NON-LINEAR EDDY-VISCOSITY MODELS – 1

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Linear eddy-viscosity models are known to fail in a number of flow situations.

Nevertheless, they are still widely used in industrial calculations.

As it becomes feasible to tackle more complex flows, and remove numerical errors, the need for more accurate models becomes apparent.

Although stress transport models (to be considered in later lectures) offer a route to account for much more of the flow physics, they do require greater computational resources.

Non-linear EVM’s have received significant interest, because they offer the potential of returning better predictions than linear EVM’s, but with only a moderate increase in required computing resources.
In this first lecture we look at

- Linear EVM’s and some of their weaknesses.
- How non-linear effects can be introduced to overcome some of the linear models’ weaknesses.

In a second lecture alternatives and more advanced non-linear EVM’s will be considered.

- Explicit algebraic Reynolds stress models (EARSMS’s).
- Near-wall modelling.
- Three equation NLEVM’s.

The modelling approaches will be illustrated by developments from UMIST and other groups, and some example applications will also be given.
Linear EVM’s

These employ a linear relation between Reynolds stresses and mean strains:

\[
\overline{u_i u_j} = -\nu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + 2k/3\delta_{ij}
\]

The turbulent viscosity is typically written as \( \nu_t = c_\mu k^2/\varepsilon \) for a constant \( c_\mu \).

In a \( k-\varepsilon \) model, transport equations are solved for \( k \) and \( \varepsilon \):

\[
\frac{Dk}{Dt} = P_k - \varepsilon + \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_t/\sigma_k \right) \frac{\partial k}{\partial x_j} \right]
\]

\[
\frac{D\varepsilon}{Dt} = c_{\varepsilon 1} \frac{\varepsilon P_k}{k} - c_{\varepsilon 2} \frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_t/\sigma_\varepsilon \right) \frac{\partial \varepsilon}{\partial x_j} \right]
\]

with the turbulence energy generation rate being \( P_k = -\overline{u_i u_j} \partial U_i/\partial x_j \).
Linear EVM Failures: Impingement – 1

- Linear EVM’s overpredict turbulence energy levels in stagnation regions.
- For example, in an impinging jet.

\[ y \]
\[ x \]
\[ z \]

\[ H/D=2 \quad Re=23000 \]

\[ \text{\textit{\textendash \textendash \textendash : Launder-Sharma } k-\varepsilon} \]

\[ \text{\textit{\textendash \textendash : \textit{\textendash \textendash \textendash}} \]

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The failure is linked to misrepresentation of the normal stresses. For a plane jet (with non-zero $\partial U/\partial x = -\partial V/\partial y$) we have

$$P_k = -(\bar{u}^2 - \bar{v}^2) \frac{\partial U}{\partial x}$$

Whereas in reality $\bar{u}^2 - \bar{v}^2$ is likely to be small near the wall (and can even be negative), the linear EVM formulation results in

$$\bar{u}^2 = 2k/3 - 2\nu_t (\partial U/\partial x) \quad \bar{v}^2 = 2k/3 + 2\nu_t (\partial U/\partial x)$$

giving

$$P_k = 4\nu_t \left( \frac{\partial U}{\partial x} \right)^2$$

which leads to excessive generation of $k$. 

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Linear EVM Failures: Curvature – 1

- Even weak curvature can have a strong effect on turbulence energy levels.

- For example, fully-developed flow in a curved channel.

- A linear EVM does not reproduce the asymmetry in the velocity profile.
Linear EVM Failures: Curvature – 2

- The reason for this failure lies in the stress-strain relation, which gives
  \[ \overline{uv} = -\nu_t \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \]

- In a weakly curved flow, \( \partial V/\partial x << \partial U/\partial y \), so the shear stress is insensitive to the curvature.

- We shall see later (when considering stress transport models) how the secondary strain should affect the shear stress.
Even in a shear flow, the linear EVM has a number of weaknesses. The normal stresses are (incorrectly) predicted as isotropic,

\[ u^2 = v^2 = w^2 = \frac{2}{3} k \]

In such flows one can characterise the rate of strain in a non-dimensional parameter, \( S \), defined as

\[ S = \frac{k}{\varepsilon} \frac{dU}{dy} \]

The linear EVM then gives \( \overline{uv}/k = -c_\mu S \).

So for a constant \( c_\mu \), the ratio \( |\overline{uv}/k| \) increases linearly with strain rate \( S \).

This is not what is found in experiments and DNS.
The variation of $\frac{\overline{uv}}{k}$ with strain rate suggests that $c_\mu$ should not be constant, but should decrease when there are high strain rates.

Near a wall (where $S$ does become large) low-Re models take $\nu_t = c_\mu f_\mu k^2/\varepsilon$, with $f_\mu$ typically a function of the turbulent Reynolds number. The effect of this is to reduce the turbulent viscosity as the wall is approached.

However, DNS studies show that near-wall channel flow (at around $y^+ = 100$) has a very similar structure to homogeneous shear flow at a similar straining rates. This suggests at least some of the “near-wall” damping should actually be due to high strain rates, instead of viscosity or wall-proximity.
One (relatively simple) modelling solution is to make $c_\mu$ a function of the strain parameter $S$. Suga (1995) proposed

$$c_\mu = \frac{0.3}{1 + 0.35\eta^{3/2}} \left[ 1 - \exp \left( \frac{-0.36}{\exp(-0.75\eta)} \right) \right]$$

where $\eta = \max(S, \Omega)$.

$$S = \frac{k}{\varepsilon} \left[ \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)^2 \right]^{1/2}$$

$$\Omega = \frac{k}{\varepsilon} \left[ \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right)^2 \right]^{1/2}$$
Functional Form of $c_\mu - 3$

- This form gives good agreement for the shear stress with free shear flow data.

- Suga also found that less viscous damping was needed with this model for near-wall regions. The corresponding $f_\mu$ function he proposed was:

$$f_\mu = 1 - \exp \left( -\left( \frac{R_t}{90} \right)^{1/2} - \left( \frac{R_t}{400} \right)^2 \right)$$
Non-Linear Stress-Strain Relations

Although making $c_\mu$ a function of the strain rate does give a better representation of $\overline{uv}$ in non-equilibrium shear flows, it cannot solve the problem of the linear EVM returning isotropic normal stresses.

In a non-linear EVM, additional terms are introduced into the stress-strain relation, making the Reynolds stresses a more general function of mean velocities and vorticities.

This reflects the fact that turbulence is a highly non-linear phenomena.

The idea of a non-linear stress-strain relation goes back at least to the proposal of Pope (1975). However, such models have recently become popular, being seen as a possible low-cost route to getting better predictions than with a linear EVM.
A Quadratic Stress-Strain Relation

One approach to constructing a non-linear model is to initially include all tensor forms satisfying the required symmetry and contraction properties, and then tune model coefficients to a range of flows.

The resulting model should satisfy $u_i u_j = u_j u_i$ and $u_i u_i = 2k$.

Including all possible terms \textit{quadratic} in the mean velocity gradients, we can write

\[ a_{ij} = \frac{u_i u_j}{k} - 2/3 \delta_{ij} = -\nu_t / k S_{ij} + c_1 \frac{\nu_t}{\varepsilon} (S_{ik} S_{jk} - 1/3 \delta_{ij} S_{mk} S_{mk}) \]

\[ + c_2 \frac{\nu_t}{\varepsilon} (\Omega_{ik} S_{kj} + \Omega_{jk} S_{ki}) \]

\[ + c_3 \frac{\nu_t}{\varepsilon} (\Omega_{ik} \Omega_{jk} - 1/3 \Omega_{lk} \Omega_{lk} \delta_{ij}) \]

where $S_{ij} = \partial U_i / \partial x_j + \partial U_j / \partial x_i$ and $\Omega_{ij} = \partial U_i / \partial x_j - \partial U_j / \partial x_i$.

The coefficients $c_1, c_2, c_3$ can then be tuned by reference to suitable flows.
**Simple Shear Flow**

In a simple shear flow, with velocity gradient $dU/dy$, this quadratic form gives

\[
\bar{u}^2 = \frac{2}{3}k + c_\mu k S^2 \left( \frac{1}{3}c_1 + \frac{1}{3}c_3 + 2c_2 \right)
\]
\[
\bar{v}^2 = \frac{2}{3}k + c_\mu k S^2 \left( \frac{1}{3}c_1 + \frac{1}{3}c_3 - 2c_2 \right)
\]
\[
\bar{w}^2 = \frac{2}{3}k + c_\mu k S^2 \left( -2/3c_1 - 2/3c_3 \right)
\]
\[
\bar{uv} = -c_\mu k S
\]

A suitable choice of coefficients can give, at least qualitatively, the correct anisotropy ($\bar{u}^2 > \bar{w}^2 > \bar{v}^2$).

Note that $\bar{uv}$ is not affected by the quadratic terms.
Quadratic NLEVM’s

A number of quadratic models have been proposed:

<table>
<thead>
<tr>
<th>model</th>
<th>$c_\mu$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>Additional terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speziale (1987)</td>
<td>0.09</td>
<td>-0.15</td>
<td>0</td>
<td>0</td>
<td>$-0.3 \nu/\epsilon (\hat{S}<em>{ij} - 1/3 \hat{S}</em>{kk} \hat{S}_{jj})$</td>
</tr>
<tr>
<td>Nisizima-Yoshizawa (1987)</td>
<td>0.09</td>
<td>-0.76</td>
<td>0.18</td>
<td>1.04</td>
<td></td>
</tr>
<tr>
<td>Rubinstein-Barton (1990)</td>
<td>0.0845</td>
<td>0.68</td>
<td>0.14</td>
<td>-0.56</td>
<td></td>
</tr>
<tr>
<td>Myong-Kasagi (1990)</td>
<td>0.09</td>
<td>0.28</td>
<td>0.24</td>
<td>0.05</td>
<td>$W_j$</td>
</tr>
<tr>
<td>Shih-Zhu-Lumley (1993)</td>
<td>$\frac{2/3}{1.25 + S + 0.9\Omega}$</td>
<td>$\frac{0.75/\epsilon}{1000 + S^2}$</td>
<td>$\frac{3.8/\epsilon}{1000 + S^2}$</td>
<td>$\frac{4.8/\epsilon}{1000 + S^2}$</td>
<td></td>
</tr>
</tbody>
</table>

- There is little agreement between them on coefficient values, which gives little confidence in their applicability over a wide range of flows.
- Even with these quadratic terms, streamline curvature and swirl effects cannot be accounted for.
One route forward is to include higher order terms in the stress-strain relation. If all possible cubic terms are included, one arrives at an expression

\[
\overline{u_i u_j} = \frac{2}{3} k \delta_{ij} - \nu_t S_{ij} + c_1 \nu_t \frac{k}{\varepsilon} \left( S_{ik} S_{jk} - \frac{1}{3} S_{kl} S_{kl} \delta_{ij} \right)
+ c_2 \nu_t \frac{k}{\varepsilon} \left( \Omega_{ik} S_{kj} + \Omega_{jk} S_{ki} \right)
+ c_3 \nu_t \frac{k}{\varepsilon} \left( \Omega_{ik} \Omega_{jk} - \frac{1}{3} \Omega_{lk} \Omega_{lk} \delta_{ij} \right)
+ c_4 \nu_t \frac{k^2}{\varepsilon^2} \left( S_{ki} \Omega_{lj} + S_{kj} \Omega_{li} \right) S_{kl}
+ c_5 \nu_t \frac{k^2}{\varepsilon^2} \left( \Omega_{il} S_{mj} + S_{il} \Omega_{mj} - 2/3 S_{ln} \Omega_{mn} \delta_{ij} \right) \Omega_{lm}
+ c_6 \nu_t \frac{k^2}{\varepsilon^2} S_{ij} S_{kl} S_{kl} + c_7 \nu_t \frac{k^2}{\varepsilon^2} S_{ij} \Omega_{kl} \Omega_{kl}
\]
A Cubic NLEVM – 2

The addition of cubic terms results in a model which can show sensitivity to curvature and swirl.

Suga (1995) devised such a cubic model, optimizing the coefficients over a range of flows including simple shear, impinging, curved and swirling flows (see Craft et al, 1996). He arrived at the set of model coefficients:

\[
\begin{array}{cccccccc}
  c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\
  -0.1 & 0.1 & 0.26 & -10c_\mu^2 & 0 & -5c_\mu^2 & 5c_\mu^2 \\
\end{array}
\]

with \( c_\mu \) and \( f_\mu \) as given earlier.
The \( k \) and \( \varepsilon \) equations proposed by Suga are similar to those in the linear Launder-Sharma model:

\[
\frac{Dk}{Dt} = P_k - \varepsilon + \frac{\partial}{\partial x_j} \left[ (\nu + \nu_t/\sigma_k) \frac{\partial k}{\partial x_j} \right]
\]

\[
\frac{D\tilde{\varepsilon}}{Dt} = c_{\varepsilon 1} \frac{\tilde{\varepsilon} P_k}{k} - c_{\varepsilon 2} \frac{\tilde{\varepsilon}^2}{k} + \frac{\partial}{\partial x_j} \left[ (\nu + \nu_t/\sigma_{\varepsilon}) \frac{\partial \tilde{\varepsilon}}{\partial x_j} \right] + E + Y
\]

where \( \varepsilon = \tilde{\varepsilon} + \nu (\partial k^{0.5}/\partial x_j)^2 \), the near-wall source term

\[
E = 0.0022 \frac{\nu_t \tilde{S} k^2}{\tilde{\varepsilon}} \left( \frac{\partial^2 U_i}{\partial x_j \partial x_k} \right)^2 \quad \text{for } R_t < 250
\]

and \( Y \) is the Yap lengthscale correction:

\[
Y = 0.83 \frac{\tilde{\varepsilon}^2}{k} \max \left[ (l/l_e - 1)(l/l_e)^2, 0 \right] \quad \text{with } l = k^{3/2}/\varepsilon, \ l_e = 2.55y
\]
Craft et al (1999) proposed some modifications to Suga’s cubic NLEVM. In a separated shear layer, the functional form of $c_\mu$ showed too strong a dependence on the mean strain.

An alternative $c_\mu$ function was proposed:

$$c_\mu = \min \left[ 0.09, \frac{1.2}{1 + 3.5\eta + f_{RS}} \right]$$

where

$$f_{RS} = 0.235 \left[ \max(0, \eta - 3.333) \right]^2 \left[ \exp(-\tilde{R}_t/400) + \sqrt{S_I^2} \right]$$

and

$$S_I = S_{ij}S_{jk}S_{ki}/(S_{nl}S_{nl}/2)^{3/2}$$
Further NLEVM Refinements – 2

The Yap term in the $\varepsilon$ equation explicitly involves the wall distance, which can be difficult to define in complex geometries.

Craft et al replaced this with a correction based on lengthscale gradients, based on a proposal of Iacovides & Raisee (1997):

$$ Y_{dc} = c_w \frac{\bar{\varepsilon}^2}{k} \max \left[ F(F + 1)^2, 0 \right] $$

where

$$ F = \left[ \left( \frac{\partial l}{\partial x_j} \frac{\partial l}{\partial x_j} \right)^{1/2} - d l_e d y \right] / c_l $$

and $d l_e d y$ is the ‘equilibrium lengthscale gradient’, obtained from differentiating the Wolfshtein 1-equation model formulation:

$$ d l_e d y = c_l \left[ 1 - \exp(-B_\varepsilon \tilde{R}_t) \right] + B_\varepsilon c_l \tilde{R}_t \exp(-B_\varepsilon \tilde{R}_t) $$

with coefficients $c_l = 2.55$, $B_\varepsilon = 0.1069$, and $c_w = 0.83$
Separation/Impingement

Pipe Expansion

Impinging Jet (—: NLEVM)

\[ Re = 23000 \]

\[ Re = 70000 \]

large symbol, solid line: \( Re = 17000 \)
small symbol, broken line: \( Re = 40000 \)
Summary

- Linear EVM’s fail in non-equilibrium shear flows, impinging flows, and flows with streamline curvature.

- The addition of non-linear terms (in $c_\mu$ and the stress-strain relation) can bring about improvements.

- Although the NLEVM outlined does result in significant improvements in a number of flows, it is by no means perfect.

- In particular, although near-wall anisotropy of the normal stresses is qualitatively correct, quantitatively the levels of anisotropy are too low.

- In the next lecture we will consider alternative ways of deriving NLEVMS, and methods of improving their predictive performance.
References