NON-LINEAR EDDY-VISCOSITY MODELS – 2

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Introduction

In the first lecture we outlined some predictive improvements that NLEVM’s could produce.

We followed the development of a form of NLEVM which had been tuned to a range of challenging flows.

The model has generally performed fairly well, but produces too little stress anisotropy in near-wall regions.

In this lecture we address two major issues:
- An alternative method of deriving NLEVM’s.
- How to improve the near-wall anisotropy predictions.
In the first lecture we derived NLEVM’s by writing a general tensor form and subsequently tuning coefficients to a range of flows.

The idea of an EARSM is to obtain the non-linear stress-strain relation from the solution of a simplified set of stress transport equations.

As we shall see in later lectures, one can obtain transport equations for the stresses, which can be written in the form:

\[
\frac{D\bar{u}_i\bar{u}_j}{Dt} = P_{ij} + \phi_{ij} - \varepsilon_{ij} + d_{ij}
\]

where \(d_{ij}\) is the diffusion, \(P_{ij}\) the generation and \(\phi_{ij}\) and \(\varepsilon_{ij}\) are processes which are modelled in terms of Reynolds stresses, mean strains, \(\varepsilon\), etc.

One could, of course, solve this set of differential equations numerically: an approach to be looked at in later lectures. But this requires significantly greater computing resources than eddy-viscosity models.
Algebraic Stress Models

- One alternative is to approximate the advection of the stresses by, for example, modelling it in terms of the advection of $k$:

$$\frac{Du_i u_j}{Dt} - d_{ij} \approx \frac{u_i u_j}{k} \left( \frac{Dk}{Dt} - d_k \right) = \frac{u_i u_j}{k} (P_k - \varepsilon)$$

- Substituting this into the stress transport equations results in a set of algebraic equations for the stresses.

- Traditional algebraic stress models (ASM’s) then solve these algebraic equations numerically (typically using an iterative algorithm).

- Unfortunately, the system of equations can often be stiff, resulting in poor convergence rates.
Explicit algebraic Reynolds stress models (EARSM’s) attempt to obtain explicit solutions to the set of algebraic equations outlined above.

These explicit solutions can be obtained by:

- Writing a general tensor form for the stresses (as was done in the previous lecture).
- Substituting these expressions into the algebraic form of the transport equations.
- Solving for the coefficients which appear in the original tensor representation of the stresses.

The coefficients used in the subsequent NLEVM thus depend on the model coefficients that appear in the underlying stress transport model.

Some additional approximations are often introduced in solving for the coefficients, in order to avoid singularities which lead to the stiffness often associated with traditional ASM’s.
In principle, therefore, EARSM’s embody at least some of the physics built into the full stress transport model.

However, the approximation of advective terms usually implies some kind of ‘equilibrium’ conditions. Thus EARSM’s are unlikely to perform as well as the underlying stress transport model in flows where, for example, the ratios $\frac{u_i u_j}{k}$ change rapidly in space or time.

Another weakness of EARSM’s is that (to date) only underlying transport models that are linear in the Reynolds stresses have been employed. The analysis using non-linear models (of $\phi_{ij}$, for instance) becomes ‘challenging’!

As will be seen in later lectures, non-linear stress transport models can offer significant improvements over linear ones.

The restriction to linear stress transport models also means that the EARSM will not generally produce the high levels of stress anisotropy found in near-wall regions (since stress transport models often achieve these via non-linear terms).
Resolving Near-Wall Anisotropy with NLEVM’s

- Both methods of devising NLEVM’s introduced so far result in stress anisotropy being driven by mean strains.

- One of the problems in the near-wall region is that the strain by itself is not sufficient to bring about the desired stress anisotropy.

- For example, in a channel flow:

- At higher Reynolds numbers, significant $S$ variation occurs only very close to the wall.

- Some other quantity is needed, which can be used to give stress anisotropy at larger distances from the wall.
The Abe et al EARS M

- They then included additional terms in the stress-strain relation, dependent on the wall-distance, to force greater levels of stress anisotropy near a wall.

Forms of the model have been applied to a number of separated and impinging flows, with reasonable success.

However, the scheme does rely on the wall distance which, as indicated earlier, is not an ideal parameter to use in flows with complex geometries.
Use of Anisotropy Invariants

As we shall see, advanced stress transport models oftan make use of invariants of the stress anisotropy tensor to sensitize the scheme to different turbulence structures.

Some parameters employed include

\[ A_2 \equiv a_{ij}a_{ij} \quad A_3 \equiv a_{ij}a_{jk}a_{ki} \quad A \equiv 1 - (9/8)(A_2 - A_3) \]

The quantity \( A \) (often referred to as Lumley’s flatness parameter) is unity in isotropic turbulence, and is zero in 2-component turbulence (when one normal stress vanishes).

One option is simply to include \( A_2, A_3 \) and \( A \) in the NLEVM coefficients, computing their values from the predicted stresses.

Unfortunately, this does not work. Calculating the invariants from the algebraic stress-strain relation does not enable them to force the desired levels of anisotropy.
A Third Transport Equation

From their definitions, exact transport equations can be derived for the invariants $A_2$, $A_3$ and $A$.

An alternative route is to solve one of these transport equations providing a quantity which gives some information on the turbulence structure, and to then use this as a parameter in the algebraic stress-strain relation.

This is the route followed by Suga (1995) as part of his PhD studies.

Lumley’s flatness parameter $A$ might initially appear to be an ideal quantity to solve for – since it varies from 0 at the wall, to 1 in isotropic turbulence.

But the $A$ transport equation is rather complex.

Instead, Suga investigated solving an equation for $A_2$, which gives a measure of how anisotropic the turbulence is.
The $A_2$ Transport Equation

The $A_2$ equation can be obtained from

\[
\frac{DA_2}{Dt} = 2a_{ij} \frac{Da_{ij}}{Dt} = 2 \frac{a_{ij}}{k} \frac{D}{Dt}(u_iu_j) - 2 \frac{A_2}{k} \frac{Dk}{Dt}
\]

Substituting from the stress and $k$ transport equations gives

\[
\frac{DA_2}{Dt} = 2 \frac{a_{ij}}{k} P_{ij} - 2 \frac{A_2}{k} P_k + 2 \frac{a_{ij}}{k} d_{ij} - 2 \frac{A_2}{k} d_k - 2 \frac{a_{ij}}{k} \varepsilon_{ij} + 2 \frac{A_2}{k} \varepsilon + 2 \frac{a_{ij}}{k} \phi_{ij}
\]

Production terms do not need modelling.

Diffusion terms are modelled via a generalized gradient diffusion scheme – although the combined $d_{ij}$ and $d_k$ term is not strictly diffusive.

$\varepsilon_{ij}$ and $\phi_{ij}$ can be taken from second-moment closure modelling.
Suga used models for $\phi_{ij}$ and $\varepsilon_{ij}$ taken from a two-component limit stress transport model (which will be examined in a later lecture).

These models do not involve the wall distance.

Although the model for $\phi_{ij}$ appears quite complex, it is only an algebraic expression. Moreover, when contracted with $a_{ij}$ (as required in the $A_2$ transport equation) a number of the terms drop out.
Use of $A_2$ in the Stress-Strain Relation

The function $r_\eta$, defined as

$$r_\eta = 1 + \left\{ 1 - \exp(-8A_2^3) \right\} \left\{ 1 + 4\sqrt{\exp(-\tilde{R}_t/20)} \right\}$$

is then used to sensitize the stress-strain relation to $A_2$.

The non-zero NLEVM coefficients are now functions of $S$, $\Omega$ and $r_\eta$.

<table>
<thead>
<tr>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$c_6$</th>
<th>$c_7$</th>
<th>$f_q$</th>
<th>$f_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.05f_q/f_\mu$</td>
<td>$0.11f_q/f_\mu$</td>
<td>$0.21f_qS/f_\mu(S+\Omega)/2$</td>
<td>$-0.8f_c$</td>
<td>$-0.5f_c$</td>
<td>$0.5f_c$</td>
<td>$r_\eta/\sqrt{(1+0.0086\eta^2)}$</td>
<td>$r_\eta^2/(1+0.45\eta^{2.5})$</td>
</tr>
</tbody>
</table>

where $\eta = \max(\tilde{S}, \tilde{\Omega})$.

c$_\mu$ and $f_\mu$ are also sensitised to $A_2$:

$$c_\mu = \frac{0.667r_\eta}{1 + 1.8\eta} \left[ 1 - \exp\left( -0.145/\exp\left( -1.3\eta^{5/6} \right) \right) \right]$$

$$f_\mu = 1.1\sqrt{\tilde{\varepsilon}/\varepsilon} \left[ 1 - 0.8\exp(-\tilde{R}_t/30) \right] / \left[ 1 + 0.6A_2 + 0.2A_2^{3.5} \right]$$
The $\varepsilon$ Equation

- A transport equation is solved for the ‘homogeneous dissipation’, $\bar{\varepsilon}$

$$\frac{D\bar{\varepsilon}}{Dt} = -c_{\varepsilon 2} \frac{\bar{\varepsilon}^2}{k} - \frac{(\varepsilon - \bar{\varepsilon})\bar{\varepsilon}}{k} \exp(-\tilde{R}_t^2/4) + c_{\varepsilon 1} \frac{\bar{\varepsilon}}{k} P_k + P_{\varepsilon 3} + S_\varepsilon + d_\varepsilon$$

- As in the second-moment closure from which $\phi_{ij}$ and $\varepsilon_{ij}$ were taken, $c_{\varepsilon 1}$ is reduced from its standard value, and $c_{\varepsilon 2}$ is made a function of the stress invariants $A_2$ and $A$.

- $P_{\varepsilon 3}$ represents near-wall source terms based on Rodi & Mansour’s (1993) work:

$$P_{\varepsilon 3} = c_{\varepsilon 3} \nu \nu_t \left( \frac{\partial^2 U_i}{\partial x_k \partial x_j} \right)^2 + c_{\varepsilon 4} \nu \frac{\nu_t}{k} \frac{\partial k}{\partial x_j} \frac{\partial U_i}{\partial x_l} \frac{\partial^2 U_i}{\partial x_j \partial x_l}$$
Additional $S_\varepsilon$ Source Term

An additional near-wall term $S_\varepsilon$ is included in the $\tilde{\varepsilon}$ equation:

$$S_\varepsilon = c_5 \left( \frac{\varepsilon - \tilde{\varepsilon}}{\varepsilon} \right) k \left[ \left( \frac{\partial U_k}{\partial x_p} \frac{\partial l}{\partial x_k} \frac{\partial l}{\partial x_p} \right) \left( \frac{\partial U_n}{\partial x_r} \frac{\partial l}{\partial x_n} \frac{\partial l}{\partial x_r} \right) \right]^{1/2}$$

where $l = k^{3/2}/\varepsilon$.

The term is mainly influential near stagnation points, where it acts to increase the dissipation, and hence reduce turbulence levels and lengthscales.

The factor $(\varepsilon - \tilde{\varepsilon})/\varepsilon$ ensures that its influence is restricted to the near-wall sublayer.

The Yap (or similar) lengthscale correction is not included.
Plane Channel Flow

$Re_b = 14000$

DNS data: Kim et al (1987)
Bypass Transition

Computed using an elliptic solver, starting upstream of the leading edge.

--- L-S, ----: NLEVMM

- - - - L-S, ----: NLEVFM
Since the near-wall stress are well-predicted, a generalised gradient diffusion hypothesis (GGDH) heat-flux model can be employed:

\[
\overline{u_i \theta} = -c_\theta \overline{u_i u_j} \frac{k}{\tilde{e}} \frac{\partial T}{\partial x_j}
\]

The coefficient \(c_\theta\) takes the functional form

\[
c_\theta = \left(0.3 + 0.2 \sqrt{\frac{\varepsilon - \tilde{\varepsilon}}{\varepsilon}}\right) \left(1 + 0.5 A_2^{1/2} + 0.07 A_2^3\right)
\]

to improve agreement with DNS data in the near-wall region.
Impinging Jet

Data: Cooper et al (1993)

\[(v^2 + V^2)^{1/2}/U_b\]

\[\nu'/U_b\]

\[-10\overline{uv}/U_b^2\]

\[N_u/(Re^{0.7}Pr^{0.4})\]

\[H/D = 2 \ Re = 23000\]

\[H/D = 2 \ Re = 70000\]

Data: Baughn et al (1992)
Turbine Blade Cascade

Exit Mach number $M_a = 0.78$.

Computations: Suga (1996)

$Re = 1.11 \times 10^6$

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3-eq. NLEVM
Launder-Sharma

$a_i$,
$Re_{\text{ex}} = 1.670 \times 10^6$

$a_i$,
$Re_{\text{ex}} = 1.113 \times 10^6$

$a_i$,
$Re_{\text{ex}} = 0.557 \times 10^6$
Summary

- EARSM’s can be derived by making assumptions regarding the advection terms in a stress transport model, and obtaining explicit solutions to the resulting algebraic equations for the stresses.

- To improve near-wall stress anisotropy, one needs to introduce some parameter other than the strain rate.

- The EARSM of Abe et al employs wall distance, and achieves good near-wall stress anisotropy.

- The 3-equation NLEVM of Suga solves an additional transport equation and employs this to force greater near-wall stress anisotropy.

- This latter model does not include wall distance, and has given reasonably good predictions in a range of flows.
References


