THE ORBIT METHOD FOR THE JACOBI GROUP

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1 Introduction

Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$. We may identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ (cf. [Vo3], Proposition 2.7). More precisely the real valued symmetric bilinear form $<\ ,\ >: \mathfrak{g} \times \mathfrak{g} \longrightarrow$ defined by

$$(1.1) \qquad < X, Y >= \mathrm{Re}\,\mathrm{tr}\,(XY), \quad X, Y \in \mathfrak{g}$$

is nondegenerate and hence there exists a G-equivariant linear isomorphism

$$\mathfrak{g}^* \longrightarrow \mathfrak{g}, \quad \lambda \mapsto X_\lambda$$

characterized by $\lambda(Y) =< X_\lambda, Y >,\ Y \in \mathfrak{g}$. Therefore the coadjoint G-orbits in $\mathfrak{g}^*$ may be identified with adjoint G-orbits in $\mathfrak{g}$. The philosophy of the orbit method says that we may attach the irreducible unitary representations of G to the coadjoint orbits in $\mathfrak{g}^*$. Historically the orbit method that was first initiated by A.A. Kirillov (cf. [K]) early in the 1960s in a real nilpotent Lie group worked beautifully. Thereafter the orbit method was extended nicely to a solvable Lie group of type I by Auslander and Kostant (cf. [A-K]). Their proof was based on the existence of complex polarizations satisfying a positivity condition. Unfortunately Kirillov's work fails to be generalized in some ways to the case of compact Lie groups or semisimple Lie groups. Relatively simple groups like $SL(2, \mathbb{R})$ have irreducible unitary representations that do not correspond to any symplectic homogeneous space. Conversely, P. Torasso [T] found that the double cover of $SL(3, \mathbb{R})$ has a homogeneous symplectic manifold corresponding to no unitary representations. The orbit method for reductive Lie groups is a kind of a philosophy but not a theorem. Many large families of orbits correspond in comprehensible ways to unitary representations, and provide a clear geometric picture of these representations. The coadjoint orbits for a reductive Lie group are classified into three kinds of orbits, namely, hyperbolic, elliptic and nilpotent ones. The hyperbolic orbits are related to the unitary representations obtained by the parabolic induction and on the other hand, the elliptic ones are related to the unitary representations obtained by the cohomological induction. However, we still have no idea of attaching unitary representations to nilpotent orbits. It is known that there are only finitely many nilpotent orbits. In a certain case, some nilpotent orbits are corresponded to the so-called *unipotent representations*. For instance, a minimal nilpotent orbit is attached to a minimal representation. In fact, the notion of unipotent representations is not still well defined. The investigation of unipotent representations is now under way.

In this paper, we study the orbit method for the Jacobi group G^J. The Jacobi group G^J is a semidirect product of the symplectic group and the Heisenberg group. For a detail, we refer to Section 3 in this paper. G^J is *not* a reductive Lie group. The real-valued symmetric bilinear form $< \, , \, >: \mathfrak{g}^J \times \mathfrak{g}^J \longrightarrow \mathbb{R}$ defined by $< X, Y >= \mathrm{tr}\,(XY), \ X, Y \in \mathfrak{g}^J$ (cf. (1.1)) is highly degenerate. Therefore in the Jacobi group G^J, we can not identify the Lie algebra $\mathfrak{g}^J$ of G^J with its dual $(\mathfrak{g}^J)^*$. We may not expect nice properties which are obtained in the reductive case. In fact, there are infinitely many nilpotent G^J-orbits and there is no correspondence like the Kostant-Sekiguchi correspondence occurring in the reductive case. The paper is organized as follows. In Section 2, we review the orbits and the Kostant-Sekiguchi correspondence for the group $SL(2,\mathbb{R})$. In Section 3, we investigate the adjoint nilpotent G^J-orbits in $\mathfrak{g}^J$ explicitly. In particular, we provide an injective mapping from the set of nilpotent G^J-orbits in $\mathfrak{g}$ to the set of nilpotent $K^J_{\mathbb{C}}$-orbits in $\mathfrak{p}^J_{\mathbb{C}}$. In the final section, we investigate the coadjoint G^J-orbits in $(\mathfrak{g}^J)^*$ explicitly. We review the unitary representations of $SL(2,\mathbb{R})$ and G^J. And then we attach to these orbits unitary representations of G^J .

NOTATIONS: We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. We denote by $\mathbb{R}^\times$ and $\mathbb{C}^\times$ the set of nonzero real numbers and the set of nonzero complex numbers respectively. We denote by $\mathbb{Z}^+$ (resp. $\mathbb{Z}_{\geq 0}$) the set of all positive (resp. nonnegative) integers, by $F^{(k,l)}$ the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. We denote the identity matrix of degree k by I_k.

2 THE KOSTANT-SEKIGUCHI CORRESPONDENCE FOR $SL(2,\mathbb{R})$

In this section, we review the Kostant-Sekiguchi correspondence for the special linear group $SL(2,\mathbb{R})$.

For brevity, we write $G = SL(2,\mathbb{R})$ and let $K = SO(2)$ be a maximal compact subgroup of G. The Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}. \tag{2.1}$$

We put

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.2}$$

Then the set $\{X, Y, Z\}$ forms a basis for $\mathfrak{g}$. We define an element $F(x,y,z) \in \mathfrak{g}$ by

$$F(x,y,z) := xX + yY + zZ = X = \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}. \tag{2.3}$$

Then we have the relations

$$X^2 + Y^2 - Z^2 = 3I_2, \quad [X,Y] = 2Z, \quad [X,Z] = 2Y, \quad [Y,Z] = -2X. \tag{2.4}$$

It is easy to see that X and Y are hyperbolic elements and Z is an elliptic element. For a nonzero real number α, the G-orbit of αX is represented by the one-sheeted hyperboloid

$$x^2 + y^2 - z^2 = \alpha^2. \tag{2.5}$$

The G-orbit of $\alpha Y\ (\alpha \in \mathbb{R}^\times)$ is also represented by the hyperboloid (2.5). The G-orbit of $\alpha Z\ (\alpha \in \mathbb{R}^\times)$ is represented by two-sheeted hyperboloids

$$x^2 + y^2 - z^2 = -\alpha^2. \tag{2.6}$$

Since

$$F(x,y,z)^2 = (x^2 + y^2 - z^2) \cdot I_2,$$

we have for any $k \in \mathbb{Z}^+$,

$$F(x,y,z)^{2k} = (x^2 + y^2 - z^2)^k \cdot I_2.$$

Thus we see that $F(x,y,z)$ is nilpotent if and only if $x^2 + y^2 - z^2 = 0$. Therefore the set $\mathcal{N}_\mathbb{R}$ of all nilpotent elements in $\mathfrak{g}$ is given by

$$\mathcal{N}_\mathbb{R} = \left\{ F(x,y,z) = \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix} \,\Big|\, x^2 + y^2 - z^2 = 0 \right\}. \tag{2.7}$$

We put

$$S = \frac{1}{2}(Y+Z) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T = \frac{1}{2}(Y-Z) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

Obviously S and T are nilpotent elements in $\mathcal{N}_\mathbb{R}$. The the G-orbit of $\alpha S\ (\alpha \in \mathbb{R}^\times)$ is represented by the cone

$$x^2 + y^2 - z^2 = 0, \quad (x,y,z) \neq (0,0,0) \tag{2.9}$$

depending on the sign of α.

If $\alpha > 0$, the G-orbit of αS is characterized by the one-sheeted cone

$$x^2 + y^2 - z^2 = 0, \quad z > 0. \tag{2.10}$$

If $\alpha < 0$, the G-orbit of αS is characterized by the one-sheeted cone

$$x^2 + y^2 - z^2 = 0, \quad z < 0. \tag{2.11}$$

The G-orbits of $\alpha T\ (\alpha > 0)$ are characterized by the one-sheeted cone (2.11) and the G-orbits of $\alpha T\ (\alpha < 0)$ are characterized by the one-sheeted cone (2.10).

We define the G-orbits $\mathcal{N}_\mathbb{R}^+$ and $\mathcal{N}_\mathbb{R}^-$ by

$$\mathcal{N}_\mathbb{R}^+ = G \cdot S \quad \text{and} \quad \mathcal{N}_\mathbb{R}^- = G \cdot T. \tag{2.12}$$

Then we obtain

$$\mathcal{N}_\mathbb{R} = \mathcal{N}_\mathbb{R}^+ \cup \{0\} \cup \mathcal{N}_\mathbb{R}^-. \tag{2.13}$$

According to (2.5), (2.6) and (2.13), we see that there are infinitely many hyperbolic orbits and elliptic orbits, and on the other hand there are only three nilpotent orbits in $\mathfrak{g}$.

Let
$$K_{\mathbb{C}} = SO(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\Big|\, a^2+b^2=1,\ a,b\in\mathbb{C} \right\}$$
be the complexification of K. The complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ has the Cartan decomposition
$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}},$$
where
$$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix} \,\Big|\, z\in\mathbb{C} \right\}$$
and
$$\mathfrak{p}_{\mathbb{C}} = \left\{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \,\Big|\, x,y\in\mathbb{C} \right\}.$$
The set $\mathcal{N}_\theta$ of all nilpotent elements in $\mathfrak{p}_{\mathbb{C}}$ is given by
$$\mathcal{N}_\theta = \left\{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \in \mathfrak{p}_{\mathbb{C}} \,\Big|\, x^2+y^2=0 \right\} \subset \mathfrak{p}_{\mathbb{C}}. \tag{2.14}$$
We put
$$H_\theta = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad X_\theta = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad Y_\theta = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}. \tag{2.15}$$
Then the set $\{H_\theta,\, Y_\theta,\, X_\theta\}$ is a normal triple, that is, a triple $\{H_\theta,\, Y_\theta,\, X_\theta\}$ satisfies properties (2.16) and (2.17):
$$H_\theta \in \mathfrak{k}_{\mathbb{C}}, \quad X_\theta,\, Y_\theta \in \mathfrak{p}_{\mathbb{C}} \tag{2.16}$$
and
$$[H_\theta, Y_\theta] = 2Y_\theta, \quad [H_\theta, X_\theta] = -2X_\theta, \quad [Y_\theta, X_\theta] = H_\theta. \tag{2.17}$$
Moreover a triple $\{H_\theta,\, Y_\theta,\, X_\theta\}$ satisfies the following property
$$\sigma_0(H_\theta) = -H_\theta, \quad \sigma_0(X_\theta) = Y_\theta, \quad \sigma_0(Y_\theta) = X_\theta, \tag{2.18}$$
where σ_0 denotes the complex conjugation on $\mathfrak{g}_{\mathbb{C}}$.

We note that $K_{\mathbb{C}}$ acts on $\mathcal{N}_\theta$. Now we define the $K_{\mathbb{C}}$-orbits $\mathcal{N}_\theta^+$ and $\mathcal{N}_\theta^-$ by
$$\mathcal{N}_\theta^+ = K_{\mathbb{C}} \cdot X_\theta \quad \text{and} \quad \mathcal{N}_\theta^- = K_{\mathbb{C}} \cdot Y_\theta. \tag{2.19}$$
Then we see that
$$\mathcal{N}_\theta = \mathcal{N}_\theta^+ \cup \{0\} \cup \mathcal{N}_\theta^-. \tag{2.20}$$
The $K_{\mathbb{C}}$-orbit $\mathcal{N}_\theta^+$ is characterized by the straight line
$$y = ix, \quad x \in \mathbb{C} - \{0\}. \tag{2.21}$$
On the other hand, The $K_{\mathbb{C}}$-orbit $\mathcal{N}_\theta^-$ is characterized by the straight line
$$y = -ix, \quad x \in \mathbb{C} - \{0\}. \tag{2.22}$$
It is easily seen that the $K_{\mathbb{C}}$-orbits of $\alpha H_\theta\ (\alpha \in \mathbb{C}^\times)$ are represented by complex hyperboloids and that there are infinitely many hyperbolic and elliptic orbits in $\mathfrak{g}_{\mathbb{C}}$. However there are only three nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$ which are $\mathcal{N}_\theta^+$, $\{0\}$ and $\mathcal{N}_\theta^-$.

The correspondence between the G-nilpotent orbits in $\mathcal{N}_{\mathbb{R}}$ and the $K_{\mathbb{C}}$-nilpotent orbits in $\mathcal{N}_\theta$ given by

$$\mathcal{N}_{\mathbb{R}}^{+} \mapsto \mathcal{N}_{\theta}^{+}, \quad \{0\} \mapsto \{0\}, \quad \mathcal{N}_{\mathbb{R}}^{-} \mapsto \mathcal{N}_{\theta}^{-} \tag{2.23}$$

is the so-called Kostant-Sekiguchi correspondence (cf. [S-V], [Se], [Ve], [Vo1-3]). M. Vergne [Ve] proved that each of the mappings

$$\mathcal{N}_{\mathbb{R}}^{+} \longrightarrow \mathcal{N}_{\theta}^{+}, \quad \{0\} \longrightarrow \{0\}, \quad \mathcal{N}_{\mathbb{R}}^{-} \longrightarrow \mathcal{N}_{\theta}^{-} \tag{2.24}$$

is a K-equivariant diffeomorphism.

3 The Adjoint Orbits for the Jacobi Group

We consider the Heisenberg group

$$H_{\mathbb{R}} = \left\{ (\lambda, \mu, \kappa) \,\middle|\, \lambda, \mu, \kappa \in \mathbb{R} \right\}$$

with the following multiplication law

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \lambda'\mu). \tag{3.1}$$

We let

$$G^J = SL(2, \mathbb{R}) \ltimes H_{\mathbb{R}}$$

be the semidirect product of $SL(2, \mathbb{R})$ and $H_{\mathbb{R}}$ endowed with the multiplication law

$$(M, (\lambda, \mu, \kappa)) \cdot (M', (\lambda', \mu', \kappa')) = (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu', \kappa + \kappa' + \tilde{\lambda}\mu' - \lambda'\tilde{\mu})) \tag{3.2}$$

with $M, M' \in SL(2, \mathbb{R}), (\lambda, \mu, \kappa), (\lambda', \mu', \kappa') \in H_{\mathbb{R}}$ and $(\tilde{\lambda}, \tilde{\mu}) := (\lambda, \mu)M'$. The group G^J is called the *Jacobi group*.

Let $\mathbb{H}$ be the Poincaré upper half plane. We see that G^J acts a nonsymmetric homogeneous space $\mathbb{H} \times \mathbb{C}$ transitively by

$$(M, (\lambda, \mu, \kappa)) \cdot (\tau, z) = (M < \tau >, (z + \lambda\tau + \mu)(c\tau + d)^{-1}), \tag{3.3}$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),\ (\lambda, \mu, \kappa) \in H_{\mathbb{R}}$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ and

$$M < \tau >= (a\tau + b)(c\tau + d)^{-1}. \tag{3.4}$$

It is easily seen that the stabilizer K^J of G^J at $(i, 0)$ under the action (3.3) is given by

$$K^J = \{(k, (0, 0, \kappa)) |\ k \in K,\ \kappa \in \mathbb{R}\} \cong K \times \mathbb{R}, \tag{3.5}$$

where $K = SO(2)$ is a maximal compact subgroup of $G = SL(2, \mathbb{R})$. We observe that the Jacobi group G^J is embedded in the symplectic group $Sp(4, \mathbb{R})$ via

$$(M, (\lambda, \mu, \kappa)) \mapsto \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.6}$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. The Lie algebra $\mathfrak{g}^J$ of G^J is given by

$$\mathfrak{g}^J = \{(X,(p,q,r)) \mid X \in \mathfrak{g},\ p,q,r \in \mathbb{R}\} \tag{3.7}$$

with the bracket

$$[(X_1,(p_1,q_1,r_1)),\ (X_2,(p_2,q_2,r_2))] = (\tilde{X},(\tilde{p},\tilde{q},\tilde{r})), \tag{3.8}$$

where

$$X_1 = \begin{pmatrix} x_1 & y_1 \\ z_1 & -x_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} x_2 & y_2 \\ z_2 & -x_2 \end{pmatrix} \in \mathfrak{g}$$

and

$$\begin{aligned} \tilde{X} &= X_1X_2 - X_2X_1, \\ \tilde{p} &= p_1x_2 + q_1z_2 - p_2x_1 - q_2z_1, \\ \tilde{q} &= q_2x_1 + p_1y_2 - q_1x_2 - p_2y_1, \\ \tilde{r} &= 2(p_1q_2 - p_2q_1). \end{aligned}$$

Indeed, an element $(X,(p,q,r))$ in $\mathfrak{g}^J$ with $X = \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix} \in \mathfrak{g}$ may be identified with the matrix

$$G(x,y,z,p,q,r) := \begin{pmatrix} x & 0 & y+z & q \\ p & 0 & q & r \\ y-z & 0 & -x & -p \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.9}$$

in the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ of $Sp(4,\mathbb{R})$.

Lemma 3.1. If $G(x,y,z,p,q,r)$ is an element in $\mathfrak{g}^J$ given by (3.9), then for a positive integer $k \in \mathbb{Z}^+$,

$$G(x,y,z,p,q,r)^{2k} = (x^2+y^2-z^2)^{k-1}G(x,y,z,p,q,r)^2. \tag{3.10}$$

Proof. By a direct computation, we obtain

$$G(x,y,z,p,q,r)^4 = (x^2+y^2-z^2)G(x,y,z,p,q,r)^2.$$

The formula (3.10) follows immediately from (3.11). □

According to Lemma 3.1, the set $\mathcal{N}_{\mathbb{R}}^J$ of all nilpotent elements in $\mathfrak{g}^J$ is given by

$$\mathcal{N}_{\mathbb{R}}^J = \left\{ G(x,y,z,p,q,r) \in \mathfrak{g}^J \mid x^2+y^2-z^2 = 0 \right\}. \tag{3.12}$$

We have the adjoint action of G^J on $\mathfrak{g}^J$ given by

$$g \cdot X = Ad(g)X = gXg^{-1}, \quad g \in G^J,\ X \in \mathcal{N}_{\mathbb{R}}^J. \tag{3.13}$$

According to (3.6), we may write $g = (M, (\lambda, \mu, \kappa)) \in G^J$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ as

$$g = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.14}$$

Then the inverse of g is given by

$$g^{-1} = \begin{pmatrix} d & 0 & -b & -\mu \\ c\mu - d\lambda & 1 & b\lambda - a\mu & -\kappa \\ -c & 0 & a & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.15}$$

Lemma 3.2. If $g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu, \kappa) \right)$ is an element of G^J, then the action of g on $G(x, y, z, p, q, r)$ is given by

$$g \cdot G(x, y, z, p, q, r) = G(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r}), \tag{3.16}$$

where

$$\begin{aligned} \tilde{x} =& (ad + bc)x - ac(y + z) + bd(y - z), \\ \tilde{y} + \tilde{z} =& -2abx + a^2(y + z) - b^2(y - z), \\ \tilde{y} - \tilde{z} =& 2cdx - c^2(y + z) + d^2(y - z), \\ \tilde{p} =& d\left\{\lambda x + \mu(y - z) + p\right\} + c\left\{\mu x - \lambda(y + z) - q\right\}, \\ \tilde{q} =& -b\left\{\lambda x + \mu(y - z) + p\right\} - a\left\{\mu x - \lambda(y + z) - q\right\}, \\ \tilde{r} =& -2\lambda\mu x + \lambda^2(y + z) - \mu^2(y - z) - 2p\mu + 2q\lambda + r. \end{aligned}$$

In particular, $\mathcal{N}_{\mathbb{R}}^J$ is stable under the action of G^J.

Proof. The proof of the first part follows from a direct computation. Let $G(x, y, z, p, q, r)$ be an element of $\mathcal{N}_{\mathbb{R}}^J$. Since

$$\tilde{x}^2 + \tilde{y}^2 - \tilde{z}^2 = (ad - bc)^2(x^2 + y^2 - z^2) = 0,$$

we see that $g \cdot \mathcal{N}_{\mathbb{R}}^J \subset \mathcal{N}_{\mathbb{R}}^J$ for all $g \in \mathcal{N}_{\mathbb{R}}^J$. □

We set

$$\begin{aligned} X^J =& G(1, 0, 0, 0, 0, 0), \\ Y^J =& G(0, 1, 0, 0, 0, 0), \\ Z^J =& G(0, 0, 1, 0, 0, 0), \\ P^J =& G(0, 0, 0, 1, 0, 0), \\ Q^J =& G(0, 0, 0, 0, 1, 0), \\ R^J =& G(0, 0, 0, 0, 0, 1). \end{aligned}$$

Obviously the set $\{X^J, Y^J, Z^J, P^J, Q^J, R^J\}$ forms a basis for $\mathfrak{g}^J$. So we have

$$G(x,y,z,p,q,r) = xX^J + yY^J + zZ^J + pP^J + qQ^J + rR^J.$$

We note that X^J and Y^J are hyperbolic elements, Z^J is an elliptic element and P, Q, R are nilpotent elements.

Let $\alpha \in \mathbb{R}$ be a fixed nonzero real number. According to Lemma 3.2, the G^J-orbit $\Pi(\alpha X^J)$ or $\Pi(\alpha Y^J)$ of αX^J or αY^J is represented by the equations

$$x^2 + y^2 - z^2 = \alpha^2, \quad 2pqx - (p^2 - q^2)y - (p^2 + q^2)z - \alpha^2 r = 0. \tag{3.17}$$

The G^J-orbit $\Pi(\alpha Z^J)$ of αZ^J is represented by the equations

$$x^2 + y^2 = z^2 - \alpha^2, \quad 2pqx - (p^2 - q^2)y + (p^2 + q^2)z + \alpha^2 r = 0. \tag{3.18}$$

The G^J-orbit $\Pi(\alpha P^J)$ or $\Pi(\alpha Q^J)$ of αP^J or αP^J is represented by the equations

$$x = y = z = 0, \quad (p,q) \neq (0,0), \; r \in \mathbb{R}. \tag{3.19}$$

The G^J-orbit $\Pi(\alpha R^J)$ of αR^J is just a single point, that is,

$$\Pi(\alpha R) = G^J \cdot (\alpha R^J) = \{\alpha R^J\}. \tag{3.20}$$

We define the nilpotent elements S^J and T^J by

$$S^J = \frac{1}{2}(Y^J + Z^J) \quad \text{and} \quad T^J = \frac{1}{2}(Y^J - Z^J). \tag{3.21}$$

The G^J-orbit $\Pi(\alpha S^J)$ or $\Pi(\alpha T^J)$ of αS^J or αT^J is represented by

$$x^2 + y^2 = z^2 > 0, \quad xr - pq = 0. \tag{3.22}$$

If $\alpha > 0$, the G^J-orbit $\Pi(\alpha S^J)$ of αS^J is represented by the variety

$$x^2 + y^2 = z^2, \; z > 0, \quad xr - pq = 0, \; r \geq 0. \tag{3.23}$$

If $\alpha < 0$, the G^J-orbit $\Pi(\alpha S^J)$ of αS^J is represented by the variety

$$x^2 + y^2 = z^2, \; z < 0, \quad xr - pq = 0, \; r \leq 0. \tag{3.24}$$

If $\alpha > 0$, the G^J-orbit $\Pi(\alpha T^J)$ of αT^J is represented by the variety (3.24) and if $\alpha < 0$, the G^J-orbit $\Pi(\alpha T^J)$ of αT^J is represented by the variety (3.23). We see that there are infinitely many nilpotent G^J-orbits, that is, $\Pi(S^J)$, $\Pi(T^J)$, $\Pi(\alpha R^J)\,(\alpha \in \mathbb{R}^\times)$, $\{0\}$.

In summary, we obtain the following theorem.

Theorem 3.3. We have a disjoint union

$$\mathcal{N}_{\mathbb{R}}^J = \Pi(S^J) \cup \Pi(T^J) \cup \{0\} \cup \Pi(P^J) \cup \left(\cup_{\alpha\in\mathbb{R}^\times} \left\{\Pi(\alpha R^J)\right\}\right).$$

In particular, there are infinitely many nilpotent G^J-orbits in $\mathcal{N}_{\mathbb{R}}^J \subset \mathfrak{g}^J$.

We let

$$\mathcal{N}_{\mathbb{R}}^{J,2} = \left\{G(0,0,0,p,q,r)|\ (p,q,r) \in \mathbb{R}^3\right\}.$$

Then G^J acts on $\mathcal{N}_{\mathbb{R}}^{J,2}$ and the orbit space $\mathcal{N}_{\mathbb{R}}^{J,2}/\!\!/G^J$ is infinite. Precisely we have the following disjoint union

$$\mathcal{N}_{\mathbb{R}}^{J,2} = \Pi(P^J) \cup \{0\} \cup \left(\cup_{\alpha\in\mathbb{R}^\times} \left\{\Pi(\alpha R^J)\right\}\right).$$

For $x, y, p, q, r \in \mathbb{C}$, we set

$$H(x,y,p,q,r) := \begin{pmatrix} x & 0 & y & q \\ p & 0 & q & r \\ y & 0 & -x & -p \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.25}$$

Let $\mathfrak{p}_{\mathbb{C}}^J$ be the vector space consisting of all $H(x,y,p,q,r)$ $(x,y,z,p,q,r \in \mathbb{C})$. Then the Cartan decomposition of the complexification $\mathfrak{g}_{\mathbb{C}}^J$ of $\mathfrak{g}^J$ is

$$\mathfrak{g}_{\mathbb{C}}^J = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}^J, \tag{3.26}$$

where

$$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{C} \right\}$$

is the complexification of the Lie algebra $\mathfrak{k}$ of K. Let $\mathcal{N}_\theta^J$ be the set of all nilpotent elements in $\mathfrak{p}_{\mathbb{C}}$. Then

$$\mathcal{N}_\theta^J = \left\{H(x,y,p,q,r) \in \mathfrak{p}_{\mathbb{C}}^J|\ x^2+y^2 = 0\right\}. \tag{3.27}$$

Indeed, (3.27) follows from the fact that

$$H(x,y,p,q,r)^{2k} = (x^2+y^2)^{k-1} H(x,y,p,q,r)^2 \quad \text{for all } k \in \mathbb{Z}^+. \tag{3.28}$$

We set

$$\begin{aligned} X_\theta^J &= \frac{1}{2} H(1,i,0,0,0), \\ Y_\theta^J &= \frac{1}{2} H(1,-i,0,0,0), \\ P_\theta^J &= H(0,0,1,0,0), \\ Q_\theta^J &= H(0,0,0,1,0), \\ R_\theta^J &= H(0,0,0,0,1). \end{aligned}$$

Obviously the set $\left\{X_\theta^J,\, Y_\theta^J,\, P_\theta^J,\, Q_\theta^J,\, R_\theta^J\right\}$ forms a basis for a complex vector space $\mathfrak{p}_{\mathbb{C}}^J$.

Proposition 3.4. $K^J_{\mathbb{C}}$ acts on $\mathfrak{p}^J_{\mathbb{C}}$ preserving $\mathcal{N}^J_\theta$.

Proof. An element k^J of $K^J_{\mathbb{C}}$ is of the form

$$k^J = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & \kappa \\ -b & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a, b, \kappa \in \mathbb{C},\ a^2 + b^2 = 1. \tag{3.29}$$

We obtain

$$k^J \cdot H(x, y, p, q, r) = H(x^*, y^*, p^*, q^*, r^*),$$

where

$$\begin{aligned} x^* =& (a^2 - b^2)x + 2aby, \\ y^* =& -2abx + (a^2 - b^2)y, \\ p^* =& ap + bq, \\ q^* =& aq - bp, \\ r^* =& r. \end{aligned} \tag{3.30}$$

If $H(x, y, p, q, r) \in \mathcal{N}^J_\theta$, then

$$(x^*)^2 + (y^*)^2 = (a^2 + b^2)^2 (x^2 + y^2) = 0.$$

Therefore $k^J \cdot \mathcal{N}^J_\theta \subset \mathcal{N}^J_\theta$ for each element $k^J \in K^J_{\mathbb{C}}$. □

We define the $K^J_{\mathbb{C}}$-orbits $\mathcal{N}^{J,+}_\theta$ and $\mathcal{N}^{J,-}_\theta$ by

$$\mathcal{N}^{J,+}_\theta = K^J_{\mathbb{C}} \cdot X^J_\theta \quad \text{and} \quad \mathcal{N}^{J,-}_\theta = K^J_{\mathbb{C}} \cdot Y^J_\theta.$$

It is easily checked that $\mathcal{N}^{J,+}_\theta$ and $\mathcal{N}^{J,-}_\theta$ are given by

$$\mathcal{N}^{J,+}_\theta = \{ H(x, ix, 0, 0, 0) \mid x \neq 0,\ x \in \mathbb{C} \} \tag{3.31}$$

and

$$\mathcal{N}^{J,-}_\theta = \{ H(x, -ix, 0, 0, 0) \mid x \neq 0,\ x \in \mathbb{C} \}. \tag{3.32}$$

We define, for a nonzero complex number $\delta \in \mathbb{C}$,

$$\mathcal{N}^{J,P}_\theta(\delta) = K^J_{\mathbb{C}} \cdot \delta P^J_\theta \quad \text{and} \quad \mathcal{N}^{J,Q}_\theta(\delta) = K^J_{\mathbb{C}} \cdot \delta Q^J_\theta.$$

Then

$$\mathcal{N}^{J,P}_\theta(\delta) = \mathcal{N}^{J,Q}_\theta(\delta) = \left\{ H(0, 0, p, q, 0) \mid p^2 + q^2 = \delta^2,\ p, q \in \mathbb{C} \right\}. \tag{3.33}$$

We define for a nonzero complex number $c \in \mathbb{C}$

$$\mathcal{N}^{J,R}_\theta(c) = K^J_{\mathbb{C}} \cdot (c R^J_\theta).$$

Then

$$\mathcal{N}^{J,R}_\theta(c) = \left\{ c R^J_\theta \right\} = \{ H(0, 0, 0, 0, c) \}. \tag{3.34}$$

Lemma 3.5. *$H(x, y, \delta, 0, r)$ and $H(\tilde{x}, \tilde{y}, \delta, 0, r)$ lie in the same $K_{\mathbb{C}}$-orbit in $\mathfrak{p}^J_{\mathbb{C}}$ if and only if $\tilde{x} = x,\ \tilde{y} = y$.*

Proof. By (3.30), we have $a = 1$ and $b = 0$. Hence $\tilde{x} = x,\ \tilde{y} = y$. □

Lemma 3.6. Let $(x,y) \in \mathbb{C}^2$ with $(x,y) \neq (0,0)$. Then $H(x,y,\delta,0,r)$ and $H(x,y,\tilde{\delta},0,\tilde{r})$ lie in the same $K^J_{\mathbb{C}}$-orbit in $\mathfrak{p}^J_{\mathbb{C}}$ if and only if $\tilde{\delta} = \pm\delta,\ \tilde{r} = r$.

Proof. According to (3.30), $\tilde{r} = r,\ b = 0$ and so $a = \pm 1$. Since $\tilde{\delta} = a\delta,\ \tilde{\delta} = \pm\delta$.

Lemma 3.7. Suppose $H(x,y,p,q,r) \in \mathcal{N}^J_\theta$ with $y = \xi x$, where $\xi = i$ or $-i$. If $H(\tilde{x},\tilde{y},\tilde{p},\tilde{q},\tilde{r})$ is in the $K_{\mathbb{C}}$-orbit of $H(x,y,p,q,r)$ in $\mathfrak{p}^J_{\mathbb{C}}$, then $\tilde{y} = \xi\tilde{x}$.

Proof. According to (3.30), we obtain

$$\tilde{x} = (a+\xi b)^2 x, \quad \tilde{y} = \xi(a+\xi b)^2 x.$$

Hence $\tilde{y} = \xi\tilde{x}$. □

Lemma 3.8. Let $x,\delta,r \in \mathbb{C}$ with $x \neq 0$. We denote by $\mathcal{N}^{J,+}_\theta(x,\delta,r)$ and $\mathcal{N}^{J,-}_\theta(x,\delta,r)$ the $K_{\mathbb{C}}$-orbits of $H(x,ix,\delta,0,r)$ and $H(x,-ix,\delta,0,r)$ respectively. Then $\mathcal{N}^{J,+}_\theta(x,\delta,r)$ and $\mathcal{N}^{J,-}_\theta(x,\delta,r)$ are given by

$$\mathcal{N}^{J,+}_\theta(x,\delta,r) = \left\{ H(z,iz,p,q,r) \mid z,p,q \in \mathbb{C},\ p^2+q^2 = \delta^2 \right\} \tag{3.35}$$

and

$$\mathcal{N}^{J,-}_\theta(x,\delta,r) = \left\{ H(z,-iz,p,q,r) \mid z,p,q \in \mathbb{C},\ p^2+q^2 = \delta^2 \right\}. \tag{3.36}$$

Proof. If $H(\tilde{x},\tilde{y},\tilde{p},\tilde{q},\tilde{r})$ is an element of $\mathcal{N}^{J,+}_\theta(x,\delta,r)$, then there exist $a,b \in \mathbb{C}$ with $a^2+b^2 = 1$ satisfying

$$\tilde{x} = (a+ib)^2 x,\ \tilde{y} = i(a+ib)^2 x,\ \tilde{p} = a\delta,\ \tilde{q} = -b\delta,\ \tilde{r} = r.$$

Thus $\tilde{y} = i\tilde{x},\ \tilde{p}^2+\tilde{q}^2 = (a^2+b^2)\delta^2 = \delta^2$. Hence we obtain the formula (3.35). In a similar way, we get the formula (3.36). □

According to (3.31)-(3.34), Lemma 3.5 - Lemma 3.8, we obtain the following theorem.

Theorem 3.9. We have the following disjoint union

$$\begin{aligned}\mathcal{N}^J_\theta = & \mathcal{N}^{J,+}_\theta \cup \mathcal{N}^{J,-}_\theta \cup \left(\bigcup_{\substack{x\in\mathbb{C}^\times \\ \delta\in\mathbb{C}^\times/\mathbb{Z}^2 \\ r\in\mathbb{C}}} \mathcal{N}^{J,+}_\theta(x,\delta,r) \right) \\ & \cup \left(\bigcup_{\substack{x\in\mathbb{C}^\times \\ \delta\in\mathbb{C}^\times/\mathbb{Z}^2 \\ r\in\mathbb{C}}} \mathcal{N}^{J,-}_\theta(x,\delta,r) \right) \cup \{0\} \\ & \cup \left(\cup_{\delta\in\mathbb{C}^\times} \mathcal{N}^{J,P}_\theta(\delta) \right) \cup \left(\cup_{c\in\mathbb{C}^\times} \mathcal{N}^{J,R}_\theta(c) \right).\end{aligned}$$

In particular, there are infinitely many nilpotent $K^J_{\mathbb{C}}$-orbits in $\mathcal{N}^J_\theta \subset \mathfrak{p}^J_{\mathbb{C}}$.

Remark 3.10. It is known that if G is a real reductive Lie group, there are only finitely many nilpotent orbits and that there is the so-called Kostant-Sekiguchi correspondence between the set of all nilpotent G-orbits in $\mathfrak{g}$ and the set of all nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$, where $K_{\mathbb{C}}$ is the complexification of a maximal compact subgroup K of G and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$ is the Cartan decomposition of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. But according to Theorem 3.3 and Theorem 3.9, we see that in the Jacobi group there is no correspondence like the Kostant-Sekiguchi correspondence occurring in the reducive case. We may define the injective mapping

$$\Psi : \mathcal{N}_{\mathbb{R}}^J /\!\!/ G^J \longrightarrow \mathcal{N}_{\theta}^J /\!\!/ K_{\mathbb{C}}^J \tag{3.37}$$

by

$$\Psi([\Pi([S^J]) = [\mathcal{N}_{\theta}^{J,+}]), \quad \Psi([\Pi([T^J]) = [\mathcal{N}_{\theta}^{J,-}], \quad \Psi([\{0\}]) = [\{0\}],$$

$$\Psi([\Pi(P^J)]) = [\mathcal{N}_{\theta}^{J,P}(1)], \quad \Psi([\Pi(\alpha R)]) = [\mathcal{N}_{\theta}^{J,R}(\alpha)].$$

Here if Ω is a G^J-orbit in $\mathcal{N}_{\mathbb{R}}^J$ or a $K_{\mathbb{C}}^J$-orbit in $\mathcal{N}_{\theta}^J$, $[\Omega]$ denotes its corresponding point in the orbit space $\mathcal{N}_{\mathbb{R}}^J /\!\!/ G^J$ or $\mathcal{N}_{\theta}^J /\!\!/ K_{\mathbb{C}}^J$.

4 The Coadjoint Orbits for the Jacobi Group

First of all, we note that we may identify the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ of $Sp(4,\mathbb{R})$ with its dual $\mathfrak{sp}(4,\mathbb{R})^*$. Indeed, there exists a $Sp(4,\mathbb{R})$-equivariant linear isomorphism

$$\mathfrak{sp}(4,\mathbb{R})^* \longrightarrow \mathfrak{sp}(4,\mathbb{R}), \quad \lambda \mapsto X_\lambda$$

characterized by

$$\lambda(Y) = \mathrm{Tr}(X_\lambda Y), \quad Y \in \mathfrak{sp}(4,\mathbb{R}). \tag{4.1}$$

For a detail, we refer to [Vo3], Proposition 2.7. Then the dual $(\mathfrak{g}^J)^*$ of $\mathfrak{g}^J$ consists of matrices of the form

$$M(x,y,z,p,q,r) := \begin{pmatrix} x & p & z & 0 \\ 0 & 0 & 0 & 0 \\ y & q & -x & 0 \\ q & r & -p & 0 \end{pmatrix}, \quad x,y,z,p,q,r \in \mathbb{R}. \tag{4.2}$$

Proposition 4.1. If the element g of G^J is given by

$$g^{-1} = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then the coadjoint action of g on $M(x,y,z,p,q,r)$ is given by

$$M(\tilde{x},\tilde{y},\tilde{z},\tilde{p},\tilde{q},\tilde{r}) = Ad^*(g) M(x,y,z,p,q,r), \tag{4.3}$$

where

$$\begin{aligned}
\tilde{x} =&(ad+bc)x+bdy-acz+\{2ac\mu-(ad+bc)\lambda\}\,p\\
&+\{(ad+bc)\mu-2bd\lambda\}\,q+r(a\mu-b\lambda)(c\mu-d\lambda),\\
\tilde{y} =&2cdx+d^2y-c^2z+2(c\mu-d\lambda)(cp+dq)+r(c\mu-d\lambda)^2,\\
\tilde{z} =&-2abx-b^2y+a^2z-2(a\mu-b\lambda)(ap+bq)-r(a\mu-b\lambda)^2,\\
\tilde{p} =&ap+bq+(a\mu-b\lambda)r,\\
\tilde{q} =&cp+dq+(c\mu-d\lambda)r,\\
\tilde{r} =&r.
\end{aligned}$$

Proof. Since $Ad^*(g)M(x,y,z,p,q,r) = g^{-1}M(x,y,z,p,q,r)g$, we obtain the formula (4.3) from a direct computation. ☐

We put

$$\begin{aligned}
X_* =&M(1,0,0,0,0,0),\\
Y_* =&M(0,1,1,0,0,0),\\
Z_* =&M(0,-1,1,0,0,0),\\
P_* =&M(0,0,0,1,0,0),\\
Q_* =&M(0,0,0,0,1,0),\\
R_* =&M(0,0,0,0,0,1).
\end{aligned}$$

The set $\{\,X_*,\,Y_*,\,Z_*,\,P_*,\,Q_*,\,R_*\,\}$ forms a basis for $(\mathfrak{g}^J)^*$. For an element of $(\mathfrak{g}^J)^*$, we use the following new coordinate

$$\begin{pmatrix} x & p & y+z & 0\\ 0 & 0 & 0 & 0\\ y-z & q & -x & 0\\ q & r & -p & 0 \end{pmatrix} = xX_*+yY_*+zZ_*+pP_*+qQ_*+rR_*. \tag{4.4}$$

According to Proposition 4.1, we obtain the following orbits explicitly. The coadjoint orbit $\Omega(\alpha X_*)$ of αX_* $(\alpha\in\mathbb{R}^\times)$ is represented by the one-sheeted hyperboloid

$$x^2+y^2-z^2=\alpha^2,\quad p=q=r=0. \tag{4.5}$$

The coadjoint orbit $\Omega(\alpha Y_*)$ of αY_* $(\alpha\in\mathbb{R}^\times)$ is also represented by the one-sheeted hyperboloid (4.5). The coadjoint orbit $\Omega(\alpha Z_*)$ of αZ_* $(\alpha\in\mathbb{R}^\times)$ is represented by the two-sheeted hyperboloids

$$x^2+y^2=z^2-\alpha^2>0,\quad p=q=r=0. \tag{4.6}$$

The coadjoint orbit $\Omega(\alpha P_*)$ of αP_* $(\alpha\in\mathbb{R}^\times)$ is represented by the variety

$$2pqx+(q^2-p^2)y+(p^2+q^2)z=0,\ (p,q)\in\mathbb{R}^2-\{(0,0)\}\,,\ r=0 \tag{4.7}$$

in $\mathbb{R}^6$. The coadjoint orbit $\Omega(\alpha Q_*)$ of αQ_* $(\alpha\in\mathbb{R}^\times)$ is represented by the variety (4.7) in $\mathbb{R}^6$. We note that the orbits $\Omega(\alpha P_*)$ and $\Omega(\alpha Q_*)$ are independent of the choice of $\alpha\in\mathbb{R}^*$. We remark that the G^J-orbit of $\alpha P_*+\beta Q_*,\ (\alpha,\beta)\neq(0,0)$ is also given by (4.7).

If $\alpha > 0$, the coadjoint orbit $\Omega(\alpha R_*)$ of αR_* is represented by the variety

$$x^2 + y^2 = z^2,\ z > 0,\ x = \alpha^{-1} pq,\ y + z = -\alpha^{-1} p^2,\ y - z = \alpha^{-1} q^2,\ r = \alpha \tag{4.8}$$

in $\mathbb{R}^6$. If $\alpha < 0$, the coadjoint orbit $\Omega(\alpha R_*)$ of αR_* is represented by the variety

$$x^2 + y^2 = z^2,\ z < 0,\ x = \alpha^{-1} pq,\ y + z = -\alpha^{-1} p^2,\ y - z = \alpha^{-1} q^2,\ r = \alpha \tag{4.9}$$

in $\mathbb{R}^6$.

We put

$$S_* = \frac{1}{2}(Y_* + Z_*) \quad \text{and} \quad T_* = \frac{1}{2}(Y_* - Z_*). \tag{4.10}$$

Then S_* and T_* nilpotent elements. If $\alpha > 0$, the coadjoint orbit $\Omega(\alpha S_*)$ of αS_* is represented by the the cone

$$x^2 + y^2 = z^2,\ z > 0, \quad p = q = r = 0. \tag{4.11}$$

in $\mathbb{R}^6$. If $\alpha < 0$, the coadjoint orbit $\Omega(\alpha S_*)$ of αS_* is represented by the the cone

$$x^2 + y^2 = z^2,\ z < 0, \quad p = q = r = 0. \tag{4.12}$$

in $\mathbb{R}^6$. If $\alpha > 0$, the coadjoint orbit $\Omega(\alpha T_*)$ of αT_* is represented by the cone (4.12) and on the other hand if $\alpha < 0$, the coadjoint orbit $\Omega(\alpha T_*)$ of αT_* is represented by the cone (4.11).

Now we are going to attach to the above coadjoint orbits the unitary representations of G^J. For brevity, we write $G = SL(2,\mathbb{R})$. We follow the notations in Section 2. Let us denote by $\mathfrak{h}$ the Lie algebra of the Heisenberg group $H_{\mathbb{R}}$.

First we shall give lists of the irreducible finite-dimensional representations, the irreducible unitary representations, and the nonunitary principal series of G (cf. [B], [D], [Kn], [L]). We define four subgroups $\bar{N}$, M, A, and N of G by

$$\bar{N} = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right\},\ M = \{\pm I_2\},\ A = \left\{ \begin{pmatrix} |a| & 0 \\ 0 & |a|^{-1} \end{pmatrix} \right\},\ \text{and } N = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right\}.$$

We observe that if $a \neq 0$ and $\epsilon = \mathrm{sgn}(a)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} |a| & 0 \\ 0 & |a|^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}. \tag{4.13}$$

This decomposition is unique and the product $\bar{N}MAN$ is a dense open submanifold of G. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a typical element of G.

(I) Irreducible finite-dimensional representations Φ_n of dimension $n+1$ ($n \geq 0,\ n \in \mathbb{Z}$) :

The representation space $\mathcal{F}_n$ of Φ_n is the space consisting of polynomials on $\mathbb{C}$ of degree $\leq n$. Φ_n is given by

$$\left(\Phi_n(g) f\right)(z) = (-bz + d)^n f\left(\frac{az - c}{-bz + d}\right), \quad f \in \mathcal{F}_n. \tag{4.14}$$

(II) Principal series $\mathcal{P}^{+,i\alpha}$ and $\mathcal{P}^{-,i\alpha}$ ($\alpha \in \mathbb{R}$) :

Fix a real number $\alpha \in \mathbb{R}$. The representation space is the Hilbert space $L^2(\mathbb{R})$ with the usual norm in $L^2(\mathbb{R})$. If $f \in L^2(\mathbb{R})$,

$$(4.15)\quad (\mathcal{P}^{\epsilon,i\alpha}(g)f)(x) = \begin{cases} |-bx+d|^{-1-i\alpha} f\left(\frac{az-c}{-bz+d}\right) & \text{if } \epsilon = +, \\ \operatorname{sgn}(-bx+d)\,|-bx+d|^{-1-i\alpha} f\left(\frac{az-c}{-bz+d}\right) & \text{if } \epsilon = -. \end{cases}$$

These representations are irreducible except for $\mathcal{P}^{-,0}$. Unitary equivalences

$$(4.16)\qquad \mathcal{P}^{+,i\alpha} \cong \mathcal{P}^{+,-i\alpha} \quad \text{and} \quad \mathcal{P}^{-,i\alpha} \cong \mathcal{P}^{-,-i\alpha}$$

are implemented by analytic continuations of intertwining operator $A(\sigma, i\alpha)$ for some irreducible representation σ of M. We refer to [D], (6.7) for more detail $A(\sigma, i\alpha)$. Indeed, $\mathcal{P}^{\pm,i\alpha} = \mathrm{Ind}_{MAN}(\sigma \otimes e^{i\alpha} \otimes 1)$ for the character $e^{i\alpha}$ of A defined by

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mapsto e^{i\alpha t}.$$

(III) Complementary series $\mathcal{C}^s$ $(0 < s < 1)$:

The representation space $C(s)$ of $\mathcal{C}^s$ is given by

$$C(s) = \left\{ f : \mathbb{R} \longrightarrow \mathbb{C} \,\middle|\, \|f\|_s^2 := \int_{\mathbb{R}^2} \frac{f(x)\overline{f(y)}}{|x-y|^{1-s}} dxdy < \infty \right\}.$$

$\mathcal{C}^s$ is given by

$$(4.17)\qquad (\mathcal{C}^s(g)f) = |-bx+d|^{-1-s} f\left(\frac{az-c}{-bz+d}\right), \quad f \in C(s).$$

These $\mathcal{C}^s$ $(0 < s < 1)$ are irreducible unitary. They arise from certain non-unitary principal series by refining the inner product.

(IV) Discrete series $\mathbb{D}_n^+$ and $\mathbb{D}_n^-$ $(n \geq 2,\ n \in \mathbb{Z})$:

Fix a positive integer $n \geq 2$. For $z \in \mathbb{H}$, we write $z = x + iy\,(x, y \in \mathbb{R})$. Let $L^2_{n,+}(\mathbb{H})$ be the Hilbert space consisting of holomorphic functions $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying the following property

$$(4.18)\qquad \|f\|^2 := \int_{\mathbb{R}^2} |f|^2 y^{n-2} dxdy < \infty.$$

$\mathbb{D}_n^+$ is defined by

$$(4.19)\qquad (\mathbb{D}_n^+(g)f)(z) = (-bx+d)^n f\left(\frac{az-c}{-bz+d}\right), \quad f \in L^2_{n,+}(\mathbb{H}).$$

$L^2_{n,+}(\mathbb{H})$ is nonempty because it contains $(z+i)^{-n}$. $\mathbb{D}_n^+$ is irreducible, unitary and square-integrable. The representation space $L^2_{n,-}(\mathbb{H})$ for $\mathbb{D}_n^-$ is the complex conjugation of $L^2_{n,+}(\mathbb{H})$ and

$$(4.20)\qquad (\mathbb{D}_n^-(g)f)(z) = \overline{(-bx+d)^n} f\left(\frac{az-c}{-bz+d}\right), \quad f \in L^2_{n,-}(\mathbb{H}).$$

(V) Limits of discrete series $\mathbb{D}_1^+$ and $\mathbb{D}_1^-$:

The spaces $L^2_{1,+}(\mathbb{H})$ and $L^2_{1,-}(\mathbb{H})$ are analogs of the discrete series with the different norm

$$\|f\|^2_\infty = \sup_{y>0} \int_\infty^\infty |f(x+iy)|^2 dx. \tag{4.21}$$

The action of $\mathbb{D}_1^+$ (resp. $\mathbb{D}_1^-$) is given as in (4.19) (resp. (4.20)) with $n=1$.

(VI) Nonunitary principal series $\mathcal{P}^{\epsilon,w}$ $(\epsilon=\pm,\ w\in\mathbb{C})$:

Let $w\in\mathbb{C}$. The representation space of $\mathcal{P}^{\epsilon,w}$ is the Hilbert space $L^2(\mathbb{R},(1+x^2)^{\mathrm{Re}\,w}dx)$ with the G-action

$$(\mathcal{P}^{\epsilon,w}(g)f)(x) = \begin{cases} |-bx+d|^{1-w} f\left(\frac{az-c}{-bz+d}\right) & \text{if } \epsilon=+, \\ \mathrm{sgn}(-bx+d)\,|-bx+d|^{1-w} f\left(\frac{az-c}{-bz+d}\right) & \text{if } \epsilon=-. \end{cases} \tag{4.22}$$

$\mathcal{P}^{\epsilon,w}$ is not unitary unless w is purely imaginary. When $0<w<1$, it becomes unitary by refining the norm.

(VII) The trivial representation.

Reducibility:

$$\begin{aligned} \mathcal{F}_n &\subseteq \mathcal{P}^{+,-(n+1)} \quad \text{if } n \text{ is even},\\ \mathcal{F}_n &\subseteq \mathcal{P}^{-,-(n+1)} \quad \text{if } n \text{ is odd},\\ \mathbb{D}_n^+\oplus\mathbb{D}_n^- &\subseteq \mathcal{P}^{+,(n-1)} \quad \text{if } n \text{ is even},\\ \mathbb{D}_n^+\oplus\mathbb{D}_n^- &\subseteq \mathcal{P}^{-,(n-1)} \quad \text{if } n \text{ is odd}. \end{aligned}$$

In particular, $\mathcal{P}^{-,0}\cong\mathbb{D}_1^+\oplus\mathbb{D}_1^-$. There is no other reducibility.

Next we shall give a list of the irreducible unitary representations of G^J (cf. [B-S]). We recall that the Schrödinger representation U_m $(m\in\mathbb{R})$ of $H_\mathbb{R}$ is given by

$$(U_m(\lambda,\mu,\kappa)f)(x) = e^{2\pi i m\{\kappa+(2x+\lambda)\mu\}} f(x+\lambda), \quad f\in L^2(\mathbb{R}). \tag{4.23}$$

See [Y1],(2.18), p. 313. U_m is an irreducible unitary representation of $H_\mathbb{R}$ with the central character $\sigma_m(\kappa) := e^{2\pi i m\kappa}$ $(\kappa\in\mathbb{R})$. G acts on $H_\mathbb{R}$ by conjugation inside G^J

$$g\star(\lambda,\mu,\kappa) := g(\lambda,\mu,\kappa)g^{-1} = ((\lambda,\mu)g^{-1},\kappa), \quad g\in G. \tag{4.24}$$

See (3.2). In particular, $g\star(0,0,\kappa)=(0,0,\kappa)$ for all $\kappa\in\mathbb{R}$. Since the irreducible unitary representation $U_m^{[g]}$ of $H_\mathbb{R}$ on $L^2(\mathbb{R})$ defined by

$$U_m^{[g]}(h) := U_m(ghg^{-1}), \quad h\in H_\mathbb{R} \tag{4.25}$$

has the same central character σ_m as U_m, according to the Stone-von Neumann theorem, $U_m^{[g]}$ is unitarily equivalent to U_m. In other words, there exists a unitary operator $\Phi_{W,m}(g) : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ such that

$$U_m^{[g]}(h) = \Phi_{W,m}(g)U_m(h)\Phi_{W,m}(g)^{-1}, \quad h \in H_{\mathbb{R}}. \tag{4.26}$$

By Schur's lemma, there is a map $c_m : G \times G \longrightarrow U(1)$ satisfying the property

$$\Phi_{W,m}(g_1g_2) = c_m(g_1, g_2)\Phi_{W,m}(g_1)\Phi_{W,m}(g_2), \quad g_1, g_2 \in G. \tag{4.27}$$

The the map c_m satisfies the cocycle condition. Let $\tilde{G}$ be the metaplectic group with the multiplier c_m, that is, as a set $\tilde{G} = G \times \{\pm 1\}$, $\tilde{G}$ has the following multiplication

$$(g, \epsilon) \cdot (g', \epsilon') := (gg', c_m(g, g')\epsilon\epsilon'). \tag{4.28}$$

The map $\pi_W^{[m]} : \tilde{G} \longrightarrow GL(L^2(\mathbb{R}))$ defined by

$$\pi_W^{[m]}((g, \epsilon)) := \Phi_{W,m}(g)\epsilon, \quad (g, \epsilon) \in \tilde{G} \tag{4.29}$$

is an ordinary representation of $\tilde{G}$, which is the so-called *Weil representation* of G. $\pi_W^{[m]}$ is decomposed into

$$\pi_W^{[m]} = \pi_W^{[m],+} \oplus \pi_W^{[m],-}, \tag{4.30}$$

where $\pi_W^{[m],+}$ (resp. $\pi_W^{[m],-}$) is the irreducible representation of $\tilde{G}$ on the space of even (resp. odd) functions in $L^2(\mathbb{R})$.

We define the map $\pi_{SW}^{[m]} : G^J \longrightarrow GL(L^2(\mathbb{R}))$ by

$$\pi_{SW}^{[m]}(hg) := U_m(h)\pi_W^{[m]}(g), \quad g \in G,\ h \in H_{\mathbb{R}}. \tag{4.31}$$

Then $\pi_{SW}^{[m]}$ is a projective representation of G^J with the multiplier c_m which is extended canonically to the representation of $\tilde{G}^J$. $\pi_{SW}^{[m]}$ is called the *Schrödinger-Weil representation* or simply the *Weil representation* of the Jacobi group G^J with the character σ_m. We observe that $\tilde{G}^J$ is isomorphic to the semidirect product of $\tilde{G}$ and $H_{\mathbb{R}}$.

We use the notation (3.9) and let $i = \sqrt{-1}$. We define the elements of $\mathfrak{g}_{\mathbb{C}}^J$ by

$$\begin{aligned}
\hat{Z} &= -\,i\,G(0, 0, 1, 0, 0, 0), \\
\hat{Z}_0 &= -\,i\,G(0, 0, 0, 0, 0, 1), \\
X_+ &= \frac{1}{2}\,G(1, i, 0, 0, 0, 0), \\
X_- &= \frac{1}{2}\,G(1, -i, 0, 0, 0, 0), \\
Y_+ &= \frac{1}{2}\,G(0, 0, 0, 1, i, 0), \\
Y_- &= \frac{1}{2}\,G(0, 0, 0, 1, -i, 0).
\end{aligned}$$

Then we have the commutation relations

$$[\hat{Z}_0, \mathfrak{g}_{\mathbb{C}}^J] = 0, \quad [\hat{Z}, Y_\pm] = \pm Y_\pm, \quad [\hat{Z}, X_\pm] = \pm 2X_\pm.$$

Therefore for each irreducible representation π of $\mathfrak{g}_{\mathbb{C}}^J$, its representation space V decomposes as $V = \sum_{k\in\mathbb{Z}} V_k$ with

$$\pi(\hat{Z}_0)V_k = \mu V_k, \quad \pi(\hat{Z})V_k = \rho_k V_k, \quad \pi(Y_\pm)V_k \subseteq V_{k\pm1}, \quad \pi(X_\pm)V_k \subseteq V_{k\pm2},$$

where μ and ρ_k are complex numbers.

According to [B-S], p. 33, if $m > 0$, the infinitesimal representation of $\pi_{SW}^{[m]}$ is a lowest weight representation of $\mathfrak{g}_{\mathbb{C}}^J$ operating on the space $V = < v_j >_{j\in\mathbb{Z}_{\geq 0}}$ by

$$\begin{aligned} \hat{Z}_0 v_j =& \mu v_j, \quad Y_+ v_j = v_{j+1}, \quad Y_- v_j = -\mu j v_{j-1}, \\ \hat{Z} v_j =& (j + \frac{1}{2})\, v_j, \quad X_+ v_j = -\frac{1}{2\mu}\, v_{j+2}, \quad X_- v_j = \frac{\mu}{2}\, j(j-1) v_{j-2}, \end{aligned}$$

where $\mu = 2\pi m$ and $v_{-1} = v_{-2} = 0$. If $m < 0$, $\pi_{SW}^{[m]}$ is a highest weight representation with the space $V = < v_{-j} >_{j\in\mathbb{Z}_{\geq 0}}$, the action given by

$$\hat{Z}_0 v_{-j} = \mu v_{-j}, \quad Y_- v_{-j} = v_{-(j+1)}, \quad Y_+ v_{-j} = -\mu j v_{-(j-1)},$$

$$\hat{Z} v_{-j} = -(j + \frac{1}{2})\, v_{-j}, \quad X_- v_{-j} = \frac{1}{2\mu}\, v_{-(j+2)}, \quad X_+ v_{-j} = -\frac{\mu}{2}\, j(j-1) v_{-(j-2)}$$

(with $v_1 = v_2 = 0$ understood).

(VIII) Principal series $\pi_{\alpha,\nu}$ ($\alpha \in \mathbb{C} \setminus \{\mathbb{Z} + \frac{1}{2}\},\ \nu = \frac{1}{2}$):

The infinitesimal representation of $\pi_{\alpha,\nu}$ is given by

$$\hat{Z} w_l = \left(l - \frac{1}{2} \right) w_l, \quad X_\pm w_l = \frac{1}{2} \left(\alpha + 1 \pm \left(l - \frac{1}{2} \right) \right) w_{l\pm2}$$

acting on the representation space

$$W_{\alpha,\nu} = < w_l >, \quad l \in 2\mathbb{Z} + \nu + \frac{1}{2}.$$

(IX) Discrete series $\pi_{k_0}^\pm$ ($k_0 \in \mathbb{Z} + \frac{1}{2}$):

The infinitesimal representations of $\pi_{k_0}^\pm$ are given by

$$\begin{aligned} \hat{Z} w_{\pm l} =& \pm (k_0 + l)\, w_{\pm l}, \\ X_\pm w_{\pm l} =& w_{\pm(l+2)}, \\ X_\mp w_{\pm l} =& -\frac{l}{2} \left(k_0 + \frac{l}{2} - 1 \right) w_{\pm(l-2)} \end{aligned}$$

acting on the representation space

$$W_{k_0}^\pm = < w_{\pm l} >, \quad l \in 2\mathbb{Z}_{\geq 0}.$$

Berndt and Schmidt [B-S] gave lists of irreducible unitary representations of G^J as follows:

(J1) The representations π of G^J where $\pi|_G$ is an irreducible unitary representation of G and $\pi|_{H_{\mathbb{R}}}$ is trivial.

(J2) The induced representations $\mathrm{Ind}_{G^J_\psi}^{G^J}\tau$, where $\psi : H_{\mathbb{R}} \longrightarrow U(1)$ is the character of $H_{\mathbb{R}}$ defined by $\psi(\lambda,\mu,\kappa) = e^{2\pi i\lambda}$, G^J_ψ is the subgroup of G^J defined by

$$G^J_\psi = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} h \,\middle|\, c \in \mathbb{R},\ h \in H_{\mathbb{R}} \right\}$$

and τ runs over the irreducible unitary representation of G^J_ψ such that $\tau|_{H_{\mathbb{R}}}$ is a multiple of ψ.

(J3) The principal series of G^J :

$$\pi_{m,\alpha,\nu} = \pi^{[m]}_{SW} \otimes \pi_{\alpha,\nu}, \quad m \in \mathbb{R}^\times,\ \alpha \in i\mathbb{R},\ \nu = \pm\frac{1}{2}.$$

(J4) The complementary series of G^J :

$$\pi_{m,\alpha,\nu} = \pi^{[m]}_{SW} \otimes \pi_{\alpha,\nu}, \quad m \in \mathbb{R}^\times,\ \alpha \in \mathbb{R},\ \alpha^2 < \frac{1}{4},\ \nu = \pm\frac{1}{2}.$$

(J5) The positive discrete series of G^J :

$$\pi^+_{m,k} = \pi^{[m]}_{SW} \otimes \pi^+_{k-\frac{1}{2}}, \quad m \in \mathbb{R}^\times,\ k \in \mathbb{Z}^+.$$

(J6) The negative discrete series of G^J :

$$\pi^-_{m,k} = \pi^{[m]}_{SW} \otimes \pi^-_{k-\frac{1}{2}}, \quad m \in \mathbb{R}^\times,\ k \in \mathbb{Z}^+.$$

The only equivalences between these above representations are

$$\pi_{m,\alpha,\nu} \cong \pi_{m,-\alpha,\nu}.$$

All other representations are inequivalent.

The principal series $\mathcal{P}^{+,i\alpha}(\cong \mathcal{P}^{+,-i\alpha})$ of G corresponds to the G-orbit $\Omega(\alpha X)$ of αX, where α is a positive real number. This orbit is represented by (2.5). On the other hand, $\mathcal{P}^{-,i\alpha}(\cong \mathcal{P}^{-,-i\alpha})$ of G corresponds to the G-orbit $\Omega(\alpha X)$ of αX, where α is a negative real number.

The discrete series $\mathbb{D}^+_n$ $(n \in \mathbb{Z}^+)$ of G corresponds to the irreducible component of the G-orbit $\Omega(nZ)$ of nZ given by

$$x^2 + y^2 - z^2 = -n^2, \quad z > 0 \quad (\text{cf. } (2.6)). \tag{4.32}$$

The discrete series $\mathbb{D}^-_n$ $(n \in \mathbb{Z}^+)$ of G corresponds to the irreducible component of the G-orbit $\Omega(nZ)$ of nZ given by

$$x^2 + y^2 - z^2 = -n^2, \quad z < 0 \quad (\text{cf. } (2.6)). \tag{4.33}$$

The nilpotent orbit $\mathcal{N}^+_{\mathbb{R}}$ (resp. $\mathcal{N}^-_{\mathbb{R}}$) is attached to the even part $\pi^{[0],+}_W$ (resp. the odd part $\pi^{[0],-}_W$) of the Weil representation $\pi^{[0]}_W$ of G (cf. (2.10)-(2.12)). The trivial representation of G corresponds to the zero (nilpotent) G-orbit. There are no coadjoint G-orbits which correspond to the complimentary series $\mathcal{C}^s$ $(0 < s < 1)$.

The principal series $\pi_{m,i\alpha,\frac{1}{2}}(\cong \pi_{m,-i\alpha,\frac{1}{2}}),\ m\in\mathbb{R}^\times,\ \alpha\in\mathbb{R}$ of G^J corresponds to the coadjoint G^J-orbit $\Omega(mR_*+\alpha X_*)$ of $mR_*+\alpha X_*$, where α is a positive real number. In this case, the G^J-orbit $\Omega(mR_*+\alpha X_*),\ \alpha>0$ is characterized by the variety

$$x^2+y^2-(z^2+\alpha^2)=\frac{2}{m}pqx+\frac{1}{m}(q^2-p^2)y+\frac{1}{m}(p^2+q^2)z,\quad r=m. \tag{4.34}$$

On the other hand, $\pi_{m,i\alpha,-\frac{1}{2}}(\cong \pi_{m,-i\alpha,-\frac{1}{2}}),\ m\in\mathbb{R}^\times,\ \alpha\in\mathbb{R}$ of G^J corresponds to the coadjoint G^J-orbit $\Omega(mR_*+\alpha X_*)$ of $mR_*+\alpha X_*$, where α is a negative real number. This orbit is also characterized by (4.34).

The discrete series $\pi^+_{m,k}\,(m\in\mathbb{R}^\times,\ k\in\mathbb{Z}^+)$ corresponds to the irreducible component of the G^J-orbit $\Omega(mR_*+kZ_*)$ of mR_*+kZ_* represented by the variety

$$x^2+y^2-(z^2-k^2)=\frac{2}{m}pqx+\frac{1}{m}(q^2-p^2)y+\frac{1}{m}(p^2+q^2)z,\ z>0,\ r=m. \tag{4.35}$$

The discrete series $\pi^-_{m,k}\,(m\in\mathbb{R}^\times,\ k\in\mathbb{Z}^+)$ corresponds to the irreducible component of the G^J-orbit $\Omega(mR_*+kZ_*)$ of mR_*+kZ_* represented by the variety

$$x^2+y^2-(z^2-k^2)=\frac{2}{m}pqx+\frac{1}{m}(q^2-p^2)y+\frac{1}{m}(p^2+q^2)z,\ z<0,\ r=m. \tag{4.36}$$

There are no coadjoint G^J-orbits which correspond to the complimentary series $\pi_{m,\alpha,\nu}\,(m\in\mathbb{R}^\times,\ \alpha\in\mathbb{R},\ \alpha^2<\frac{1}{2},\ \nu=\pm\frac{1}{2})$. There are no unitary representations of G^J corresponding to the G^J-orbits of $\alpha P_*+\beta Q_*$ with $(\alpha,\beta)\neq(0,0)$.

The family $\Omega(\alpha R_*)(\alpha\in\mathbb{R}^\times)$ have the following properties (Ω1)-(Ω2):

(Ω1) Under the natural projection of $(\mathfrak{g}^J)^*$ onto $\mathfrak{h}^*$, the coadjoint orbit $\Omega(\alpha R_*)$ goes to the orbit which corresponds to the irreducible unitary representation U_α of the Heisenberg group $H_\mathbb{R}$, namely, the Schrödinger representation of $H_\mathbb{R}$.

(Ω2) Under the projection on $\mathfrak{g}^*=\mathfrak{sl}(2,\mathbb{R})^*$, the coadjoint orbit $\Omega(\alpha R_*)$ goes to $\Omega_{\mathrm{sgn}(\alpha)}$, where $\Omega_+=\mathcal{N}^+_\mathbb{R}$ and $\Omega_-=\mathcal{N}^-_\mathbb{R}$.

In fact, there is an irreducible unitary representation $\pi_\alpha\,(\alpha\in\mathbb{R}^\times)$ of G^J or (its universal cover) with the properties

$$\mathrm{Res}^{G^J}_{H_\mathbb{R}}\pi_\alpha\cong U_\alpha,\quad \mathrm{Res}^{G^J}_{G}\pi_\alpha\cong\pi_{\mathrm{sgn}(\alpha)}, \tag{4.37}$$

where π_+ (resp. π_-) is an irreducible representation of $\tilde{G}$ corresponding to the nilpotent orbit $\mathcal{N}^+_\mathbb{R}$ (resp. $\mathcal{N}^-_\mathbb{R}$) (cf. (2.12)). Indeed, $\pi_\pm$ are two irreducible components $\pi_W^{[\alpha],\pm}$ of the Weil representation $\pi_W^{[\alpha]}$ of G (cf. (4.30)) and π_α is the irreducible components $\pi_{SW}^{[\alpha],\pm}$ of the Schrödinger-Weil representations $\pi_{SW}^{[\alpha]}$ of G^J (cf. (4.31)). Again we remind that the G^J-orbit $\Omega(\alpha R_*)$ is characterized by (4.8) or (4.9) depending on the sign of α.

Remark 4.2. The principal series, discrete series of G^J are attached to hyperbolic or elliptic orbits, while the non-zero nilpotent orbits are corresponded to the irreducible components $\pi_\pm$ of the Weil representation of G or the irreducible components $\pi_{SW}^{[\alpha],\pm}$ of the Schrödinger-Weil representations $\pi_{SW}^{[\alpha]}$ of G^J. In fact, $\pi_\pm$ and $\pi_{SW}^{[\alpha],\pm}$ are minimal representations and may be the so-called *unipotent* representations of G^J. See [Vo1-2] for a detail on unipotent representations in the reductive case.

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