is not continuous at prices where payoff matrices drop in rank
- If payoffs of two assets are almost collinear (e.g. $(1,1)$ and $(1+\epsilon,1)$), there equilibrium prices are almost the same and agents equilibrium portfolios can get very large. For example, to support payoff of a portfolio $(1,0)$ one needs to hold $1/\epsilon$ of asset 2 and $-1/\epsilon$ of asset 1.
- Portfolio demand explodes as asset payoffs become collinear

How to fix it

- Generic existence:
  In the example, if we perturb dividends by arbitrarily small ϵ, GEI equilibrium will exist.
  Equilibrium exists generically in individual endowments
- Radner: Impose short-sale constraints on assets which pay in other assets

Assumptions for existence

- Assume that utility is strictly increasing and strictly concave and that endowments are strictly positive
- Also assume that there exists some $K > 0$ such that whenever at σ an asset j pays in an other asset next period, agents face the exogenous constraint

$$\theta^h(\sigma) \geq -K$$

- Assume that asset payoffs are non-negative, i.e. $d(\sigma) \geq 0$ for all $\sigma \in \Sigma$.

Outline of existence proof

- Agents' best responses are uhc, convex valued and non-empty ?
  Easy to get with constraints on trades and positive good prices, $p \geq \epsilon$.
- Price player chooses prices of assets and commodities such that at a fixed point we are at a GEI equilibrium ?
  What is the right set of prices to choose from ?

Outline of existence proof (cont.)

- At each node σ, normalize

$$p(\sigma) + \sum_{j \in \mathcal{J}} q_j(\sigma) = 1$$

- Introduce M different price players (one for each node).
- Each σ solves

$$\max_{(p,q) \in \Delta^J} p \sum_{h \in \mathcal{H}} (c^h(\sigma) - e^h(\sigma)) + q \cdot \sum_{h \in \mathcal{H}} \theta^h(\sigma) \quad p \geq \epsilon$$

Outline of existence proof (cont.)

Each household h solves

$$\max_{c,\theta} u^h(c) \text{ s.t. } c \geq 0$$
$$-\kappa \leq \theta_j(\sigma) \leq \kappa \quad j \in \mathcal{J}, \quad \text{non-terminal } \sigma \in \Sigma$$
$$p(\sigma_0)(c(\sigma_0) - e^h(\sigma_0)) \leq -q(\sigma_0)\theta(\sigma_0)$$
$$p(\sigma)(c(\sigma) - e^h(\sigma)) \leq -q(\sigma)\theta(\sigma) + (q(\sigma) + d(\sigma))\theta(\sigma_-), \sigma \notin \mathcal{N}_T \cup \mathcal{N}_0$$
$$p(\sigma)(c(\sigma) - e^h(\sigma)) \leq (q(\sigma) + d(\sigma))\theta(\sigma_-), \text{ terminal } \sigma \in \Sigma$$

Outline of last part of existence proof

Fix some $\kappa > 0$. For each $\epsilon > 0$, there exist $(p(\sigma), q(\sigma), (c^h(\sigma), \theta^h(\sigma))_{h \in \mathcal{H}})_{\sigma \in \Sigma}$ such that

- For all σ, $p(\sigma), q(\sigma)$ solves

$$\max_{(p,q) \in \Delta^J} p \sum_{h \in \mathcal{H}} (c^h(\sigma) - e^h(\sigma)) + q \cdot \sum_{h \in \mathcal{H}} \theta^h(\sigma) \quad p \geq \epsilon$$

and...

$(c^h(\sigma), \theta^h(\sigma))_{\sigma \in \Sigma}$ solves

$$\max_{c,\theta} u^h(c) \text{ s.t. } c \geq 0$$
$$-\kappa \leq \theta_j(\sigma) \leq \kappa \quad j \in \mathcal{J}, \quad \text{non-terminal } \sigma \in \Sigma$$
$$p(\sigma_0)(c(\sigma_0) - e^h(\sigma_0)) \leq -q(\sigma_0)\theta(\sigma_0)$$
$$p(\sigma)(c(\sigma) - e^h(\sigma)) \leq -q(\sigma)\theta(\sigma) + (q(\sigma) + d(\sigma))\theta(\sigma_-), \sigma \notin \mathcal{N}_T \cup \mathcal{N}_0$$
$$p(\sigma)(c(\sigma) - e^h(\sigma)) \leq (q(\sigma) + d(\sigma))\theta(\sigma_-), \text{ terminal } \sigma \in \Sigma$$

How does equilibrium follow ? (cont.)

- Consider sequence of fixed points as $\epsilon \to 0$
- For sufficiently small $\epsilon > 0$, the constraint $p(\sigma) \geq \epsilon$ cannot be binding for any σ.
- By agents' budget constraints we must have

$$p(\sigma_0) \sum_{h \in \mathcal{H}} (c^h(\sigma_0) - e^h(\sigma_0)) + q(\sigma_0) \cdot \sum_{h \in \mathcal{H}} \theta^h(\sigma_0) = 0$$

and therefore

$$\sum_{h \in \mathcal{H}} (c^h(\sigma_0) - e^h(\sigma_0)) = 0$$

$$\sum_{h \in \mathcal{H}} \theta^h(\sigma_0) = 0$$

How does equilibrium follow ? (cont.)

By induction, if for all $\sigma' \in \mathcal{N}_{t-1}$

$$\sum_{h\in\mathcal{H}} \theta^h(\sigma') = 0$$

the budget constraints imply for all $\sigma \in \mathcal{N}_t$,

$$p(\sigma) \sum_{h\in\mathcal{H}} (c^h(\sigma) - e^h(\sigma)) + q(\sigma) \cdot \sum_{h\in\mathcal{H}} \theta^h(\sigma) = 0$$

and therefore

$$\sum_{h\in\mathcal{H}} c^h(\sigma) - e^h(\sigma) = 0$$

$$\sum_{h\in\mathcal{H}} \theta^h(\sigma) = 0$$

Representative agent

- In many models in macro there is only one agent (e.g. stochastic growth model, Lucas asset pricing model)
- This makes solving the model MUCH easier
- But general equilibrium theorists hate it...

A formal representative agent

Given an economy $((u^h, e^h)_{h=1}^H, A)$ where all agents maximize time-separable expected utility of the form

$$U^h(c) = E \sum_{t=0}^{T} \beta^t v^h(c(s^t))$$

(the expectation is taken under some common probability measure). If the GEI equilibrium is Pareto efficient there exists a utility function $v^R : \mathbb{R}_+ \to \mathbb{R}$ such that the economy with only one agent, $((v^R, \sum_{h \in \mathcal{H}} e^h), A)$ has an GEI equilibrium with identical equilibrium asset prices.

Why do we care about the price agent ?

- While it is difficult to derive the representative agent's utility from utilities of actual agents, it seems unlikely that his risk aversion is much higher than some average of individual risk aversions
- Asset prices which cannot be rationalized by a representative agent cannot be rationalized by a GEI economy (as long as markets are complete)
- Does not say anything about comparative statics...