Mean-Variance versus Expected Utility in Dynamic Investment Analysis*

Yonggan Zhao

Nanyang Business School

Nanyang Technological University, Singapore 639798

Email: aygzhao@ntu.edu.sg

William T. Ziemba

Faculty of Commerce and Business Administration

University of British Columbia, Vancouver, BC, Canada, V6T 1Z2

Email: ziemba@interchange.ubc.ca

January, 2002

JEL classification: C61 D92 G11 G12

*This research was partially supported by Natural Sciences and Engineering Research Council of Canada.

Mean-Variance versus Expected Utility in Dynamic Investment Analysis

Abstract

This paper extends Merton's continuous time (instantaneous) mean-variance analysis and the mutual fund separation theory in which the growth optimal portfolio can be chosen as the risky fund. Given the existence of a Markovian state price density process, the CAPM holds without assuming log-normality for prices. The optimal investment policies for the case of HARA utility function are analytically derived. It is proved that only the quadratic utility exhibits global mean-variance efficiency among the family of HARA utility functions. The (global) efficient frontier for the dynamic model is linear in the space of standard deviation and expected return of the portfolio. A numerical comparison is made between the growth optimal portfolio and mean-variance analysis for the case of log-normally distributed assets. We discuss the optimal choice of target return that maximizes the probability that mean-variance analysis outperforms the expected utility approach. Finally, we discuss how to control portfolio's downside losses using a put option on the market portfolio.

I. Introduction

Markowitz (1952) mean-variance analysis blends elegance and simplicity. Compared to expected utility models, it offers an intuitive explanation for diversification and a relatively simple computational procedure. However, most discussions of mean-variance analysis are restricted to static models. Hence, investors can only make decisions at the beginning and must wait for the results without adjusting the portfolio weights until the end of the horizon. This is awkward for mean-variance analysis compared to versatile dynamic (multiperiod or continuous time) models that maximize expected utility. This paper investigates and develops a dynamic version of mean-variance analysis. Tobin (1958) showed that the mean-variance model is consistent with the von Neumann-Morgenstern postulates of rational behavior if the utility of wealth is quadratic. Merton (1973, 1992) developed an analog of the continuous time (instantaneous) mean-variance analysis and concluded that the mutual fund separation theory applies for the case of log-normal prices. This paper extends these results to a more general case in which the existence of a Markovian state price density process is the only assumption.

Alternative approaches are sought for the mean-variance criterion to apply in dynamic investment analysis. Merton (1973, 1992) discussed the approach of applying the Markowitz static model instantaneously at each time, i.e., minimizing the instantaneous standard deviation for a given target instantaneous mean return rate. By doing this, Merton was able to develop the intertemporal CAPM and mutual fund separation theory by applying stochastic control methodology. He showed that growth optimal portfolio is instantaneously mean-variance efficient when asset prices are log-normal. However, the log-normal assumption is only a sufficient condition. Therefore, this result can be extended to a more broad dynamic setting in which the asset price behavior mechanism is determined by a Markovian state price density process. With this setting, we can derive an instantaneous mean-variance efficient frontier as in the

Markowitz-Tobin model. The intertemporal CAPM holds as the growth optimal portfolio acts as the "market portfolio." We prove that all portfolios constructed by maximizing expected utility of terminal wealth are on the efficient frontier. The growth optimal portfolio can be chosen as the risky mutual fund. All investors are indifferent between investing in the two mutual funds, riskless portfolio and the growth optimal portfolio, and the combination of market primary assets.

Investors can use mean-variance analysis as the investment criterion under which investors minimize the variance of the total portfolio return by setting the portfolio expected return to a prescribed target as in the static case. The difference is that we allow the portfolio to be traded dynamically which is more realistic. In this way, the global efficient frontier can be developed. It is shown that the global efficient frontier is a straight line in mean-standard deviation of the portfolio return space as in the static case. We argue that this criterion for investment is better than instantaneous mean-variance analysis by showing that only quadratic utilities exhibit the (global) mean-variance efficiency, even for the case of log-normal asset prices. The optimal portfolio policies can be identified as a function of the state price and time variable by solving the associated partial differential equations. The solution techniques follow Cox and Huang (1989). The well known two fund separation theorem still holds and investors invest solely in the riskless asset and the growth optimal portfolio. However, the growth optimal portfolio is not on the global efficient frontier.

Investors might be interested in knowing the advantages of mean-variance analysis compared to the expected utility approach, since the mean-variance dominates the expected utility approach in mean-standard deviation space. However, there is no absolute dominance. It is well known that the growth optimal strategy (log utility) will outperform any essentially different strategy in the long run; see Breiman (1961) and Algoet and Cover (1988). The comparison provided in this paper shows that the mean-variance criterion achieves a better performance if the outcome of the market state price is near its mean value. The expected utility approach has superior performance when the outcomes are in the tails of the state price which accommodates the investor's risk aversion. To apply mean-variance analysis, the probability that a mean-variance model outperforms an expected utility model is a good criterion for identifying the optimal target mean level. We provide a general method for calculating this probability and a closed form solution in the case of log-normal prices and logarithmic utility. A numerical example compares the two approaches.

A critical problem of mean-variance approach is that it allows for arbitrarily large negative realizations of terminal wealth. This renders the practical importance of the analysis quite limited. Furthermore, in the continuous time setting such a concern is of particular importance. Dybvig and Huang (1988) use a nonnegative wealth constraint for ruling out the arbitrage opportunity. The martingale approach for portfolio selection problem is useful in dealing with portfolio constraints on the level of terminal wealth, for example, see Cox and Huang (1989). We discuss this issue to accommodate practical needs.

II. Asset Price Dynamics and Intertemporal CAPM

Assume a complete probability space $(\Omega, \mathcal{F}, P)$ and a time horizon $[0, T]$, where T is a strictly positive real number. Let $z_t = (z_{1t}, \cdots, z_{nt})^\top$ denote an n-dimensional standard Brownian motion, which generates a filtration $\mathbf{F} = \{\mathcal{F}_t \subseteq \mathcal{F}; t \in [0, T]\}$. A stochastic process W_t is called adapted to $\mathbf{F}$ if W_t is measurable with respect to $\mathcal{F}_t$. Assume the market is arbitrage free and continuous trading occurs without friction. Furthermore, the market has a state price density process described as the following Markovian diffusion process

$$\frac{d\xi_t}{\xi_t} = \alpha(t, \xi_t)\ dt + \beta(t, \xi_t)^\top\ dz_t, \qquad \xi_0 = 1, \tag{1}$$

where $\alpha(t, \xi)$ and $\beta(t, \xi)$ are at most functions of t and ξ and β satisfys the Novikov condition for the purpose of stochastic integrability. The assumption of the Markovian property for the state price density process ξ_t is credible, because it at least contains the case of log-normal asset prices as a subset. We will use α_t and β_t for short whenever there is no confusion. Throughout this paper, we make the following assumptions:

Assumption 1. *For every asset price (portfolio) process W_t, $\xi_t W_t$ is a martingle terminated at $\xi_T W_T$.*

Assumption 2. *If $\xi_t W_t$ is a martingale, then W_t is a valid portfolio process, i.e., it can be replicated using market primary assets.*

It can be proved that the martingale assumption with the positivity of the state price process ξ_t entirely rules out the arbitrage opportunities. So, ξ_t is restricted to a positive Markovian process. Following Harrison and Kreps (1979) and Harrison and Pliska (1981), Assumption 1 implies that the expected value of a security or a portfolio should be the same as its current value after being adjusted by the state prices over time. Assumption 2 should be viewed as an equivalent hypothesis to market completeness, but it is a weaker version than that in the literature.

Let W_t be a portfolio value (asset price) process. By Assumption 2, $\xi_t W_t$ is a martingale. Hence, by the martingale representation theorem, one can derive

Proposition 1. *For any asset (portfolio) in the market, there exists a predictable stochastic process ϕ_t such that the asset price (portfolio value) process W_t follows the stochastic differential* *equation*

$$\frac{dW_t}{W_t} = \left[\beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t\right] dt + \left[\phi_t^\top - \beta_t^\top\right] \ dz_t. \tag{2}$$

Proof, see Appendix A. Proposition 1 characterizes the dynamics of stock prices by the state price density process. There are some advantages of modelling the stock price dynamics as such an equilibrium representation for a given state price density process. We can be certain that the state prices are implied by the overall market performance but not individual stocks. On the contrary, equilibrium prices of individual stocks should follow Equation (2) to ensure that the market is arbitrage free. A criterion for admissible portfolio process is given in

Proposition 2. *If the stochastic process*

$$\frac{dW_t}{W_t} = \mu_t \ dt + \sigma_t^\top \ dz_t. \tag{3}$$

represents an admissible portfolio process, then μ_t *and* σ_t *satisfy the linear equation:*

$$\alpha_t + \mu_t + \sigma_t^\top \beta_t = 0. \tag{4}$$

Conversely, if (4) *holds and* σ_t *satisfies the Novikov condition of integrability, then Equation* (3) *represents an admissible portfolio process.*

Equation (4) determines how the mean and standard deviation of asset returns are related. Some special cases are worth of examination. If $\phi_t = \beta_t$ with probability one, then Equation (2) implies that

$$\frac{dW_t}{W_t} = -\alpha_t \ dt$$

which represents the riskless asset. Hence, the short term interest rate is $-\alpha_t$ (note that α_t should be negative by the definition of the state price density process). If $\phi_t = 0$, then Equation

(2) implies that

$$\frac{dW_t}{W_t} = (\beta_t^\top \beta_t - \alpha_t)\ dt - \beta_t^\top\ dz_t$$

which represents the rate of return of the growth optimal portfolio, i.e., the process ξ_t^{-1} which is equivalent to log utility maximization.

Let $\mu_{\xi t}$ and $\sigma_{\xi t}^\top \sigma_{\xi t}$ denote the mean and variance of the instantaneous rate of return of the growth optimal portfolio. By Equation (4), the following equation must hold for any asset price or portfolio value W_t with instantaneous mean return μ_t and instantaneous standard deviation σ_t

$$\mu_t + \alpha_t = \frac{\sigma_t^\top \beta_t}{\sigma_{\xi t}^\top \beta_t}\ (\mu_{\xi t} + \alpha_t). \tag{5}$$

Since $\mu_{\xi t} = \beta_t^\top \beta_t - \alpha_t$ and $\sigma_{\xi t} = -\beta_t$, then $-\sigma_t^\top \beta$ is the covariance of the return rates of W_t and ξ_t^{-1}. Also, $-\sigma_{\xi t}^\top \beta_t = \beta_t^\top \beta_t$ is the variance of the return rate of ξ_t^{-1}. The above argument has established the intertemporal CAPM, which is the continuous time version of the static CAPM model of Sharpe (1964) and Lintner (1965).

III. Mean Variance Analysis and Mutual Fund Separation

Merton (1973, 1992) developed the continuous time analog of the static mean-variance analysis. He concluded that, if asset prices follow a log-normal process, the growth optimal portfolio is (instantaneous) mean-variance efficient. Now, we extend this theory to a more general case.

The (instantaneous) mean-variance analysis is an optimization model that minimizes the instantaneous variance for a given mean of the rate of return, which can be represented as, by

Equation (2)

$$
\begin{aligned}
\min_{\phi_t} \quad & (\phi_t - \beta_t)^\top (\phi_t - \beta_t) \\
s.t. \quad & \beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t = \mu_t.
\end{aligned}
\tag{6}
$$

Theorem 1. *The instantaneous mean-variance efficient frontier is a straight line in mean and standard deviation space with a slope of* $\sqrt{\beta_t^\top \beta_t}$, *i.e., the optimal mean rate of return* μ_t *and its standard deviation* $\sqrt{\sigma_t^\top \sigma_t}$ *have the relation*

$$
\frac{\mu_t + \alpha_t}{\sqrt{\sigma_t^\top \sigma_t}} = \sqrt{\beta_t^\top \beta_t}, \quad \mu_t > -\alpha_t. \tag{7}
$$

For the proof of Theorem 1, see Appendix A. Since $-\alpha_t$ is the return rate of the riskless asset, Equation (7) is the continuous time version of the Markowitz (1952) static model.

Now we derive the conditions for a portfolio to be mean-variance efficient. Equation (4) and Equation (7) imply that

$$
-\sigma_t^\top \beta_t = \sqrt{\sigma_t^\top \sigma_t}\sqrt{\beta_t^\top \beta_t}. \tag{8}
$$

One can observe that the growth optimal portfolio ξ_t^{-1} (with $\sigma_t = -\beta_t$) is on the efficient frontier. This result does not require the assumption of log-normal prices. Obviously the case of log-normal prices is included in our general setting.

Proposition 3. *A portfolio is instantaneous mean-variance efficient if and only if*

$$
\sigma_t = -a_t \; \beta_t,
$$

where a_t *is a positive scalar.*

For the proof, see Appendix A.

Mutual fund separation theory is about alternative approaches for allocating wealth to primary assets. An immediate advantage of the theory is that investment decisions can be divided into two parts by the establishment of two financial intermediaries (mutual funds) to hold all individual assets and to issue shares of their own for purchase by individual investors. The separation is executable because mutual fund managers are instructed to hold the proportions of the individual assets independent of investors' preferences and wealth distribution. If asset returns are normally distributed or the investor's utility is quadratic, all investors can alternatively invest in two mutual funds that are constructed using the primary assets. Merton (1973, 1992) derived a two mutual fund separation for fixed investment opportunities (constant μ_t and σ_t and a three mutual fund separation for stochastic interest rate (stochastic α_t but fixed μ_t and σ_t). Here we extend this result to the general case in which the state price density process is Markovian.

Let $U(x)$ be a strictly increasing and concave utility function. Assume investors' decisions are based on the maximization of the expected utility of the terminal wealth (we ignore consumption for simplicity). The optimization model is

$$\begin{aligned} \max_{W_T} \quad & E\left[U(W_T)\right] \\ s.t. \quad & E\left[\xi_T W_T\right] = W_0. \end{aligned} \tag{9}$$

Now, we state our main results as

Theorem 2. *If market state price density process is Markovian, then*

i.) *All investment portfolios that maximize the expected utility of terminal wealth are instantaneous mean-variance efficient.*

ii.) *The two fund separation theorem applies. The growth optimal portfolio can be chosen as the risky fund. Let $F(t, \xi_t)$ be the optimal wealth at time t and the proportion invested in the growth optimal portfolio $\theta_{\xi t}$. Then*

$$\theta_{\xi t} = -\frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}, \tag{10}$$

where $\frac{\partial F}{\partial \xi}$ *is the partial derivative of* $F(t, \xi)$ *with respect to* ξ.

For the proof, see Appendix A.

Theorem 2 indicates that investors are indifferent between investing in two mutual funds and a combination of market tradable assets. The growth optimal portfolio can be chosen as the risky fund. However, the optimal portfolio weights are functions of calender time and the value of the risky fund. In other words, there are no incentives for investors to know the performance of the individual stocks, but managers of the risky fund fully take the responsibility of managing the fund. This theory is consistent with the operation of financial investment. The log-normal assumption is a special case of the setting that the state price density process is Markovian. Hence, Theorem 2 extends the mutual fund separation theory to a broader setting.

IV. The Global Mean-Variance Model

In continuous time, investors may be also interested in seeking sound investment decisions by a mean-variance criterion as in the static case where the portfolio risk is minimized for a given expected return. Unlike instantaneous mean-variance analysis, investors are interested not only in the expected value of the portfolio return but also in monitoring the portfolio's risk measured by its standard deviation. Let $\tilde{R} = \frac{W_T}{W_0}$ be the portfolio return, and denote the expected value and the standard deviation of $\tilde{R}$ by R and V, respectively. Since $\xi_t W_t$ is a martingale, $E[\xi_T \tilde{R}] = 1$. Considering the opportunity of dynamic trading, investors wish to solve the global mean-variance model

$$\begin{aligned} V^2 = \min_{\tilde{R}} \;\; & E[\tilde{R}^2] - R^2 \\ s.t. \;\; & E[\tilde{R}] = R \\ & E[\xi_T \tilde{R}] = 1. \end{aligned} \tag{11}$$

The objective function is equal to the variance of the portfolio value for a given mean return R.

Theorem 3. *The optimal portfolio return* $\tilde{R}$ *is a linear function of the state price* ξ_T *with a negative slope if the target return is greater than the riskless rate*

$$\tilde{R} = \frac{1}{2}\lambda - \frac{1}{2}\rho\, \xi_T, \tag{12}$$

where λ *and* ρ *are the multipliers on the constraints in model* (11)*. The mean-variance efficient frontier is also linear and determined by*

$$V = \begin{cases} \frac{R_\xi}{V_\xi}(R - R_\xi^{-1}), & R \geq R_\xi^{-1}, \\ -\frac{R_\xi}{V_\xi}(R - R_\xi^{-1}), & R < R_\xi^{-1}, \end{cases} \tag{13}$$

where R_ξ *and* V_ξ *are the expected value and the standard deviation of* ξ_T*, respectively.*

For the proof, see Appendix A.

Figure 1 depicts the feasible region of portfolio policies as the area between the two lines in mean-standard deviation space. The upward straight line (with positive slope $\frac{V[\xi]}{E[\xi]}$) is the

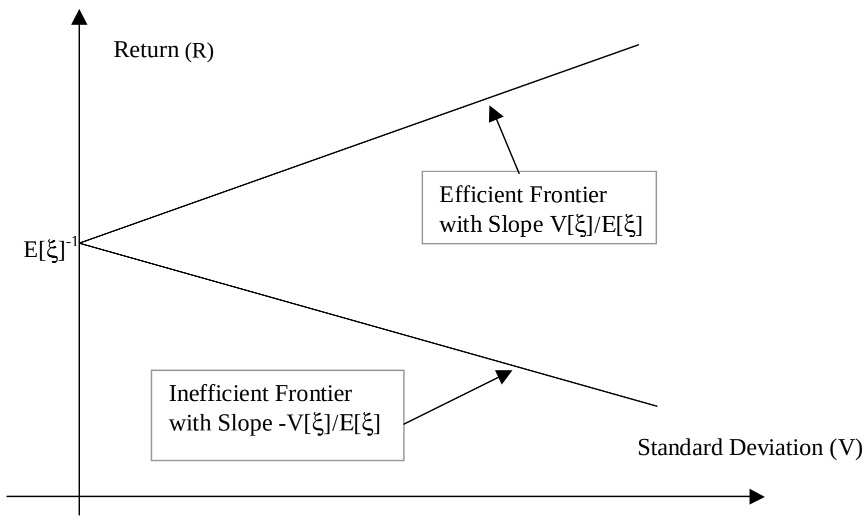

Figure 1: The Dynamic Mean-Variance Efficient Frontier

efficient frontier and the downward straight line (with negative slope $-\frac{V[\xi]}{E[\xi]}$) is the inefficient frontier as in the static model. The global mean-variance model has a similar shape but with a larger slope than the static mean-variance efficient frontier, since the set of continuous trading strategy contains the set of static strategies as a subset. Furthermore, the mean and the standard deviation of the contingent state price jointly determine the mean-variance efficient frontier by (13).

Which portfolios are global mean-variance efficient? We have a quite satisfactory answer for the case of static model and its instantaneous version. However, for our setting of the dynamic equilibrium market, only the portfolio obtained by maximizing a quadratic utility is global mean-variance efficient, even for the case of log-normal asset prices. We give this result as

Proposition 4. *Assume $U(x)$ is a HARA utility function. If $U(x)$ is quadratic and concave, then the optimal portfolio obtained from maximizing $E[U(W_T)]$ is global mean-variance efficient. Therefore, global mean-variance analysis is consistent with the utility maximization. Conversely, if the portfolio obtained from maximizing $E[U(W_T)]$ is global mean-variance efficient, then $U(x)$ must be a quadratic function with a negative second order derivative.*

For the proof, see Appendix A.

So far, the optimal portfolio is characterized as a function of the state price density process. How is this portfolio replicated by using market traded securities?

Suppose we have a market of $m+1$ "continuously" tradable securities. One of them is riskless and denoted by B_t,

$$\frac{dB_t}{B_t} = -\alpha_t \, dt,$$

and the other m securities follow the following processes:

$$\frac{dS_{it}}{S_{it}} = (-\sigma_{it}^\top \beta_t - \alpha_t) \, dt + \sigma_{it}^\top dz_t, \quad i = 1, \cdots, m.$$

Equation (4) demonstrates that the above formulation is correct. Denote $\Sigma_t = (\sigma_{1t}, \cdots, \sigma_{mt})^\top$, and assume $\Sigma_t \Sigma_t^\top$ is invertible $(m \leq n)$ which accommodates the completeness of the market.

The wealth at time t is,

$$W_t = \xi_t^{-1} E\left[\xi_T \left(\tfrac{1}{2}\lambda - \tfrac{1}{2}\rho\ \xi_T\right) | \mathcal{F}_t\right].$$

Since ξ_t is Markovian and, by the two-fund separation theorem (Theorem 2), any optimal portfolio obtained from utility maximization is equivalent to a portfolio rule investing only in two mutual funds, the riskless asset and the growth optimal portfolio. Without proving the equivalence of the global mean variance efficiency to a quadratic utility maximization as given in the static case, we directly derive the optimal portfolio rule. Let $W_t = F(t, \xi_t)$ be the optimal portfolio value, where $F(t, \xi)$ is differentiable with respect to t and twice differentiable with respect to ξ. By Itô's formula

$$\begin{aligned} \frac{dW_t}{W_t} &= \frac{1}{W_t}\left(\frac{\partial F}{\partial t}\ dt + \frac{\partial F}{\partial \xi}\ d\xi_t + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\ (d\xi_t)^2\right) \\ &= \frac{1}{W_t}\left(\frac{\partial F}{\partial t} + \alpha_t \xi_t \frac{\partial F}{\partial \xi} + \tfrac{1}{2}\xi_t^2 \beta_t^\top \beta_t \frac{\partial^2 F}{\partial \xi^2}\right)\ dt + \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}\beta_t^\top dz_t. \end{aligned} \tag{14}$$

The equilibrium condition (4) implies

$$\frac{\partial F}{\partial t} + \alpha_t \xi_t \frac{\partial F}{\partial \xi} + \tfrac{1}{2}\xi_t^2 \beta_t^\top \beta_t \frac{\partial^2 F}{\partial \xi^2} + \alpha_t\ F + \xi_t \beta_t^\top \beta_t \frac{\partial F}{\partial \xi} = 0. \tag{15}$$

Hence, $F(t, \xi)$ is the solution to the partial differential equation

$$\frac{\partial F}{\partial t} + \alpha_t \xi \frac{\partial F}{\partial \xi} + \tfrac{1}{2}\xi^2 \beta_t^\top \beta_t \frac{\partial^2 F}{\partial \xi^2} + \alpha_t\ F + \xi \beta_t^\top \beta_t \frac{\partial F}{\partial \xi} = 0 \tag{16}$$

where α_t and β_t are viewed as functions of (t, ξ), with the boundary condition

$$F(T, \xi) = \tfrac{1}{2}\lambda - \tfrac{1}{2}\rho\ \xi.$$

An analytical solution is available when α_t and β_t are constant as shown in

Theorem 4. *Let θ_{it} be the proportion of wealth allocated in the ith risky asset. Denote $\theta_t = (\theta_{1t}, \cdots, \theta_{nt})^\top$. Then*

$$\theta_t = \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi} \cdot (\Sigma_t \Sigma_t^\top)^{-1}\Sigma_t \beta_t. \tag{17}$$

If the parameters α_t and β_t in the state price density process are constant, i.e., $\alpha_t = \alpha$, $\beta_t = \beta$, and $\Sigma_t = \Sigma$, then the partial differential equation (16) *has the closed form solution,*

$$F(t, \xi) = \tfrac{1}{2}\left(\lambda e^{\alpha(T-t)} - \rho\xi\ e^{(\beta^\top\beta + 2\alpha)(T-t)}\right). \tag{18}$$

The optimal portfolio policy at time t is

$$\theta_t = \left(\tfrac{1}{2W_t}\lambda e^{\alpha(T-t)} - 1\right)(\Sigma\Sigma^\top)^{-1}\Sigma\beta, \tag{19}$$

where λ is the Lagrangian multiplier on the first constraint in the model (11)*.*

For the proof of Theorem 4, see Appendix A. An alternative proof of Theorem 4 can be approached by taking use of the results in Theorem 2 that any optimal portfolio is equivalent to a portfolio investing in the two mutual funds, the riskless asset and the growth optimal portfolio. Since the growth optimal portfolio weights (Merton ratio) are given by

$$-(\Sigma_t \Sigma_t^\top)^{-1}\Sigma_t \beta_t.$$

Combining this with (10) yields (17).

V. Mean-Variance versus Expected Utility

A. The Relation between Portfolio Returns

Much research has been focused on determining which of the expected utility approach and the mean-variance analysis is preferable in making sound investment decisions. It is well known that the optimal portfolio generated from a utility maximization is not on the mean-variance efficient frontier except in a few special instances: either a "carefully" chosen quadratic utility function is used or the asset returns are joint normally distributed; see Ross (1978), and Ziemba and Vickson (1975) for other exceptions. However, investors and academic researchers do not accept these assumptions for practical use. In Section IV., we proved that only the quadratic utility functions are global mean-variance efficient in the setting of the market. This seems to be in favor of the mean-variance approach. However, see Breiman (1961), the growth optimal portfolio (log utility) will beat any essentially different portfolio strategy with probability 1 in the long run. Grauer (1981) compared the growth optimal strategies and the mean-variance analysis in the static case. Kroll and Markowitz (1984) conducted a similar study. For users of mean-variance analysis, the following question may be asked: what is the best choice of the target return such that the portfolio return has the maximum probability of being higher than the portfolio return obtained from a utility maximization approach?

Let $\tilde{R}_m$ and $\tilde{R}_u$ be the portfolio returns for mean-variance analysis and an expected utility approach, respectively. $\tilde{R}_m$ is given by (12). The utility maximization is given by (9). Hence, the optimal return $\tilde{R}_u$ for the utility maximization portfolio is

$$\tilde{R}_u = \tfrac{1}{W_0} U_x^{-1}(\lambda_u \xi_T),$$

where λ_u is the Lagrangian multiplier on the budget constraint in model (9) and $U_x^{-1}(\cdot)$ stands for the inverse function of the marginal utility of wealth; see Cox and Huang (1989).

Assuming $U(x)$ is a strictly increasing HARA utility function, $\tilde{R}_u$ is usually a convex function of ξ_T. On the other hand $\tilde{R}_m$ is a straight line with negative slope. Therefore, there are two intersection points ξ_1 and ξ_2 in the two portfolio returns. Mean-variance analysis will be superior if the outcome of the state price ξ_T occurs around the mean value $E[\xi_T]$ of the state price, as represented by shaded area, and be inferior if the outcome is beyond one of the tails, ξ_1 or ξ_2. Also, ρ changes as a function of $\mu = E\left[\tilde{R}_m\right]$. As μ increases, $\tilde{R}_m$ shifts up, and at the same time becomes steeper. Hence, the effect of an increase of μ on the probability of outperforming the expected utility of wealth is non-monotonic. Figure 2 depicts the relation of the two optimal portfolio values in terms of the state price ξ_T.

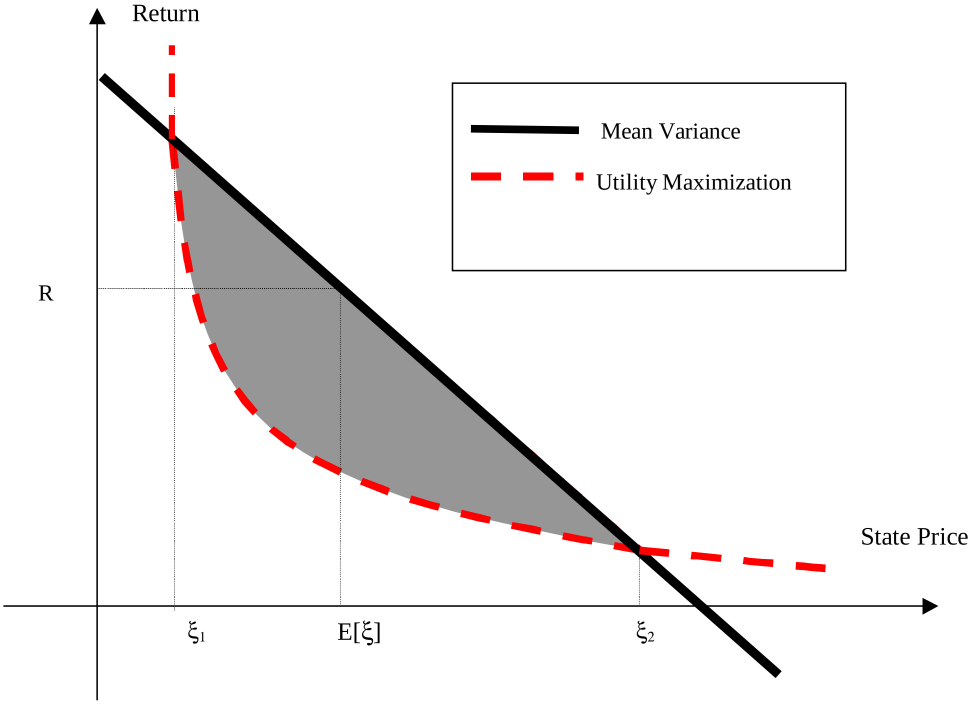

Figure 2: The Optimal Terminal Portfolio Values

B. Opportunities Superior to the Expected Utility Approach

By varying the expected portfolio return μ , investors can find the maximum probability that the mean-variance optimal portfolio will outperform the expected utility maximization portfolio. The maximum probability is given by solving

$$\max_{\mu} \quad \Pr\{\xi_1 \leq \xi_T \leq \xi_2 \mid R_m \geq E[\xi_T]^{-1}\}. \tag{20}$$

Here ξ_1 and ξ_2, $\xi_1 < \xi_2$, are the two intersection points of $\tilde{R}_u$ and $\tilde{R}_m$ which satisfy the following transcendental equation,

$$-\tfrac{2}{W_0} U_x^{-1}(\lambda_u \xi_T) + \lambda - \rho \xi_T = 0. \tag{21}$$

Since $\tilde{R}_u$ is usually a convex function of the contingent state price ξ_T and $\tilde{R}_m$ is linear in ξ_T, $\tilde{R}_u$ and $\tilde{R}_m$ intersect at exactly two points for a given $\mu < \infty$. Since λ and ρ are functions of μ, so are ξ_1 and ξ_2. Both the mean-variance analysis and expected utility approaches are considered as standard approaches for constructing optimal investment strategies. For mean-variance optimizers, an interesting question is how to set the target wealth level such that the mean-variance criterion will be superior to the expected utility approach with maximum probability. With appropriate conditions, we can calculate the optimal value μ and, therefore, the maximum probability. Let $\phi(x)$ be the density function of ξ_T. Assuming that there is a solution to (20) and that both $\xi_1(\mu)$ and $\xi_2(\mu)$ are differentiable with respect to μ, then problem (20) becomes

$$\max_{\mu} \quad \int_{\xi_1}^{\xi_2} \phi(x)dx.$$

By the first order conditions, the optimal μ is given by

$$\phi(\xi_2(\mu))\xi_2'(\mu) - \phi(\xi_1(\mu))\xi_1'(\mu) = 0, \tag{22}$$

where "$'$ " stands for the mathematical derivative.

C. Optimal Portfolio with a Lower Bound

A critical problem of mean-variance approach is that it allows for arbitrarily large negative realizations of terminal wealth as shown in Figure 2. This clearly renders the practical importance of the analysis quite limited. Dybvig and Huang (1988) considered a nonnegative wealth constraint for ruling out arbitrage opportunities, Cox and Huang (1989) consider a model of investment and consumption with nonnegative constraints and relate the optimal rules to a synthetic option strategy. For mean-variance analysis, downward control can be easily implemented.

Let W_0 be the initial wealth, $\bar{W}$ the targeted mean level of wealth, and W_l a lower bound of the wealth level. Assume investor are interested in solving the following constrained mean-variance model

$$\begin{aligned} \min \quad & E[W_T^2] - \bar{W}^2 \\ & E[W_T] = \bar{W} \\ & E[\xi_T W_T] = W_0 \\ & W_T \geq W_l, \text{in probability.} \end{aligned} \tag{23}$$

Let λ^*, ρ^*, and $\tilde{\gamma}^*$ be the multipliers on the three constraints in model (23), respectively. we use "*" to emphasize that these multipliers are different from those without the lower bound constraint. The optimal wealth is

$$W_T = \tfrac{1}{2}\lambda^* - \tfrac{1}{2}\rho^*\xi_T + \left(W_l - (\tfrac{1}{2}\lambda^* - \tfrac{1}{2}\rho^*\xi_T)\right)^+. \tag{24}$$

This expression is equivalent to a mean-variance portfolio strategy with a protective put option on the portfolio value. One can implement this portfolio policy by investing in a mean variance model and taking an insurance policy against the portfolio's decrease in value at the same time. The problem is what percentage of the wealth should be allocated for investment and how to purchase the insurance policy, given the existence of such a financial intermediary. To resolve such a concern, we devise an alternative approach that will reach the same solution. The growth optimal portfolio ξ_t^{-1} acts as the "market" portfolio, therefore, it should be conceivably available for investment if all investors hold efficient portfolios. Hence, we want to relate the insurance strategy to this portfolio.

Equation (24) can be rewritten as

$$W_T = \tfrac{1}{2}\lambda^* - \tfrac{1}{2}\rho^*\xi_T + \left(\tfrac{1}{2}\lambda^* - W_l\right)\xi_T\left(\tfrac{\rho^*}{\lambda^* - 2W_l} - \xi_T^{-1}\right)^+. \tag{25}$$

Hence, the portfolio strategy can be decomposed as a mean-variance model and put options on the growth optimal portfolio.

Let $W_0^* = E\left[\xi_T(\frac{1}{2}\lambda^* - \frac{1}{2}\rho^*\xi_T)\right]$ and $\bar{W}^* = E\left[\frac{1}{2}\lambda^* - \frac{1}{2}\rho^*\xi_T\right]$. Then investors can solve the mean-variance model with the initial investment W_0^* and a target mean level $\bar{W}^*$. Then, using $W_0 - W_0^*$ to purchase put options on the growth optimal portfolio with strike price $\frac{\rho^*}{\lambda^* - 2W_l}$. The number of puts is $\frac{2(\bar{W} - \bar{W}^*)}{\lambda^* - 2W_l}$.

D. A Numerical Example

Capital growth theory is very interesting to both academicians and practitioners. Hakansson (1971) studied capital growth and mean-variance approach to portfolio selection, MacLean et al. (1992) and MacLean and Ziemba (1999) studied the growth versus security tradeoffs using fractional Kelly strategies that are blends of the capital growth portfolio and cash. Hakansson and Ziemba (1995) surveyed this area. Here we examine the difference of the two approaches within the continuous time framework using a numerical example.

Consider an investor having one dollar to invest between a riskless asset and a risky asset. For the riskless interest rate for the period of August 2, 1999 to August 1, 2000, we use $r = 0.05$ per annum, i.e., the riskless asset price $B(t) = e^{rt}$. The S&P 500 is the risky asset. After scaling the initial index level to a dollar, Figure 3 depicts the price dynamics of the S&P 500 without dividend for this period.

Assuming that the price S_t of the S&P 500 follows the geometric Brownian motion

$$\frac{dS_t}{S_t} = bdt + \sigma dz_t,$$

with estimated $b = 0.101$ and $\sigma = 0.212$. Let the investment horizon be one year, so $T = 1$. Hence, $\alpha = -0.05$ and $\beta = -(b - r)/\sigma = -0.24$, i.e., the state price density process is given by

$$\frac{d\xi_t}{\xi_t} = -0.05dt - 0.24dz_t$$

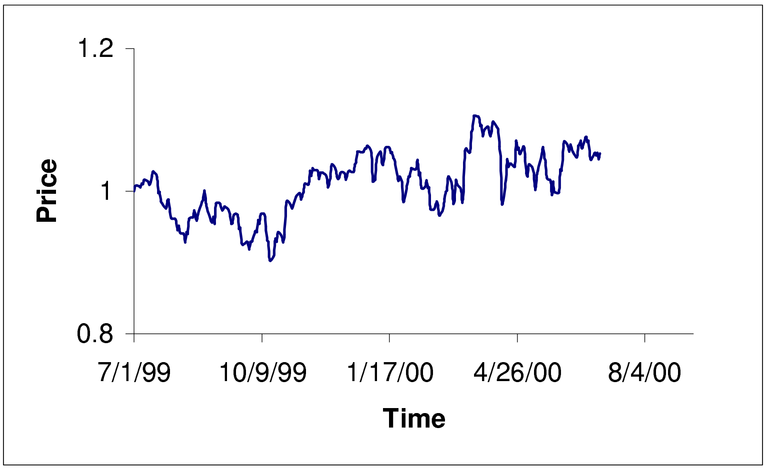

Figure 3: Scaled Index Level of S&P 500.

where z_t is a standard Brownian motion. Hence, $E[\xi_T] = 0.95$ and $V^2[\xi_T] = 0.96$. The λ and ρ are

$$\begin{cases} \lambda & = 35.69\mu - 35.42 \\ \rho & = -35.42\mu + 37.24. \end{cases}$$

The mean-variance optimal portfolio return is, by Equation (12),

$$\tilde{R}_m = (17.85\mu - 17.76) + (-17.76\mu + 18.62)\xi_T.$$

For the logarithmic utility, the optimal return is

$$R_u = \xi_T^{-1}.$$

See Cox and Huang (1989) for a derivation of this. The intersection points, ξ_1 and ξ_2, are given by the quadratic equation

$$(-17.76\mu + 18.62)\xi_*^2 + (17.85\mu - 17.76)\xi_* - 1 = 0,$$

whose solutions are

$$\begin{cases} \xi_1(\mu) & = \frac{17.85\mu - 17.76 - \sqrt{240.94 - 562.99\mu + 318.62\mu^2}}{2(17.76\mu - 18.62)} \\ \xi_2(\mu) & = \frac{17.85\mu - 17.76 + \sqrt{240.94 - 562.99\mu + 318.62\mu^2}}{2(17.76\mu - 18.62)}. \end{cases} \tag{26}$$

For given μ, the probability that the mean-variance model outperforms the growth optimal strategy is

$$\int_{\xi_1}^{\xi_2} \frac{1}{\sqrt{2\pi\beta^\top\beta}} \exp\{-\frac{(\ln x - \alpha + \frac{1}{2}\beta^\top\beta)^2}{2\beta^\top\beta}\} \cdot \frac{1}{x} dx.$$

Using the first order condition indicates that the numerical solution of the optimal μ is

$$\mu \approx 1.139.$$

This means that, for this specific investment environment, investors should set the target wealth to be about 13.9% higher than the initial wealth to maximize the probability of surpassing the growth optimal strategy (logarithmic utility). Then, the probability that the mean-variance model will beat the growth optimal strategy under the assumption of log-normal asset prices exceeds 70%. Figure 4 depicts the probabilities corresponding to different choices of μ.

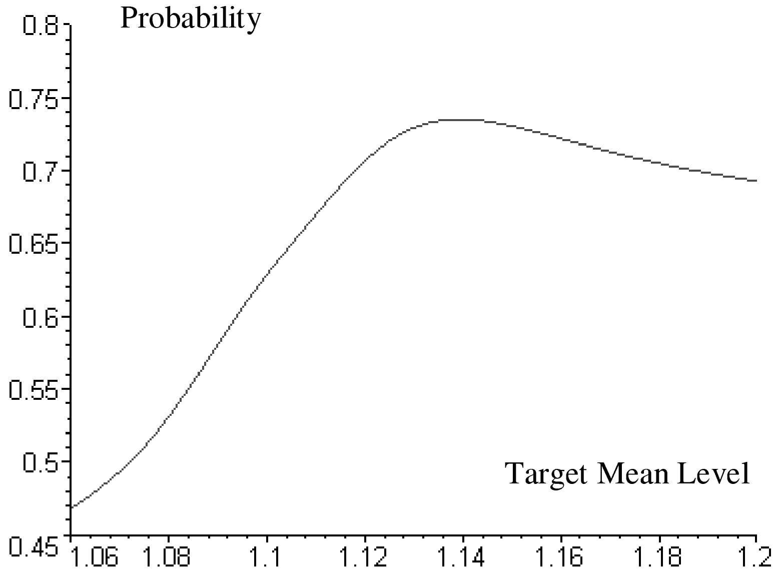

Figure 4: The Probability of Mean-Variance Superior to the Growth Optimal Strategy

Remark. Since the logarithmic utility has an expected portfolio return $E[\xi_T^{-1}] = e^{(\beta^\top\beta-\alpha)T}$ which is dominant in the long run (as $T \to \infty$), the logarithmic utility will have a higher chance of beating the mean-variance analysis for the long investment horizon. This leads to the assertion that the logarithmic utility may have a high probability of beating a mean-variance model when the market investment environments are changed to a long investment horizon and/or a moderately high market price for risk (a high $-\beta$). See the discussion on this in Hakansson and Ziemba (1995).

E. Implementation of the Mean-Variance Optimal Strategy

In the static mean-variance model, the investor needs only to choose an appropriate target mean level to find the optimal portfolio strategy by minimizing the standard deviation. This process can be completed with the calculation of the first and the second moments and a quadratic programming optimizer. However, in dynamic investment analysis, the portfolio weights are continuously changed according to the observed market asset prices. To illustrate the dynamic mean-variance analysis, we use the data of the previous example to compare the performances of mean-variance analysis and the growth optimal portfolio. The target mean return level for the mean-variance analysis is chosen as 13.9% which maximizes the probability of outperforming the growth optimal strategy (logarithmic utility). Figure 5 describes the performances of these two strategies with S&P 500 over time (note that the similar performances of the growth optimal portfolio and S&P 500 for this period show a strong support to our assumption of a Markovian state price density). While the growth optimal portfolio has a similar performance to the index for this specific data, the mean-variance analysis has a superior performance if we set the target mean level to be about 2% more than the mean return of S&P 500.

VI. Concluding Remarks

With the assumption of the existence of a Markovian state price density process and a replication argument, we have proved that the intertemporal CAPM holds. All models of utility

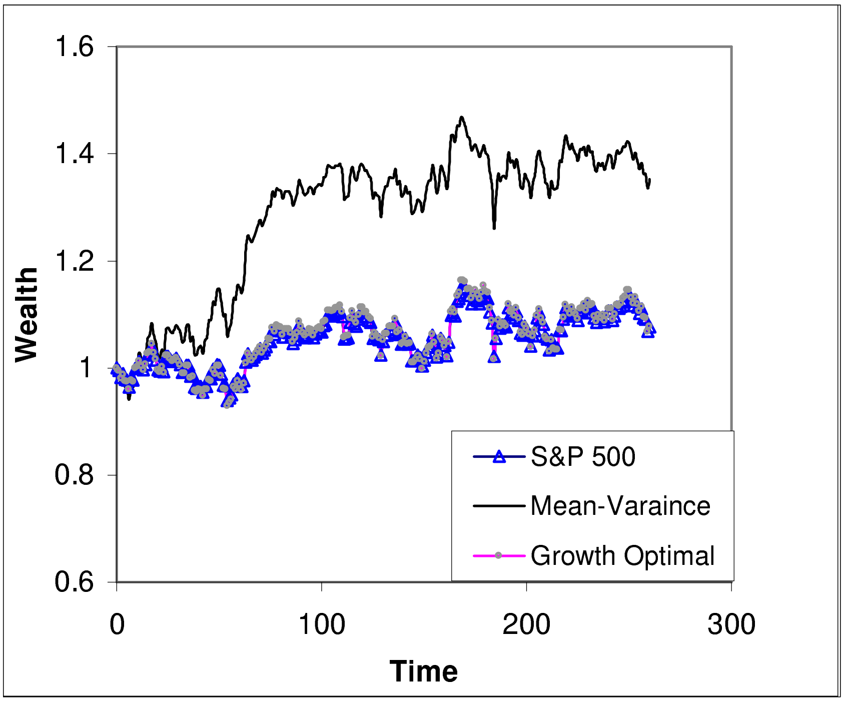

Figure 5: Wealth Level Performances over Time

maximization of terminal wealth are instantaneously mean-variance efficient. This result has extended the two fund separation theory to broader settings of the market asset returns. The growth optimal portfolio can be evidently chosen as the risky fund.

Taking mean-variance as the investment criterion, we have derived the global mean-variance efficient frontier and optimal portfolio policies for dynamic investments. The efficient frontier is uniquely determined by the mean and the standard deviation of the contingent state price. The efficient frontier is also a ray which intersects the vertical axis corresponding to the riskless rate with slope equal to the ratio of the standard deviation and the mean of the contingent price. Unlike instantaneous mean-variance analysis, no utility function except the quadratic utility is globally mean-variance efficient.

Because of the non-efficiency of the utility maximization in mean and standard deviation, investors might be interested in knowing which approach makes a sound investment decision. The optimal portfolio return for a mean-variance model is a linear function of the state price with a negative slope, while that of a (HARA) utility maximization usually appears to be a convex function of the state price. This provides us a method for calculating the probability that the mean-variance model outperforms the expected utility maximization approach. To show the important role of mean-variance analysis in making sound investment decisions, this paper compares, state by state, the optimal values obtained from both mean-variance and expected utility models. The mean-variance analysis is superior to the expected utility if the outcome of the contingent state price is near its mean and inferior to the expected utility model if the outcome is in the tails. An interesting question is: what is the optimal target wealth such that the mean-variance criterion will be superior to a given expected utility model with maximum probability? To perform the analysis, we used S&P 500 data for a one year period to calibrate the asset price model. For a one-year investment horizon, the mean-variance return will beat the growth optimal (logarithmic utility) return by more than a 70% chance if the wealth target (the mean level) is set to be about 13.9% more than the initial wealth.

A Appendix

Proof of Proposition 1:

By assumption 2, $\xi_t W_t$ is a martingale that terminates at $\xi_T W_T$. By the martingale representation theorem, see Øksendal (1995), there exists a unique adapted stochastic process ϕ_t such that

$$\frac{d\xi_t W_t}{\xi_t W_t} = \phi_t^\top \; dz_t.$$

Since the growth optimal portfolio follows the stochastic differential equation

$$\frac{d\xi_t^{-1}}{\xi_t^{-1}} = (\beta_t^{-1}\beta_t - \alpha_t) \; dt - \beta_t^\top \; dz_t,$$

by Itô's formula

$$\begin{aligned} \frac{dW_t}{W_t} &= \frac{d\xi_t^{-1}}{\xi_t^{-1}} + \frac{d\xi_t W_t}{\xi_t W_t} + \frac{d\xi_t W_t}{\xi_t W_t} \cdot \frac{d\xi_t^{-1}}{\xi_t^{-1}} \\ &= (\beta_t^{-1}\beta_t - \alpha_t) \; dt - \beta_t^\top \; dz_t + \phi_t^\top \; dz_t - \phi_t^\top \beta_t \; dt \\ &= (\beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t) \; dt + (\phi_t - \beta_t)^\top \; dz_t. \end{aligned} \tag{27}$$

Proof of Proposition 2:

By Assumption 1 and Proposition 1, there is an adapted stochastic process ϕ_t such that

$$\mu_t = \beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t, \quad \sigma_t = \phi_t - \beta_t.$$

Substituting out ϕ_t yields

$$\mu_t + \alpha_t + \sigma_t^\top \beta_t = 0.$$

Conversely, if the above equation hold, then $\xi_t W_t$ is a martingale. By assumption 2, W_t is a portfolio process.

Proof of Theorem 1:

The Lagrange multiplier of the optimization model (6) is

$$\mathcal{L}(\phi_t, \lambda) = \left[(\phi_t - \beta_t)^\top (\phi_t - \beta_t)\right] - \lambda \left(\beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t - \mu_t\right).$$

The first order conditions imply that

$$\begin{cases} 2(\phi_t - \beta_t) + \lambda \beta_t = 0 \\ \beta_t^\top \beta_t - \alpha_t) - \phi_t^\top \beta_t - \mu_t = 0, \end{cases}$$

which yields

$$(\phi_t - \beta_t)^\top (\phi_t - \beta_t) = \frac{(\mu_t + \alpha_t)^2}{\beta_t^\top \beta_t}.$$

Hence, Theorem 1 is proved.

Proof of Proposition 3:

Let W_t be a portfolio with $\sigma_t = -a_t \beta_t$, where a_t is a scalar. Thus, Equation (8) is satisfied. Hence, by Proposition 2, W_t is instantaneous mean-variance efficient. Conversely, if W_t is mean-variance efficient but $\sigma_t \neq -a_t \beta_t$ for any $a_t > 0$, then

$$(\sigma_t + a_t \beta_t)^\top (\sigma_t + a_t \beta_t) > 0,$$

which implies that

$$(\sigma_t^\top \beta_t)^2 - \sigma_t^\top \sigma_t \cdot \beta_t^\top \beta_t > 0.$$

This contradicts Theorem 1. Hence, there exists a positive scalar process a_t such that $\sigma_t = -a_t \beta_t$.

Proof of Theorem 2:

Let $U(x)$ be a strictly increasing and concave utility function. A utility maximizer solves the optimization model (9) with the Lagrange multiplier as

$$\mathcal{L}(W_T, \lambda) = E\left[U(W_T)\right] - \lambda \left(E\left[\xi_T W_T\right] - W_0\right).$$

The first order conditions are

$$\begin{cases} U_x(W_T) - \lambda \xi_T = 0 \\ \\ E\left[\xi_T W_T\right] = W_0, \end{cases}$$

where $U_x(\cdot)$ is the first order derivative. Let $U_x^{-1}(\cdot)$ is its inverse function, then

$$W_T = U_x^{-1}(\lambda \xi_t)$$

where λ is given by

$$E\left[\xi_T \cdot U_x^{-1}(\lambda \xi_T)\right] = W_0.$$

Let W_t be the portfolio value at time t. Since $\xi_t W_t$ is a martingale, then

$$\xi_t W_t = E\left[\xi_T U_x^{-1}(\lambda \xi_T) | \mathcal{F}_t\right].$$

Since ξ_t is Markovian, the wealth W_t must be a function of t and ξ_t, i.e., there exists a deterministic function $F(t, \xi)$ such that

$$W_t := F(t, \xi_t) = \xi_t^{-1}\ E\left[\xi_T \cdot U_x^{-1}(\lambda \xi_T) | \mathcal{F}_t\right].$$

By Itô's formula

$$\begin{aligned} \frac{dW_t}{W_t} &= \frac{1}{W_t}\left(\frac{\partial F}{\partial t}\ dt + \frac{\partial F}{\partial \xi}\ d\xi_t + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\ (d\xi_t)^2\right) \\ &= \frac{1}{W_t}\left(\frac{\partial F}{\partial t} + \alpha_t \xi_t \frac{\partial F}{\partial \xi} + \frac{1}{2}\xi_t^2 \beta_t^\top \beta_t \frac{\partial^2 F}{\partial \xi^2}\right)\ dt + \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}\beta_t^\top dz_t. \end{aligned} \tag{28}$$

By Theorem 1, W_t is instantaneous mean-variance efficient. This proves the first part of Theorem 2.

Equation (4)implies

$$\frac{1}{W_t}\left(\frac{\partial F}{\partial t} + \alpha_t \xi_t \frac{\partial F}{\partial \xi} + \tfrac{1}{2}\xi_t^2 \beta_t^\top \beta_t \frac{\partial^2 F}{\partial \xi^2}\right) + \alpha_t + \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}\beta_t^\top \beta_t = 0.$$

Denote the riskless asset by B_t, i.e., $dB_t = -\alpha_t\ B_t dt$. Hence, one can derive from the above equation that

$$\frac{dW_t}{W_t} = \left(1 + \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}\right) \cdot \frac{dB_t}{B_t} - \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi} \cdot \frac{d\xi_t^{-1}}{\xi_t^{-1}}. \tag{29}$$

This is equivalent to saying that the optimal portfolio can be replicated by the riskless asset and the growth optimal portfolio. The optimal portfolio rule is to invest the proportion of $-\frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}$ in the growth optimal portfolio and $(1+\frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi})$ in the riskless asset. Thus, the two fund separation theorem applies.

Proof of Theorem 3:

Applying the Kuhn-Tucker conditions yields

$$\begin{aligned} 2\tilde{R} - \lambda + \rho\xi_T &= 0 \\ E[\tilde{R}] - R &= 0 \\ E[\xi_T \tilde{R}] - 1 &= 0 \end{aligned} \tag{30}$$

which implies (11). Solving Equation (30) yields

$$\begin{cases} \lambda &= \frac{2(RE[\xi_T^2]-E[\xi_T])}{V^2[\xi_T]} \\ \rho &= \frac{2(RE[\xi_T]-1)}{V^2[\xi_T]}, \end{cases}$$

where $V^2[\xi_T]$ denotes the variance of ξ_T. Substituting λ and ρ into (30) yields

$$E[\tilde{R}^2] = \frac{(RE[\xi_T^2] - E[\xi_T])R}{V^2[\xi_T]} + \frac{(1 - RE[\xi_T])}{V^2[\xi_T]}.$$

Hence

$$V^2 = \frac{(E[\xi_T])^2}{V^2[\xi_T]}(R - E[\xi_T]^{-1})^2 \tag{31}$$

which completes the proof of Theorem 3.

Proof of Proposition 4: Let $U(x)$ be the utility function determined by

$$-\frac{U_x(x)}{U_{xx}(x)} = ax + b.$$

Then

$$\frac{U_{xxx}(x)U_x(x)}{U_{xx}^2(x)} = 1 + a.$$

The optimal terminal wealth for the utility maximization is

$$W_T = U_x^{-1}(\lambda \xi_T),$$

where λ is the Lagrangian multiplier in the model 9. Let $A_t = U_x^{-1}(\lambda \xi_t)$, then $A_T = W_T$. By Itô's formula

$$\begin{aligned} dA_t &= \frac{1}{U_{xx}(A_t)} d\lambda\xi_t - \frac{U_{xxx}(A_t)}{2U_{xx}^2(A_t)} (d\lambda\xi_t)^2 \\ &= \frac{U_x(A_t)}{U_{xx}(A_t)} (\alpha_t dt + \beta_t^\top dz_t) - \frac{U_{xxx}(A_t) U_x^2(A_t)}{2U_{xx}^3(A_t)} \beta_t^\top \beta_t \; dt \\ &= (aA_t + b) \left[\left(\tfrac{1}{2}(a+1)\beta_t^\top \beta_t - \alpha_t \right) dt - \beta_t^\top dz_t \right]. \end{aligned}$$

If $U(x)$ is quadratic, then $a = -1$, therefore

$$\frac{d(b - A_t)}{(b - A_t)} = \frac{d\xi_t}{\xi_t}.$$

Hence, there is a constant c such that $A_t = b - c\, \xi_t$. This proves that

$$W_T = b - c\, \xi_T.$$

Compared with Theorem 3, we have also proved that mean-variance analysis is consistent with quadratic utility maximization.

Now let $U(x)$ be a concave and increasing utility function (second-order differentiable) and the associated optimal portfolio return is $\tilde{R}$. If $\tilde{R}$ is global mean-variance efficient, it follows that

$$\frac{E[\tilde{R}] - E[\xi_T]^{-1}}{V[\tilde{R}]} = \frac{V[\xi_T]}{E[\xi_T]}$$

which implies that

$$E[\tilde{R}]\; E[\xi_T] - V[\tilde{R}]\; V[\xi_T] = 1.$$

Hence, the covariance of $\tilde{R}$ and ξ_T is

$$\mathrm{Cov}[\tilde{R}, \xi_T] = -V[\tilde{R}]\; V[\xi_T],$$

i.e., $\tilde{R}$ and ξ_T are perfectly negatively correlated. This means that $\tilde{R}$ is a linear function of ξ_T with a negative slope. Hence $\tilde{R}$ is the return of a quadratic utility maximization.

Proof of Theorem 4:

Since a portfolio value process W_t with weight θ_{it} in the ith asset S_{it} at time t is represented as

$$\begin{aligned} \frac{dW_t}{W_t} &= (1 - \theta_t^\top \mathbf{1})\frac{dB_t}{B_t} + \theta_t^\top \frac{dS_t}{S_t} \\ &= -(\alpha_t + \theta_t^\top \Sigma_t)dt + \theta_t^\top \Sigma_t \, dz_t. \end{aligned} \tag{32}$$

Comparing the coefficients of the dz_t terms of (32) and (14) yields (17).

If α_t and β_t are constant over time. The first and second order derivatives of $F(t, \xi)$,

$$\begin{aligned} \frac{\partial F}{\partial t} &= -\tfrac{1}{2}\lambda\alpha e^{\alpha(T-t)} + \tfrac{1}{2}\rho\xi(\beta^\top\beta + 2\alpha)e^{(\beta^\top\beta+2\alpha)(T-t)} \\ \frac{\partial F}{\partial \xi} &= -\tfrac{1}{2}\rho e^{(\beta^\top\beta+2\alpha)(T-t)} \\ \frac{\partial^2 F}{\partial \xi^2} &= 0 \end{aligned}$$

satisfy Equation (16) and the associated boundary condition. So, $F(t, \xi)$ is the solution to the partial differential equation. Therefore, the optimal portfolio policy is given by (17).

References

Algoet, P., and T. Cover, 1988, Asymptotic Optimality and Asymptotic Equipartion Properties of Log-Optimum Investment, *Annals of Probability*, 16, 876–898.

Basak, S., 1995, A General Equilibrium Model of Portfolio Insurance, *Review of Financial Studies*, 8, 1059–1090.

Breiman, L., 1961, Optimal Gambling System for Favorable Games, *Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics and Probability*, 1, 63–68.

Cox, J., and C.-f. Huang, 1989, Optimal Consumption and Portfolio Polices When Asset Prices Follow a Diffusion Process, *Journal of Economic Theory*, 49, 33–83.

Dybvig, P. H., and C.-f. Huang, 1988, Nonnegative Wealth, Absence of Arbitrage, and Feasible Consumption Plans, *Review of Financial Studies*, 1(4), 377–401.

Grauer, R. R., 1981, A Comparison of Growth Optimal and Mean Variance Investment Policies, *Journal of Financial and Quantitative Analysis*, 16, 1–21.

Grauer, R. R., and N. H. Hakansson, 1993, On the Use of Mean-Variance and Quadratic Approximations in Implementing Dynamic Investment Strategies: A Comparison of the Returns and Investment Policies, *Management Science*, 39, 856–871.

Grossman, S., and Z. Zhou, 1996, Equilibrium Analysis of Portfolio Insurance, *Journal of Finance*, 51, 1379–1403.

Hakansson, N. H., 1971, Capital Growth and the Mean-Variance Approach to Portfolio Selection, *Journal of Financial and Quantitative Analysis*, 6, 517–557.

Hakansson, N. H., and W. T. Ziemba, 1995, Capital Growth Theory, in *Finance*, ed. by R. Jarrow, V. Maksimovic, and W. Ziemba. Elsevier, North-Holland, vol. 9, pp. 61–86.

Harrison, M., and D. Kreps, 1979, Martingale and Multiperiod Securities Markets, *Journal of Economic Theory*, 20, 382–408.

Harrison, M., and S. Pliska, 1981, Martingales and Stochastic Integrals in the Theory of Continuous Trading, *Stochastic Process Appl.*, 11, 215–260.

Kallberg, J., and W. Ziemba, 1983, Comparison of Alternative Utility Functions in Portfolio Selection Problems, *Management Science*, 29(11), 1257–1276.

Kroll, Y., H. Levy, and H. Markowitz, 1984, Mean-Varaince versus Direct Utility Maximization, *Journal of Finance*, 39, 47–75.

Levy, H., and H. Markowitz, 1979, Approximating Expected Utility by a Function of Mean and Variance, *American Economic Review*, 69, 308–317.

Lintner, J., 1965, The Valuation of Risk Assets and the Selection of Risky Investment in Stock Portfolios and Capital Budgets, *Review of Economics and Statistics*, XLVII, 13–37.

MacLean, L., and W. Ziemba, 1999, Growth Versus Security Tradeoffs in Dynamic Investment Analysis, *Annals of Operations Research*, 85, 193–225.

MacLean, L., W. Ziemba, and G. Blazenko, 1992, Growth Versus Security in dynamic investment analysis, *Management Science*, 38, 1562–1585.

Markowitz, H., 1952, Portfolio Selection, *Journal of Finance*, 7(1), 77–91.

Merton, R., 1973, An Intertermporal Capital Asset Pricing Model, *Econometrica*, 41(5), 867–887.

——— , 1992, *Continuous - Time Finance*. Blackwell, Oxford.

Øksendal, B., 1995, *Stochastic Partial Differential Equations*. Springer-Verlag New York.

Pliska, S., 1986, A Stochastic Calculus Model of Continuous Trading: Optimal Portfolio, *Mathematics of Operations Research*, 11, 371–382.

Ross, S. A., 1978, The Current Status of the Capital Asset Pricing Model (CAPM), *Journal of Finance*, 33, 885–901.

Sharpe, W., 1964, Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk, *Journal of Finance*, 19, 425–442.

Tobin, J., 1958, Liquidity Preference as Behaviour Towards Risk, *Review of Economic Statistics*, 25(2), 65–86.

Ziemba, W., and J. Mulvey, 1998, *World Wide Asset and Liability Modeling*. Cambridge University Press.

Ziemba, W., C. Parkan, and R. Brook-Hill, 1974, Calculation of Investment Portfolio with Risk Free Borrowing and Lending, *Management Science*, 21(2), 209–222.

Ziemba, W., and R. Vickson, 1975, *Stochastic Optimization Models in Finance*. Academic Press.