TOWARDS THE MAXIMAL (DOOB, BESSEL, HARDY-LITTLEWOOD, ...) INEQUALITIES: OPTIMAL STOPPING AND FREE BOUNDARY PROBLEMS METHOD

ALBERT N. SHIRYAEV

If $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, then it is known that the following *maximal inequality* holds:

$$\mathbb{E} \max_{t\leq T} |B_t| = \sqrt{\frac{\pi}{2}T}$$

for every deterministic time T. Suppose now that instead of the deterministic time T we are given some (random) *stopping time* of B. The question then arises naturally as to determine $\mathbb{E}\max_{t\leq\tau}|B_t|$. On closer inspection, however, it becomes clear that it is virtually impossible to compute this expectation for every stopping time τ of B. Thus, one can try to bound the expectation with a quantity which is easier computed. This problem leads to the following *maximal inequality*:

$$\mathbb{E} \max_{t\leq \tau} |B_t| \leq c\sqrt{\mathbb{E}\tau}$$

which is valid for all stopping times τ of B.

We show that the best constant is $c = \sqrt{2}$. Our method is based on reformulation of the problem to the solving of the following optimal stopping problem

$$V = \sup_{\tau} \mathbb{E}(\max_{t\leq\tau} |B_t| - c\tau)$$

where the supremum is taken over all stopping times of B satisfying $\mathbb{E}\tau < \infty$ and the constant $c > 0$ is given and fixed.

The solution of this problem can be reduced to a free-boundary problem. In our talk we describe formulated ways of solutions to the different kinds of maximal inequalities and different kinds of stochastic processes generated by a Brownian motion.