

Rich Product Probability Spaces, Exact Law of Large Numbers and Characterization of Insurable Risks*

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Abstract: A rich product probability space is formally introduced as an extension of the usual product space that retains the Fubini property and includes a measurable process with essentially iid random variables of uniform distribution. Simple measure-theoretic methods are applied to this framework to obtain various versions of the exact law of large numbers and their converses for a continuum of random variables or stochastic processes. A model for a large economy with individual risks is developed; and insurable risks are characterized by essential pairwise independence. The usual continuum product based on the Kolmogorov construction together with the Lebesgue measure as well as the usual finitely additive measure-theoretic framework is shown further to be not suitable for modeling individual risks. The existence of a rich product probability space is also shown via an extension of the usual continuum product.

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1 Introduction

The intuitive idea of the complete cancelation of individual risks through aggregation or diversification has been widely used in the economic literature.¹ The underlying mathematical model usually involves a continuum of random variables or stochastic processes that are stochastically independent. It is hypothesized that almost all sample functions are Lebesgue measurable with a common mean or distribution.²

As noted in [3] and [15] and [23], it is easy to construct examples of a continuum of independent random variables whose sample means or distributions are constant. However, one can also construct other examples of a continuum of independent random variables whose sample functions may not be measurable, or behave in a very “strange” way. In fact, for an iid process based on the usual continuum product via the Kolmogorov construction, one can obtain the absurd claim that almost all sample functions are essentially equal to an arbitrarily given function h on the index space (see Proposition 6.1 and the discussion afterwards); and thus, the sample mean or distribution can be undefined or completely arbitrary.³

So a natural question is: besides constructing a few examples or counterexamples in an ad hoc way, can one study a continuum of independent random variables (or stochastic processes) in a meaningful way at all? An essential difficulty is that independence and joint measurability with respect to the usual measure-theoretic product are never compatible with each other except for some trivial cases.⁴ Thus one needs to work with an enrichment of the usual measure-theoretic product for the study of processes with independent random variables. As demonstrated in [33]-[36], the Loeb product probability spaces⁵ do provide such a framework. Several versions of the exact law of large numbers as well as some new properties of independence are proven in [34] and [35].

In this paper, a rich product probability space is introduced formally as an extension of the usual product that retains the Fubini property and includes a measurable process with essentially iid random variables of uniform distribution. The advantage of this approach is that one can apply simple measure-theoretic methods directly in this framework to obtain some useful versions of the exact law of large numbers and their converses without going through some details in nonstandard models as in [34]. Large economies with individual risks can also be studied easily in this framework. Note that the Fubini type property that allows changing

¹See, for example, [5], [7], [10], [22], [29], [31], and references in [14], [34] and [36].

²In the case of a continuum of iid stochastic processes with a Lebesgue measurable index space, it is assumed that almost all empirical processes have common finite dimensional distributions.

³See also [12] and [23].

⁴See [13], p. 67, and Proposition 2.1 below.

⁵For details on Loeb (product) spaces, see [2], [21] and [28].

the order of iterated integrals is an idealization of changing the order of summation signs in the discrete setting. It should be a minimal requirement for working with a continuum of random variables whenever possible.

The rest of the paper is organized as follows. The concept of a rich product probability space is defined in Section 2. It is shown to have the symmetric property as well as the subset property in Proposition 2.4. Though one only requires the existence of a measurable process with essentially iid random variables of uniform distribution on a rich product probability space, Proposition 2.6 shows that one can actually construct processes on it with essentially pairwise independent random variables that take any given variety of distributions.

In Section 3, we develop systematically some applications of the framework of a rich product probability space. In particular, we obtain various versions of the exact law of large numbers and their converses for a continuum of random variables or stochastic processes in this framework by applying some simple measure-theoretic methods. These versions of the exact law of large numbers and their converses are perhaps all that one needs for applications in economics; and thus we give complete proofs. Such kind of results were shown earlier in [34] in the particular case of Loeb product spaces.⁶ The Loeb product framework naturally allows the Fubini property for functions of several variables while the Fubini property for a rich product probability space is only stated for functions of two variables. The Fubini property for some functions of three variables has to be specially proven in Lemmas 3.1 and 9.2 below; and the proofs for several versions of the exact law of large numbers and their converses in [34] have to be adapted to the new framework.

Malinvaud [30] considered individual risks in the context of Pareto optimal allocations for a sequence of finite markets with given finitely many types and each type having N agents with convex, continuous and complete preorderings.⁷ A full system of insurance that allows net transfers across states by individual agents was introduced to allocate individual risks. Some versions of the classical law of large numbers were used to claim that the budget of the insurance system would be approximately balanced in the absence of transaction costs when N goes to infinity.

Section 4 develops a model for a large economy with individual risks based on a rich product probability space.⁸ The model presented here goes beyond the one in [30] in several

⁶Theorems 3.4, 3.5, 3.9, 3.13, and 3.14 in this paper correspond respectively to (part of) Theorems 4.6, 7.6, 7.16, 8.2 and 8.12 in [34]. Note that the Loeb products of atomless Loeb probability spaces are rich product probability spaces; see the discussion in Section 7.2.

⁷Here is Footnote 7 in [30]: “Since we are dealing here with large economies we could dispense with convexity assumptions. But the argument would have to be more laborious; the complication does not seem advisable here.”

⁸Malinvaud held the expectation that the idea of allocating individual risks in large markets could be better

aspects. First, the restriction of fixed types with the same number of agents in each type (i.e., only a replicated economy was considered in [30]) is removed. Second, the convexity of individual preferences is not assumed. Third, it is shown that the risks are insurable if and only if they are essentially pairwise independent.⁹ We work not only with Pareto optimality but also with pseudo equilibrium and equilibrium.

Theorem 5.2 in Section 5 shows the following. Once a rich product probability space is adopted, one can prove that almost all sample functions in an essentially pairwise independent process cannot be essentially measurable with respect to any countably generated sub- σ -algebra of the σ -algebra on the index space. Thus it can be proved that almost no sample functions can be Lebesgue measurable. This is in contrast with the indeterminacy of the size of Lebesgue measurable samples in the setting of the usual continuum product based on the Kolmogorov construction.¹⁰

Section 6 has two parts. The first part argues that the usual continuum product based on the Kolmogorov construction together with the unit Lebesgue interval fails to be a good mathematical model for modeling individual risks in three aspects. First, the usual continuum product is a bad sample space that allows arbitrary claims on sample functions; second, the unit Lebesgue interval is a bad index space since an additional Fubini property would imply that almost no sample functions are Lebesgue measurable as in Theorem 5.2; third, the usual product of the unit Lebesgue interval with the usual continuum product is a bad product space since the process cannot be jointly measurable with respect to it. The second part points out that the finitely additive measure-theoretic framework is also not suitable for modeling individual risks in the sense that arbitrary claims can be made and the exact law of large numbers can fail completely in terms of sample distributions.

Section 7 presents two existence results for rich product probability spaces. One involves an extension of the usual continuum product based on the Kolmogorov construction, and the other an extension of the product of a probability space on the unit interval with itself. Since these rich product probability spaces are not Loeb product spaces themselves, the results presented in Section 3 here are more general than the corresponding results presented in [34]. Section 8 provides some concluding remarks. The proofs of some results in Sections 2 - 5 and

developed in the context of economies with a continuum of agents. He wrote ([30], p. 313) that “Economists of the Aumann-Debreu school will soon make this paper obsolete when they turn their powerful machinery in the direction that I am exploring.” This expectation has not been realized earlier due to problems with a continuum of independent random variables.

⁹Malinvaud attempted to introduce some dependence among agents. Our characterization of insurable risks shows that the condition of essential pairwise independence is the optimal condition for an insurance system to work.

¹⁰The sample size is shown to have inner and outer measures 0 and 1 respectively, and thus can be assigned any value in $[0, 1]$ in an extension; see [23] and the discussion after Proposition 6.1 in Section 6.

7 are given in the appendix. We note that standard measure-theoretic methods are used at all places except in Section 9.5 where nonstandard analysis is used for the proof of existence. The reader may choose to skip Section 9.5 by accepting the existence of a rich product probability space.

2 Rich product probability spaces

Let probability spaces $(I, \mathcal{I}, \lambda)$ and (Ω, \mathcal{F}, P) be our index and sample spaces respectively. If one prefers, I can be taken to be the unit interval $[0, 1]$. Let $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ be the usual product probability space.¹¹ The following result shows that independence and joint measurability with respect to the usual measure-theoretic product are never compatible with each other except for the trivial case that almost all random variables are essentially constant.¹² It also means that even if one chooses a very large σ -algebra on the index space I of a nontrivial independent process so that all the sample functions are measurable, the process itself is still not jointly measurable with respect to $\mathcal{I} \otimes \mathcal{F}$. We include its short proof (together with the proofs of other results in this section) in Section 9.1 for the convenience of the reader. For a function f on $I \times \Omega$, and $(i, \omega) \in I \times \Omega$, f_i represents the function $f(i, \cdot)$ on Ω , and f_ω the function $f(\cdot, \omega)$ on I .

Proposition 2.1 *Let f be a function from $I \times \Omega$ to a Polish space X (i.e., a topological space that is homeomorphic to a complete separable metric space). If f is jointly measurable on the product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$, and for λ -almost all $s \in I$, f_s is independent of f_i for λ -almost all $i \in I$ (this condition is called essential pairwise independence),¹³ then, for λ -almost all $i \in I$, f_i is a constant random variable.¹⁴*

As noted in Section 1, the Fubini type property associated with joint measurability is simply an idealization of changing the order of summation signs in the discrete setting, and this property should be retained as far as possible for a meaningful study of independent processes. Thus, all one needs is to work with a rich product probability space $(I \times \Omega, \mathcal{W}, Q)$ that extends

¹¹For the definition and properties of product measures, see [32], p. 303. Here we also use the convention that a probability space is always a complete measure space.

¹²This is Proposition 1 in [35]. See [13] (p. 67) for the case that $(I, \mathcal{I}, \lambda)$ is a Lebesgue space and the relevant process is iid.

¹³If λ is atomless and the process f has mutually independent random variables, then for any $s \in I$, f_s is independent of f_i for any $i \neq s$ (and hence for λ -almost all $i \in I$). This means that the condition of essential pairwise independence is indeed satisfied.

¹⁴The result is still valid when λ has atoms; let A be such an atom. One can simply observe that the essential pairwise independence condition implies the essential constancy of the process f restricted to $A \times \Omega$; see the argument in the proof of Lemma 2.3.

the usual product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$, retains the Fubini type property, and admits nontrivial essentially pairwise independent processes. Here is a formal definition.

Definition 2.2 *Let $(I \times \Omega, \mathcal{W}, Q)$ be a probability space extending $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \times P)$. It is said to be a rich product probability space if it satisfies the following two conditions.*

(1) *It is a Fubini extension of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \times P)$ (which will also be called an extended Fubini product probability space). This means that for any real-valued \mathcal{W} -integrable function f on $I \times \Omega$, the two functions f_i and f_ω are integrable respectively on (Ω, \mathcal{F}, P) for λ -almost all $i \in I$ and on $(I, \mathcal{I}, \lambda)$ for P -almost all $\omega \in \Omega$; moreover, $\int_\Omega f_i dP$ and $\int_I f_\omega dP$ are integrable respectively on $(I, \mathcal{I}, \lambda)$ and on (Ω, \mathcal{F}, P) , with $\int_{I \times \Omega} f dQ = \int_I (\int_\Omega f_i dP) d\lambda = \int_\Omega (\int_I f_\omega d\lambda) dP$.¹⁵*

(2) *There is a real-valued \mathcal{W} -integrable function g on $I \times \Omega$ such that for λ -almost all $i \in I$, the random variable g_i on (Ω, \mathcal{F}, P) has uniform distribution on $[0, 1]$ and is independent of the random variable g_s for λ -almost all $s \in I$.*

To reflect the fact that the rich product probability space $(I \times \Omega, \mathcal{W}, Q)$ has $(I, \mathcal{I}, \lambda)$ and (Ω, \mathcal{F}, P) as its marginal spaces as required by the Fubini property, it will be denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

Since for λ -almost all $i \in I$, g_i has uniform distribution, P is atomless. The following simple lemma shows that λ is also atomless.

Lemma 2.3 *Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich product probability space. Then λ is atomless.*

Proof: Suppose otherwise; let A be an atom of $(I, \mathcal{I}, \lambda)$. Then for P -almost all $\omega \in \Omega$, the λ -integrable function g_ω must be essentially constant on A . Thus, for P -almost all $\omega \in \Omega$, λ -almost all $i \in A$, $g_\omega(i) = \int_A g_\omega d\lambda / \lambda(A)$. By the Fubini property, we have for λ -almost all $i \in A$, g_i is the random variable φ defined by $\int_A g_\omega d\lambda / \lambda(A)$. Thus φ also has the uniform distribution. However, the independence assumption implies that φ is independent of itself and thus a constant. This is a contradiction. ■

The following proposition shows that the property of being a rich product probability space is symmetric and can be retained for subsets. The results follow easily from Theorem 3.9 below.

Proposition 2.4 *Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich product probability space. Then we have the following.*

¹⁵The classical Fubini Theorem is only stated for the usual product measure spaces. It does not apply to integrable functions on $(I \times \Omega, \mathcal{W}, Q)$ since these functions may not be $\mathcal{I} \otimes \mathcal{F}$ -measurable. However, the conclusions of that theorem do hold for processes on the enriched product space $(I \times \Omega, \mathcal{W}, Q)$ that extends the usual product.

(1) The space $(\Omega \times I, \mathcal{F} \boxtimes \mathcal{I}, P \boxtimes \lambda)$, obtained by switching the index and sample variables, is a rich product probability space.¹⁶

(2) Let $A \in \mathcal{I}$ and $B \in \mathcal{F}$ with $\lambda(A) > 0$ and $P(B) > 0$. Define $\mathcal{I}^A = \{C \in \mathcal{I} : C \subseteq A\}$, $\mathcal{F}^B = \{D \in \mathcal{F} : D \subseteq B\}$, and $(\mathcal{I} \boxtimes \mathcal{F})^{A \times B} = \{E \in \mathcal{I} \boxtimes \mathcal{F} : E \subseteq (A \times B)\}$. Let λ^A , P^B , and $(\lambda \boxtimes P)^{A \times B}$ be the probability measures rescaled respectively from the restrictions of λ to \mathcal{I}^A , P to \mathcal{F}^B and $\lambda \boxtimes P$ to $(\mathcal{I} \boxtimes \mathcal{F})^{A \times B}$. Then $(A \times B, (\mathcal{I} \boxtimes \mathcal{F})^{A \times B}, (\lambda \boxtimes P)^{A \times B})$ is a rich product probability space with marginals $(A, \mathcal{I}^A, \lambda^A)$ and (B, \mathcal{F}^B, P^B) .

The following is a formal definition of essential pairwise independence in the setting of a rich product probability space.

Definition 2.5 Let f be a process from a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a Polish space X . The random variables f_i are said to be essentially pairwise independent if for λ -almost all $i \in I$, the random variables f_i and f_s are independent for λ -almost all $s \in I$.

Thus, a rich product probability space is endowed with a process whose random variables are essentially pairwise independent with uniform distribution on $[0, 1]$. As shown in the following proposition, such a product space has the universality property in the sense that one can construct processes on it with essentially pairwise independent random variables that take any given variety of distributions on a general Polish space.

Proposition 2.6 Let X be a Polish space and φ a measurable mapping from $(I, \mathcal{I}, \lambda)$ to the space $\mathcal{M}(X)$ of Borel probability measures on X , where $\mathcal{M}(X)$ is endowed with the topology of weak convergence of measures. If $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a rich product probability space, then there is a $\mathcal{I} \boxtimes \mathcal{F}$ -measurable process G from $I \times \Omega$ to X such that the random variables G_i are essentially pairwise independent, and the distribution $P G_i^{-1}$ is the given distribution $\varphi(i)$ for λ -almost all $i \in I$.¹⁷

Let μ be any Borel probability measure on the space $\mathcal{M}(X)$. Since $(I, \mathcal{I}, \lambda)$ is atomless by Lemma 2.3, any distribution can be generated by some measurable function on $(I, \mathcal{I}, \lambda)$. Thus, there is an \mathcal{I} -measurable function φ from I to $\mathcal{M}(X)$ whose distribution is the given μ . By applying Proposition 2.6 to this case, we know that there exists a $\mathcal{I} \boxtimes \mathcal{F}$ -measurable process G from $I \times \Omega$ to X such that the random variables G_i are essentially pairwise independent, and

¹⁶This symmetric property also implies the atomlessness of λ trivially since there is a process G such that G_ω has uniform distribution on $[0, 1]$ for P -almost all $\omega \in \Omega$.

¹⁷The measurability of φ with respect to the Borel algebra generated by the topology of weak convergence of measures on $\mathcal{M}(X)$ is equivalent to the measurability of the real-valued functions $\varphi(\cdot)(B)$ on I for all Borel sets B in X ; see, for example, Lemma 1 in [17]. It is obvious that for each Borel set B in X , $P G_i^{-1}(B) = \int_\Omega \mathbf{1}_B(G_i)(\omega) dP$ is measurable in terms of $i \in I$. Thus $P G_i^{-1}$ defines an \mathcal{I} -measurable function from I to $\mathcal{M}(X)$.

the distribution on $\mathcal{M}(X)$ generated by the \mathcal{I} -measurable function PG_i^{-1} from I to $\mathcal{M}(X)$ is the given μ . The following corollary is obvious.

Corollary 2.7 *If $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a rich product probability space, then there is a $\mathcal{I} \boxtimes \mathcal{F}$ -measurable process G from $I \times \Omega$ to a Polish X with a distribution τ such that the random variables G_i are essentially pairwise independent with common distribution τ .*

3 Exact law of large numbers and its converse

In this section, we obtain various versions of the exact law of large numbers and their converses for a continuum of random variables or stochastic processes in the framework of a rich product probability space. It is shown in Section 3.1 that essential uncorrelatedness is necessary and sufficient for the sample means to be constant in some strong sense. Section 3.2 presents the exact law of large numbers and its converse in terms of sample distributions. Section 3.3 proves the exact law of large numbers for a continuum of stochastic processes. Some of the results are proven in this section, and the rest of the proofs can be found in Section 9.2.¹⁸

3.1 Uncorrelatedness and the exact law of large numbers

The following lemma is a generalized version of the Fubini property for a rich product probability space.

Lemma 3.1 *Let g and h be real-valued square integrable processes on a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Define a real-valued function G on $I \times I \times \Omega$ by letting $G(s, i, \omega) = g(s, \omega)h(i, \omega)$ for $(s, i, \omega) \in I \times I \times \Omega$. Then*

(i) *For λ -almost all $s \in I$, λ -almost all $i \in I$, the function $G(s, i, \cdot)$ on Ω is P -integrable.*

(ii) *For λ -almost all $s \in I$, the function $\int_{\Omega} G(s, i, \omega) dP(\omega)$ in terms of $i \in I$ is λ -integrable.*

(iii) *The function $\int_I \int_{\Omega} G(s, i, \omega) dP(\omega) d\lambda(i)$ in terms of $s \in I$ is λ -integrable.*

(iv) *The function $\int_I g(s, \omega) d\lambda(s) \int_I h(i, \omega) d\lambda(i)$ in terms of $\omega \in \Omega$ is P -integrable, and*

$$\int_I \int_I \int_{\Omega} g(s, \omega) h(i, \omega) dP(\omega) d\lambda(i) d\lambda(s) = \int_{\Omega} \int_I g(s, \omega) d\lambda(s) \int_I h(i, \omega) d\lambda(i) dP(\omega).$$

Proof: (i) The Fubini property implies that for λ -almost all $s \in I$, λ -almost all $i \in I$, both the functions $g_s(\omega)$ and $h_i(\omega)$ in terms of $\omega \in \Omega$ are P -square integrable; and hence the product $G(s, i, \omega) = g(s, \omega)h(i, \omega)$ is P -integrable on Ω .

¹⁸The mathematical results stated in this section, as well as in the next two sections, are valid on any extended Fubini product probability spaces since only the Fubini property is used in the proofs. However, we choose to state them on rich product probability spaces, to ensure that we work with nontrivial processes.

(ii) Any real-valued function φ that is P -square integrable on Ω is still square integrable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$; and thus the product $\varphi(\omega)h(i, \omega)$ is integrable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

For λ -almost all $s \in I$, g_s is P -square integrable; and hence the product $g_s(\omega)h(i, \omega)$ is integrable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. The Fubini property implies that for λ -almost all $s \in I$, the function $\int_{\Omega} G(s, i, \omega)dP(\omega)$ in terms of $i \in I$ is λ -integrable, and also

$$\begin{aligned} \int_I \int_{\Omega} G(s, i, \omega)dP(\omega)d\lambda(i) &= \int_I \int_{\Omega} g(s, \omega)h(i, \omega)dP(\omega)d\lambda(i) \\ &= \int_{\Omega} g(s, \omega) \int_I h(i, \omega)d\lambda(i)dP(\omega). \end{aligned} \quad (1)$$

(iii) Since $\int_I h(i, \omega)d\lambda(i)$ is P -square integrable on Ω , the product $g(s, \omega) \int_I h(i, \omega)d\lambda(i)$ is integrable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. The Fubini property implies that the function

$$\int_{\Omega} g(s, \omega) \int_I h(i, \omega)d\lambda(i)dP(\omega)$$

in terms of $s \in I$ is λ -integrable, and also

$$\int_I \int_{\Omega} g(s, \omega) \int_I h(i, \omega)d\lambda(i)dP(\omega)d\lambda(s) = \int_{\Omega} \int_I g(s, \omega)d\lambda(s) \int_I h(i, \omega)d\lambda(i)dP(\omega). \quad (2)$$

(iv) Since both $\int_I g(s, \omega)d\lambda(s)$ and $\int_I h(i, \omega)d\lambda(i)$ are P -square integrable, their product is P -integrable. By the identities in Equations (1) and (2), we have

$$\int_I \int_I \int_{\Omega} g(s, \omega)h(i, \omega)dP(\omega)d\lambda(i)d\lambda(s) = \int_{\Omega} \int_I g(s, \omega)d\lambda(s) \int_I h(i, \omega)d\lambda(i)dP(\omega),$$

and we are done. ■

The following is a formal definition of essential uncorrelatedness in the setting of a rich product probability space.

Definition 3.2 *Let f be a real-valued square integrable process on a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. The random variables f_i are said to be essentially uncorrelated, if for λ -almost all $s \in I$, f_s is uncorrelated with f_i for λ -almost all $i \in I$*

The following proposition shows that it is very simple to obtain an exact law of large numbers in terms of sample means in the framework of a rich product probability space. In particular, it shows that if a square integrable process is essentially uncorrelated, then its sample means are essentially constant. Note that only the Fubini property is used in the computations below. The richer measure-theoretic framework simply guarantees that the processes considered can be nontrivial.

Proposition 3.3 *Let f be a real-valued square integrable process on a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. If the random variables f_i are essentially uncorrelated, then for P -almost all $\omega \in \Omega$, the sample mean $Ef_\omega = \int_I f_\omega d\lambda$ is the same as the mean of the process $Ef = \int_{I \times \Omega} f d\lambda \boxtimes P$.*

Proof: Take $g = h = f - Ef_i$ in Lemma 3.1 (iv). Then, for λ -almost all $s \in I$, λ -almost all $i \in I$, $\int_\Omega g_s(\omega)h_i(\omega)dP(\omega) = 0$. Hence

$$\int_\Omega \int_I g(s, \omega) d\lambda(s) \int_I h(i, \omega) d\lambda(i) dP(\omega) = \int_\Omega (Ef_\omega - Ef)^2 dP(\omega) = 0,$$

which implies $Ef_\omega = Ef$ for P -almost all $\omega \in \Omega$. ■

It is argued in [14], [15] and [33]-[36] that a stronger conclusion than that of Proposition 3.3 may often be important from an economic point of view. Namely, the sample means are essentially constant on all the non-negligible coalitions, i.e., for any set $A \in \mathcal{I}$ with $\lambda(A) > 0$, $\int_A f_\omega d\lambda = \int_A Ef_i d\lambda$ holds for P -almost all $\omega \in \Omega$ (it is obvious when $\lambda(A) = 0$). It turns out that this stronger conclusion is necessary and sufficient for the random variables f_i to be essentially uncorrelated. The part (ii) \implies (i) is a converse law of large numbers.

Theorem 3.4 *Let f be a real-valued square integrable process on a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Then the following are equivalent.*

(i) *The random variables f_i are essentially uncorrelated.*

(ii) *For any set $A \in \mathcal{I}$ with $\lambda(A) > 0$, $\int_A f_\omega d\lambda = \int_A Ef_i d\lambda$ holds for P -almost all $\omega \in \Omega$.*

Proof: Let $A, B \in \mathcal{I}$, $1_A, 1_B$ be their indicator functions, and $g = 1_A(f - Ef_i)$ and $h = 1_B(f - Ef_i)$ in Lemma 3.1 (iv). Then,

$$\begin{aligned} & \int_A \int_B \int_\Omega (f_s(\omega) - Ef_s)(f_i(\omega) - Ef_i) dP(\omega) d\lambda(i) d\lambda(s) \\ &= \int_\Omega \int_A (f_\omega(s) - Ef_s) d\lambda(s) \int_B (f_\omega(i) - Ef_i) d\lambda(i) dP(\omega). \end{aligned} \quad (3)$$

(i) \implies (ii) is thus obvious from this identity by taking $A = B$.¹⁹

On the other hand, assume (ii) holds. Then

$$\int_A \int_B \int_\Omega (f_s(\omega) - Ef_s)(f_i(\omega) - Ef_i) dP(\omega) d\lambda(i) d\lambda(s) = 0$$

for any $A, B \in \mathcal{I}$. By the arbitrary choice of A , the integrable function $\int_B \int_\Omega (f_s(\omega) - Ef_s)(f_i(\omega) - Ef_i) dP(\omega) d\lambda(i)$ is zero for λ -almost all $s \in I$. Hence, for λ -almost all $s \in I$, the arbitrary choice of B implies that the integrable function $\int_\Omega (f_s(\omega) - Ef_s)(f_i(\omega) - Ef_i) dP(\omega)$ is zero for λ -almost all $i \in I$. Thus (i) follows. ■

¹⁹One can also observe that the restriction f^A to $A \times \Omega$ still satisfies the condition of Proposition 3.3 when the restriction of $\lambda \boxtimes P$ to $A \times \Omega$ is rescaled to a probability measure. Thus (i) \implies (ii) follows.

3.2 The exact law of large numbers in sample distributions and its converse

The following theorem is an analog of Theorem 3.4 in terms of sample distributions.

Theorem 3.5 *Let f be a process from a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a Polish space X . Then the following are equivalent.*

(i) *The random variables f_i are essentially pairwise independent.*

(ii) *For any set $A \in \mathcal{I}$ with $\lambda(A) > 0$, the sample distribution $\lambda(f_\omega^A)^{-1}$ is the same as the distribution $(\lambda^A \boxtimes P)(f^A)^{-1}$ of the process f^A for P -almost all $\omega \in \Omega$, where f^A is the restriction of f to $A \times \Omega$, and λ^A and $(\lambda^A \boxtimes P)$ the probability measures on A and $A \times \Omega$ rescaled from λ and $(\lambda \boxtimes P)$ respectively (see Proposition 2.4 (2)).*

Proof: For (i) \implies (ii), we only consider the case $A = I$ for notational simplicity. Fix a countable open base $\{O_n\}_{n=1}^\infty$ for X so that it is closed under finite intersections. By the independence assumption, the random variables $(1_{f^{-1}(O_n)})_i$ in the process $1_{f^{-1}(O_n)}$ are essentially uncorrelated, where 1_D denotes the indicator function of the set D in some space. By Proposition 3.3, there is a P -null set S_n such that for any $\omega \notin S_n$, f_ω is \mathcal{I} -measurable and $\lambda(f_\omega^{-1}(O_n)) = (\lambda \boxtimes P)(f^{-1}(O_n))$. Let S be the union of all the S_n . Then S is still a P -null set.

For each $\omega \in \Omega$, let μ_ω be the distribution of the sample function f_ω (which exists almost surely by the Fubini property). The distribution of f as a random variable is denoted by μ . Then, for any $\omega \notin S$,

$$\mu_\omega(O_n) = \lambda(f_\omega^{-1}(O_n)) = (\lambda \boxtimes P)(f^{-1}(O_n)) = \mu(O_n)$$

for all $n \geq 1$. Since the class of all the O_n generates the Borel algebra, and is also closed under finite intersections, i.e., a π -system, it follows from the result on the uniqueness of measures (see [9], p. 45) that $\mu_\omega = \mu$ for any $\omega \notin S$.

Next, we consider (ii) \implies (i). Fix $m, n \geq 1$. As in the proof of Theorem 3.4, let $A, B \in \mathcal{I}$, and take $g = 1_A(1_{O_m}(f) - E1_{O_m}(f_i))$ and $h = 1_B(1_{O_n}(f) - E1_{O_n}(f_i))$ in Lemma 3.1 (iv). Then,

$$\int_A \int_B \int_\Omega g_s(\omega) h_i(\omega) dP(\omega) d\lambda(i) d\lambda(s) = \int_\Omega \int_A g_\omega(s) d\lambda(s) \int_B h_\omega(i) d\lambda(i) dP(\omega). \quad (4)$$

(ii) implies that the right-hand side of Equation (4) is zero. By the arbitrary choice of A and B , the same proof as in Theorem 3.4 can be used to claim that for λ -almost all $s \in I$, g_s is orthogonal to h_i for λ -almost all $i \in I$. This means that for λ -almost all $s \in I$, λ -almost all $i \in I$,

$$\int_\Omega (1_{O_m}(f))_s \cdot (1_{O_n}(f))_i dP = \int_\Omega (1_{O_m}(f))_s dP \int_\Omega (1_{O_n}(f))_i dP. \quad (5)$$

Thus, there is a set \mathcal{N} such that Equation (5) holds for all $(s, i) \notin \mathcal{N}$, and λ -almost all $s \in I$, the set \mathcal{N}_s is a λ -null set. By grouping countably many such sets together, there exists a set \mathcal{N}' such that the set \mathcal{N}'_s is a λ -null set, and

$$P(f_s^{-1}(O_m) \cap f_i^{-1}(O_n)) = P(f_s^{-1}(O_m)) \cdot P(f_i^{-1}(O_n))$$

holds for all $(s, i) \notin \mathcal{N}'$, and for all $m, n \geq 1$. Thus, for each $(s, i) \notin \mathcal{N}'$, the joint distribution $P(f_s, f_i)^{-1}$ agrees with the product of its marginals on a π -system $\{O_m \times O_n : m, n \geq 1\}$ for $X \times X$. Hence, $P(f_s, f_i)^{-1} = P f_s^{-1} \otimes P f_i^{-1}$ by a result on the uniqueness of measures (see [9], p. 45). This means that f_s and f_i are independent for all $(s, i) \notin \mathcal{N}'$. ■

The following obvious corollary is an exact law of large numbers for a continuum of random variables in the setting of sample distributions.

Corollary 3.6 *Let f be a process from a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a Polish space X . If the random variables f_i are essentially pairwise independent, then the sample distribution λf_ω^{-1} is the same as the distribution $(\lambda \boxtimes P) f^{-1}$ for P -almost all $\omega \in \Omega$.*

In Proposition 3.3, the process f is assumed to be square integrable. However, if the real-valued random variables f_i are essentially pairwise independent, we only need to assume integrability for f in order to obtain the same conclusion. The following corollary follows immediately from Corollary 3.6 by the fact that $E f_\omega = \int_{\mathbb{R}} x d\lambda f_\omega^{-1}$ and $E f = \int_{\mathbb{R}} x d\lambda \boxtimes P f^{-1}$.

Corollary 3.7 *Let f be a real-valued integrable process on a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. If the random variables f_i are essentially pairwise independent, then the sample mean $E f_\omega = \int_I f_\omega d\lambda$ is the same as the mean $E f = \int_{I \times \Omega} f d\lambda \boxtimes P$ of the process f for P -almost all $\omega \in \Omega$.*

The following corollary generalizes Corollaries 3.6 and 3.7 to the setting of conditional independence on a finite sub- σ -algebra of \mathcal{F} . It can be proved by applying the results in Corollaries 3.6 and 3.7 to the processes restricted to $I \times C_k$ for some measurable subsets C_k of Ω .

Corollary 3.8 *Let f be a process from a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a Polish space X , and \mathcal{C} a σ -algebra generated by a \mathcal{F} -measurable partition $\{C_1, \dots, C_n\}$ of Ω with $P(C_k) > 0$ for $1 \leq k \leq n$. Assume that the random variables f_i are essentially pairwise independent conditioned on \mathcal{C} , i.e., for each $1 \leq k \leq n$, the restriction f^{C_k} of f to $I \times C_k$ is a process with essentially pairwise independent random variables $f_i^{C_k}$ on the probability space*

$(C_k, \mathcal{F}^{C_k}, P^{C_k})$ rescaled from the original sample space (see Proposition 2.4 (2)). Then the following results hold.

(1) The sample distribution λf_ω^{-1} is essentially \mathcal{C} -measurable with $\lambda f_\omega^{-1}(B) = (\lambda \boxtimes P^{C_k})(f^{C_k})^{-1}$ for P -almost all $\omega \in C_k$.

(2) When f is a real-valued $\lambda \boxtimes P$ -integrable process, the sample mean $E f_\omega$ is the average $\frac{1}{P(C_k)} \int_{I \times C_k} f d\lambda \boxtimes P$ of f on $I \times C_k$ for P -almost all $\omega \in C_k$.

The following is a duality result for the random variables and sample functions in a process, which will be used to give a proof of Proposition 2.4 in the appendix.

Theorem 3.9 *Let f be a process from a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a Polish space X . Then the following are equivalent.*

(i) *The random variables f_i are essentially pairwise independent; and the f_i 's have a common distribution μ for λ -almost all $i \in I$.*

(ii) *The sample functions f_ω are essentially pairwise independent; and the f_ω 's have a common distribution μ for P -almost all $\omega \in \Omega$.*

3.3 The exact law of large numbers for a continuum of stochastic processes

Let T be a set of time parameters, which is assumed to be the set of \mathbb{Z}_+ of positive integers or an interval (starting from 0) in the set \mathbb{R}_+ of non-negative real numbers. Let \mathcal{B}_T be the power set of T when T is the countable set \mathbb{Z}_+ , and the Borel σ -algebra on T when T is an interval.

Let F be a real-valued measurable function on the mixed product measurable space $((I \times \Omega) \times T, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}_T)$. For any $i \in I$, let F_i be the function on $\Omega \times T$ with $F_i(\omega, t) = F(i, \omega, t)$; and for any $\omega \in \Omega$, let F_ω be the function on $I \times T$ with $F_\omega(i, t) = F(i, \omega, t)$. Lemma 3.10 below says that both F_i and F_ω are measurable stochastic processes. Thus, F can be viewed as a family of stochastic processes, $F_i, i \in I$, with a sample space (Ω, \mathcal{F}, P) and a time parameter space T . For $\omega \in \Omega$, F_ω is called an empirical process with the index space $(I, \mathcal{I}, \lambda)$ as the sample space. The function F itself can also be viewed as a stochastic process with sample space $I \times \Omega$ and time parameter space T .

Lemma 3.10 *Let F be a real-valued measurable function on the mixed product measurable space $((I \times \Omega) \times T, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}_T)$. Then, F_i is $\mathcal{F} \otimes \mathcal{B}_T$ -measurable for λ -almost all $i \in I$; and F_ω is $\mathcal{I} \otimes \mathcal{B}_T$ -measurable for P -almost all $\omega \in \Omega$.*

The following are the formal definitions on the independence and finite dimensional distributions of stochastic processes.

Definition 3.11 (1) Two real-valued stochastic processes φ and ψ on the same sample space with time parameter space T are said to be independent, if, for any positive integers m, n , and for any t_1^1, \dots, t_m^1 in T , and t_1^2, \dots, t_n^2 in T , the random vectors $(\varphi_{t_1^1}, \dots, \varphi_{t_m^1})$ and $(\psi_{t_1^2}, \dots, \psi_{t_n^2})$ are independent.

(2) We say that the stochastic processes $\{F_i, i \in I\}$ are essentially pairwise independent, if, for λ -almost all $s \in I$, λ -almost all $i \in I$, the stochastic processes F_s and F_i are independent.

Definition 3.12 (1) Two real-valued stochastic processes φ and ψ on some (possibly different) sample spaces with time parameter space T are said to have the same finite dimensional distributions, if, for any $t_1, \dots, t_n \in T$, the random vectors $(\varphi_{t_1}, \dots, \varphi_{t_n})$ and $(\psi_{t_1}, \dots, \psi_{t_n})$ have the same distribution.

(2) We say that the stochastic processes $\{F_i, i \in I\}$ have essentially the same finite dimensional distributions if there is a real-valued stochastic process G with time parameter space T such that for λ -almost all $i \in I$, the stochastic processes F_i and G have the same finite dimensional distributions.

When the time space T is discrete, a real-valued discrete parameter stochastic process can be viewed as a random variable taking values in \mathbb{R}^∞ . Thus, Corollary 3.6 can be used to obtain an exact law of large numbers for a continuum of discrete time processes. Similarly, for the case that T is an interval and for the stochastic processes whose paths come from some function space with a complete separable metric (for example, the continuous function space on T or the Skorokhod space as in [6]), they can be regarded as random variables in the function space. Corollary 3.6 can be applied again.

To consider more general continuous time processes, one may need a technique used in [20] that relates a continuous time process to a discrete time process. In particular, it is shown that for a continuous time process x on $(\Lambda \times T, \mathcal{A} \otimes \mathcal{B}_T)$ with a probability measure ν on (Λ, \mathcal{A}) , there exists a sequence $\{t_n\}_{n=1}^\infty$ in T and a Borel function $\psi : \mathbb{R}^\infty \times T \rightarrow \mathbb{R}$, such that for any $t \in T$, $x(q, t) = \psi(\{x(q, t_n)\}_{n=1}^\infty, t)$ for ν -almost all $q \in \Lambda$.²⁰ Thus, x has a version \hat{x} (i.e., for each t , $x_t = \hat{x}_t$ almost surely) such that for ν -almost all $q \in \Lambda$, $\hat{x}(q, t) = \psi(\{\hat{x}(q, t_n)\}_{n=1}^\infty, t)$ for all $t \in T$. In probability theory, one can always work with a version of a stochastic process with some more regularity. We shall assume that F has this property as a stochastic process on $\Lambda = I \times \Omega$, i.e., there exists a sequence $\{t_n\}_{n=1}^\infty$ in T such that for $\lambda \boxtimes P$ -almost all $(i, \omega) \in I \times \Omega$, $F(i, \omega, t) = \psi(\{F(i, \omega, t_n)\}_{n=1}^\infty, t)$ for all $t \in T$.²¹

²⁰See [20], p. 172 and [26]; denote the composition $\phi \circ \psi$ there simply by ψ . The proof in [20], which is stated for $T = \mathbb{R}_+$, works for the case that T is a general interval.

²¹This is trivially satisfied when $T = \mathbb{Z}_+$; take $t_n = n$ for any $n \geq 1$, and $\psi(\{z_n\}_{n=1}^\infty, t) = z_t$ for any

We shall now present an exact law of large numbers for a continuum of stochastic processes.²²

Theorem 3.13 *Let F be a real-valued measurable function on the mixed product measurable space $((I \times \Omega) \times T, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}_T)$. If the stochastic processes F_i are essentially pairwise independent, then, for P -almost all $\omega \in \Omega$, the empirical process F_ω on $I \times T$ and F viewed as a stochastic process have the same finite dimensional distributions.*

The following is an analog of Theorem 3.9 in the setting of a continuum of stochastic processes.

Theorem 3.14 *Let F be a real-valued measurable function on the mixed product measurable space $((I \times \Omega) \times T, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}_T)$. The stochastic processes F_i are essentially pairwise independent and have essentially the same finite dimensional distributions as a fixed stochastic process H with a sample space $(\Lambda, \mathcal{A}, \nu)$ if and only if the empirical processes F_ω satisfy the same properties.*

In particular, if H is a Markov process, then almost all empirical processes F_ω behave exactly like H since a Markov process is completely characterized by its finite dimensional distributions (see, for example, [20], p. 163). Thus, if a continuum of physical or social entities move independently of each other and each of them can be described by a Brownian motion, then the observable empirical process is also exactly governed by a Brownian motion process (see [34], p. 491 for a detailed discussion).

4 Pareto optimality, equilibrium and characterization of insurable risks

This section develops a model for a large economy with individual risks based on a rich product probability space. While a replicated economy with convex preferences was considered in [30], our model in Section 4.1 is a general economy with possibly non-convex preferences.²³ While [30] used the second welfare theorem to support a Pareto optimal equilibrium by a sure price, we work out the detailed relationship between Pareto optimality, pseudo equilibrium and equilibrium for the original economy and an averaged economy in Section 4.2. A complete insurance system was introduced in [30] and its asymptotic viability is shown under the assumption of independent (or approximately independent) agents. The primary contribution of Section 4.3

$\{z_n\}_{n=1}^\infty \in \mathbb{R}^\infty, t \in \mathbb{Z}_+$. It also holds automatically for the stochastic process F when its sample paths $F_{i\omega}$ are right continuous with left limits $\lambda \boxtimes P$ -almost surely.

²²Similar to Theorem 3.5, a converse to the result in Theorem 3.13 can also be proven.

²³[36] considered the exact law of large numbers for a continuum of correspondences and economies with random preferences.

is to show that essentially pairwise independent risks are insurable; and more surprisingly, only the essentially pairwise independent risks are insurable. The proofs of all the results in this Section except for Theorem 4.10 are given in Section 9.3.

4.1 The economic model

Let $(I, \mathcal{I}, \lambda)$ be an atomless measure space of agents with $\lambda(I) = 1$, (Ω, \mathcal{F}, P) a sample probability space modeling all the uncertainty in the economy, and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ a rich product probability space.

Let $S = \{1, 2, \dots, h\}$ be a finite set of states that all the agents are possibly in. For an individual agent i , she may find herself in some of these states, not necessarily in all of them. Let f be a measurable mapping from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S ; $f(i, \omega)$ gives the particular state that agent i may find herself in for a particular realization of ω .

We shall denote the probability of agent i being in state s , $P(f_i^{-1}(\{s\}))$ by $\pi_i(s)$. The Fubini property implies that for each $s \in S$, $\pi_i(s)$ is \mathcal{I} -measurable on I . Let $S_i = \{s \in S : \pi_i(s) > 0\}$ with n_i the number of elements in S_i . These are the only states that are relevant to agent i . Note that S_i is a nonempty subset of the finite set S for each $i \in I$. We can thus find a measurable partition $\{I^1, \dots, I^K\}$ of I such that for each $1 \leq k \leq K$, the agents i in I^k have the same S_i , which is denoted by S^k . Without loss of generality, we assume that the K sets S^1, \dots, S^K are different from each other and $\lambda(I^k) > 0$ for each k .

For simplicity, we shall work with a pure exchange economy with m commodities for consumption. For an agent $i \in I$, $x_s \in \mathbb{R}_+^m$ denotes her consumption in state $s \in S_i$. Thus the consumption space for agent i is $(\mathbb{R}_+^m)^{S_i}$, which is a product space of dimension $m \cdot n_i$. For each $i \in I$ and a consumption plan $x \in (\mathbb{R}_+^m)^{S_i}$, let $E_i x$ be the expected consumption $\sum_{s \in S_i} \pi_i(s) x(s)$. When $i \in I^k$, $(\mathbb{R}_+^m)^{S_i}$ is simply $(\mathbb{R}_+^m)^{S^k}$.

For each $1 \leq k \leq K$, let \mathcal{P}^k be the space of complete continuous preorderings \preceq , i.e., reflexive, transitive and complete binary relations on $(\mathbb{R}_+^m)^{S^k}$ whose graphs are closed subsets of $(\mathbb{R}_+^m)^{S^k} \times (\mathbb{R}_+^m)^{S^k}$. The space \mathcal{P}^k is endowed with the topology of closed convergence.²⁴

A preference-endowment mapping $\alpha = (\alpha_p, \alpha_e)$ is a mapping from the space of economic agents $(I, \mathcal{I}, \lambda)$ to the space of preference-endowment pairs $\cup_{k=1}^K \mathcal{P}^k \times (\mathbb{R}_+^m)^{S^k}$ such that (1) the restriction $\alpha|_{I^k}$ of α to I^k is a measurable mapping from I^k to $\mathcal{P}^k \times (\mathbb{R}_+^m)^{S^k}$; (2) for each $i \in I^k$, $\alpha_p(i)$ is the preference relation of agent i , which is an element of \mathcal{P}^k and also denoted by \preceq_i ; (3) for each $i \in I^k$, $\alpha_e(i)$ is the initial endowment of agent i , which is an element of $(\mathbb{R}_+^m)^{S^k}$ and also denoted by e_i or $e(i)$; (4) $\int_{I^k} e(i) d\lambda$ is finite. We shall denote the economy summarized as

²⁴See [19], p. 86 for the definition of a preference relation, p. 19 and p. 96 for the topology of closed convergence on preference relations. Proposition 4 in [16] showed that \mathcal{P}^k is a G_δ -set in the compact metric space of all preferences, and thus a Polish space in itself.

$[(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P), f, (\alpha_p, \alpha_e)]$ by \mathcal{E} . A (deterministic) price system p for the economy \mathcal{E} is a nonzero vector in \mathbb{R}^m .

An allocation ϕ for the economy \mathcal{E} is a mapping from $(I, \mathcal{I}, \lambda)$ to $\cup_{k=1}^K (\mathbb{R}_+^m)^{S^k}$ such that the restriction $\phi|_{I^k}$ of ϕ to I^k is an integrable function from I^k to $(\mathbb{R}_+^m)^{S^k}$. For each $i \in I$, $\phi(i)$ (or ϕ_i) is the consumption plan of agent i , and $\phi_i(s)$ her consumption at state s . It is clear that the expected consumption $E_i\phi(i)$ defines an integrable function on I . An allocation ϕ is said to be *feasible* if it has the same mean-expected consumption as the allocation defined by the initial endowments, i.e., $\int_I E_i\phi(i)d\lambda(i) = \int_I E_ie(i)d\lambda(i)$.

Definition 4.1 *Let ϕ be a feasible allocation for the economy \mathcal{E} .*

(1) *ϕ is Pareto optimal if there does not exist any other feasible allocation ψ such that $\phi_i \preceq_i \psi_i$ for λ -almost all $i \in I$, and $\lambda(\{i \in I : \phi_i \prec_i \psi_i\}) > 0$.*

(2) *ϕ is a pseudo-equilibrium if there is a price system p such that for λ -almost all $i \in I$, $\phi(i)$ is a minimizer for expected expenditure in the sense that for any $z \in (\mathbb{R}_+^m)^{S_i}$, $\phi(i) \preceq_i z$ implies $p \cdot E_i\phi(i) \leq p \cdot E_iz$.*

(3) *ϕ is an equilibrium if there is a price system p such that for λ -almost all $i \in I$, $\phi(i)$ is a maximal element in the expected budget set $\{z \in (\mathbb{R}_+^m)^{S_i} : p \cdot E_iz \leq p \cdot E_i\phi(i)\}$, i.e., for any $z \in (\mathbb{R}_+^m)^{S_i}$, $p \cdot E_iz \leq p \cdot E_i\phi(i)$ implies $z \preceq_i \phi(i)$.*

Corresponding to the economy \mathcal{E} , we shall now consider an averaged non-random economy $\bar{\mathcal{E}}$ as in [30]. Let the initial endowment $\bar{e}(i)$ of agent i in the averaged economy $\bar{\mathcal{E}}$ be $E_ie(i)$. It is clear that $\int_I \bar{e}(i)d\lambda$ is finite. We shall now define a preference relation for each agent i in the economy $\bar{\mathcal{E}}$. Take any $1 \leq i \leq K$, and let $i \in I^k$ be a fixed agent. Then E_i defines a mapping from $(\mathbb{R}_+^m)^{S^k}$ to \mathbb{R}_+^m . For any given $a \in \mathbb{R}_+^m$, the set $E_i^{-1}(\{a\}) = \{z \in (\mathbb{R}_+^m)^{S^k} : E_iz = a\}$ is compact since $S^k = S_i = \{s \in S : \pi_i(s) > 0\}$. Since \preceq_i is a continuous preordering, maximal elements exist on $E_i^{-1}(\{a\})$ under \preceq_i ; let $\Psi^k(i, a)$ be the set of all such maximal elements. For simplicity, we may often drop the superscript k when there is no confusion. Define a binary relation \preceq_i^- on \mathbb{R}_+^m such that for any $a, b \in \mathbb{R}_+^m$, $a \preceq_i^- b$ if and only if $x_a \preceq_i x_b$, where $x_a \in \Psi^k(i, a)$ and $x_b \in \Psi^k(i, b)$.

Lemma 4.2 (1) *The binary relation \preceq_i^- is a well-defined, continuous and complete preordering.*

(2) *For any \mathcal{I} -measurable mappings $a(\cdot)$ and $b(\cdot)$ from I to \mathbb{R}_+^m , the set $\{i \in I : a(i) \preceq_i^- b(i)\}$ is measurable.*

For any \mathcal{I} -measurable mapping $a(\cdot)$ from I to \mathbb{R}_+^m , the proof of Lemma 4.2 in the appendix also shows that $\Psi^k(i, a(i))$ defines a measurable correspondence $\Phi_a^k(\cdot)$ with a measurable

selection $\phi_a^k(\cdot)$ on I^k . For simplicity, we shall often use $\Phi_a(i)$ and $\phi_a(i)$ instead of $\Phi_a^k(i)$ and $\phi_a^k(i)$.

4.2 Pareto optimality and equilibrium

The following lemma shows that the individual consumption x_i in an equilibrium or Pareto optimal allocation in the economy \mathcal{E} is optimal in $E_i^{-1}(\{E_i x_i\})$. This kind of condition is used in Part (2) of Propositions 4.4, 4.5 and 4.6.

Lemma 4.3 *Let x be an equilibrium or Pareto optimal allocation in the economy \mathcal{E} and $a(i) = E_i x_i$, $i \in I$. Then, for λ -almost all $i \in I$, $x_i \in \Phi_a(i)$.*

In the next three propositions, we relate Pareto optimality, pseudo equilibrium and equilibrium in the economy \mathcal{E} to those in the averaged economy $\bar{\mathcal{E}}$.

Proposition 4.4 *Let x be a feasible allocation in the economy \mathcal{E} , and $a(i) = E_i x_i$. Then the following are equivalent.*

- (1) x is Pareto optimal in the economy \mathcal{E} .
- (2) The allocation a is Pareto optimal in the economy $\bar{\mathcal{E}}$; and for λ -almost all $i \in I$, $x_i \in \Phi_a(i)$.

Proposition 4.5 (1) *Let (x, p) be a pseudo-equilibrium in the economy \mathcal{E} , and $a(i) = E_i x_i$. Then (a, p) is a pseudo-equilibrium in the economy $\bar{\mathcal{E}}$.*

(2) *Let (b, p) be a pseudo-equilibrium in the economy $\bar{\mathcal{E}}$ and y a feasible allocation in \mathcal{E} such that for λ -almost all $i \in I$, $y_i \in \Phi_b(i)$. Then (y, p) is a pseudo-equilibrium in the economy \mathcal{E} .*

Proposition 4.6 (1) *Let (x, p) be an equilibrium in the economy \mathcal{E} , and $a(i) = E_i x_i$. Then (a, p) is an equilibrium in the economy $\bar{\mathcal{E}}$.*

(2) *Let (b, p) be an equilibrium in the economy $\bar{\mathcal{E}}$ and y a feasible allocation in \mathcal{E} such that $y_i \in \Phi_b(i)$ for $i \in I$. Then (y, p) is an equilibrium in the economy \mathcal{E} .*

It is easy to see that if \preceq_i is monotonic (i.e., $y \prec_i z$ if $y, z \in (\mathbb{R}_+^m)^{S_i}$ with $y \leq z$ and $y \neq z$), so is \preceq_i^- . Since the economy $\bar{\mathcal{E}}$ is a usual measure-theoretic economy without uncertainty, the following type of results are standard in the literature; see [18] and [19].

Proposition 4.7 *Assume that the preference \preceq_i is monotonic for each agent $i \in I$ and $\int_I E_i e_i d\lambda$ strictly positive. Let a be a feasible allocation in the economy $\bar{\mathcal{E}}$. Then the following are equivalent.*

- (1) The allocation a is Pareto optimal.
- (2) There is a strictly positive price system p such that (a, p) is a pseudo-equilibrium in $\bar{\mathcal{E}}$.
- (3) There is a strictly positive price system p such that (a, p) is an equilibrium in $\bar{\mathcal{E}}$.

By Propositions 4.4-4.7, the following corollary is obvious.

Corollary 4.8 *Assume that the preference \preceq_i is monotonic for each agent $i \in I$ and $\int_I E_i e_i d\lambda$ strictly positive. Let x be a feasible allocation in the economy \mathcal{E} . Then the following are equivalent.*

- (1) The allocation x is Pareto optimal.
- (2) There is a strictly positive price system p such that (x, p) is a pseudo-equilibrium in \mathcal{E} .
- (3) There is a strictly positive price system p such that (x, p) is an equilibrium in \mathcal{E} .

4.3 The insurance system

In this subsection, we consider an insurance system that allows free transfers between states by economic agents as in [30]. For agent i , $v_i : S_i \rightarrow \mathbb{R}$ is a net transfer if its expected value $\int_{\omega \in \Omega} v_i(f_i(\omega)) dP(\omega) = \sum_{s \in S_i} \pi_i(s) v_i(s) = 0$.

Under the assumptions of Corollary 4.8, a feasible allocation x is Pareto optimal if and only if there exists a strictly positive price system p such that (x, p) is an equilibrium in \mathcal{E} . The consumption plan of agent i can be viewed as a mapping $x_i(f_i(\cdot))$ on Ω . Then $E_i x_i$ is simply the expectation of $x_i(f_i(\cdot))$ on Ω . Thus, almost every agent maximizes her preference within her expected budget in an equilibrium. Only one budget condition is needed here. On the other hand, if we take any two net transfers v_i^1 and v_i^2 for agent i , and any $z \in (\mathbb{R}_+^m)^{S_i}$ that satisfies the budget conditions for the individual states in the form

$$pz(s) + v_i^2(s) \leq px_i(s) + v_i^1(s) \text{ for } s \in S_i,$$

then we still have $z \preceq_i x_i$. That is, no matter what kind of net transfers an agent chooses, if budget conditions are satisfied for individual states, then the original x_i is still optimal.

Next, we consider the behavior of individual agents in an equilibrium. Suppose that her real income at state s is $R_i(s)$. She will first choose x_i such that it is the maximal element within her expected budget. Then she will choose some net transfers $v_i(s)$ such that her expenditure $px_i(s)$ does not exceed her resources $R_i(s) + v_i(s)$ in each state s . In fact, one can take

$$v_i(s) = px_i(s) - R_i(s) + \sum_{s \in S_i} \pi_i(s) R_i(s) - pE_i x_i.$$

It is clear that $\sum_{s \in S_i} \pi_i(s) v_i(s) = 0$. Since $\sum_{s \in S_i} \pi_i(s) R_i(s) - p E_i x_i \geq 0$, we have $v_i(s) \geq p x_i(s) - R_i(s)$, which is to say $p x_i(s) \leq R_i(s) + v_i(s)$ for all $s \in S_i$.

Note that for each agent t , the transfer $v_i(f_i(\omega))$ satisfies $\int_{\omega \in \Omega} v_i(f_i(\omega)) dP = 0$. For a sample realization ω , the transfer for agent i is $v_i(f_i(\omega))$; and the average transfer for the society is $\int_{i \in I} v_i(f_\omega(i)) d\lambda(i) = 0$.

For an agent i , her net transfers v_i are defined on S_i . One can simply define v_i on the whole S by letting $v_i(s) = 0$ for $s \notin S_i$. The net transfers $v_i, i \in I$ are said to be measurable if for each $s \in S$, $v_i(s)$ is \mathcal{I} -measurable on I .

Definition 4.9 *The risks $f_i, i \in I$ are said to be insurable if for any measurable net transfers $v_i, i \in I$,*

$$\int_{i \in I} v_i(f_\omega(i)) d\lambda(i) = 0 \text{ for } P\text{-almost all } \omega \in \Omega,$$

which means that the budget of the whole insurance system is “exactly balanced” almost surely.

Theorem 4.10 *The risks $f_i, i \in I$ are insurable if and only if they are essentially pairwise independent.*

Proof: Define a process g from $I \times \Omega$ to \mathbb{R} by letting $g(i, \omega) = v_i(f(i, \omega))$. Then, g is still $\mathcal{I} \boxtimes \mathcal{F}$ -measurable; and $E g_i = 0$ for λ -almost all $i \in I$. If the f_i 's are essentially pairwise independent, the g_i 's must be essentially uncorrelated. Thus, the exact law of large numbers in Proposition 3.3 implies that

$$\int_{i \in I} v_i(f_\omega(i)) d\lambda = \int_{i \in I} g_\omega(i) d\lambda = 0$$

for P -almost all $\omega \in \Omega$. This means that the risks $f_i, i \in I$ are insurable.

On the other hand, assume that the risks $f_i, i \in I$ are insurable. Take any $A \in \mathcal{I}$ with $\lambda(A) > 0$. Take any $s_0 \in S$. Define measurable net transfers $v_i^{s_0}, i \in I$ by letting

$$v_i^{s_0}(s) = 1_A(i) \cdot (1_{\{s_0\}}(s) - \pi_i(s_0)),$$

where 1_A and $1_{\{s_0\}}$ are the indicator functions of the sets A and $\{s_0\}$. Since the risks $f_i, i \in I$ are insurable, we have

$$\int_{i \in I} v_i^{s_0}(f_\omega(i)) d\lambda(i) = \int_A (1_{\{s_0\}}(f_\omega(i)) - \pi_i(s_0)) d\lambda(i) = 0$$

for P -almost all $\omega \in \Omega$. Hence, for P -almost all $\omega \in \Omega$, $(\lambda^A)(f_\omega^A)^{-1}(\{s_0\}) = (\lambda^A \boxtimes P)(f^A)^{-1}(\{s_0\})$, which means that $(\lambda^A)(f_\omega^A)^{-1} = (\lambda^A \boxtimes P)(f^A)^{-1}$ by the arbitrary choice of s_0 . By the converse law of large numbers in Theorem 3.5, we know that the f_i 's are essentially pairwise independent.

■

5 Sample measurability in a countably generated σ -algebra

Definition 5.1 (1) A σ -algebra \mathcal{C} is said to be countably generated if it is generated by a countable collection of sets (see [9], p. 110).

(2) Let \mathcal{C} be a countably generated sub- σ -algebra of \mathcal{I} . A measurable function α on the index space $(I, \mathcal{I}, \lambda)$ is said to be essentially \mathcal{C} -measurable if there is another measurable function β on $(I, \mathcal{I}, \lambda)$ such that $\beta(i) = \alpha(i)$ for λ -almost all $i \in I$ and β is \mathcal{C} -measurable.

The following theorem shows that for a non-trivial process with essentially pairwise independent random variables, almost no sample functions are essentially measurable with respect to any fixed countably generated sub- σ -algebra of \mathcal{I} . The proof is given in Section 9.4.

Theorem 5.2 Let f be a process from a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a Polish space X , and \mathcal{C} any fixed countably generated sub- σ -algebra of \mathcal{I} . If the random variables f_i are essentially pairwise independent, then the set of all $\omega \in \Omega$ such that the sample function f_ω is essentially \mathcal{C} -measurable must have probability zero except for the trivial case that almost all the random variables f_i are constant.

Note that the Borel σ -algebra on the unit interval $[0, 1]$ is generated by the rational sub-intervals and thus countably generated. It is also well-known that a Lebesgue measurable function is essentially Borel measurable. Hence, when the index space I is taken to be $[0, 1]$, almost all sample functions of a process with essentially pairwise independent random variables cannot be Lebesgue measurable as stated in the following obvious corollary of Theorem 5.2.²⁵

Corollary 5.3 Let f be a process from a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a Polish space X . Assume that the random variables f_i are essentially pairwise independent and not essentially constant. If $I = [0, 1]$, and $(I, \mathcal{I}, \lambda)$ extends the Lebesgue interval $([0, 1], \mathcal{L}, \nu)$, then the set of those sample points ω whose corresponding sample functions f_ω are Lebesgue measurable must have probability zero.

Intuitively, the sample functions of a continuum of nontrivial independent random variables should be highly discontinuous. Since Lusin's theorem says that any Lebesgue measurable function is continuous off a set of small measure, this suggests that almost all sample functions should not be Lebesgue measurable from an intuitive point of view. The point of Corollary 5.3 is that this intuitive fact can indeed be rigorously proven in a natural and suitable analytic framework.

²⁵Thus, the indeterminacy problem mentioned in [23] and Footnote 10 is solved; see also the discussion after Proposition 6.1.

6 Further discussion on the difficulties in some earlier approaches

6.1 Difficulties with the continuum product

Let $\Lambda = \mathbb{R}^{[0,1]}$ be the space of functions on $I = [0, 1]$, \mathcal{A} the product σ -algebra generated by the cylinders of the form $\{\omega \in \Lambda : \omega(i) \in B\}$ for any Borel set B in \mathbb{R} and $i \in I$, and μ the product measure on (Λ, \mathcal{A}) constructed from a probability distribution τ on \mathbb{R} . Note that the existence of such a continuum product measure space is guaranteed by the Kolmogorov existence theorem (see [6], p. 230). The unit Lebesgue interval is denoted by (I, \mathcal{L}, ν) .

For $i \in I$, let f_i be the i -th coordinate function, i.e., $f_i(\omega) = \omega(i)$, where $\omega \in \Lambda$ is a function on I . Then, f is a real-valued process on $I \times \Lambda$ with a continuum of independent and identically distributed (iid) random variables $f_i, i \in I$ with a common distribution τ . Since the classical law of large numbers holds for a sequence of iid random variables, one may hope an exact version of the law of large numbers to be valid for f so that the distributions of almost all sample functions f_ω on (I, \mathcal{L}, ν) are the theoretical distribution τ . However, it was already shown by Doob in [12] (see Theorem 2.2, p. 113) that the sample functions may not be Lebesgue measurable. The following proposition is taken essentially from [12].

Proposition 6.1 *For any real-valued function h on $[0, 1]$, let*

$$\mathcal{M}_h = \{\omega \in \Lambda : f_\omega(i) = h(i) \text{ except for countably many } i \in I\}.$$

Then \mathcal{M}_h has μ -outer measure one.

Proof: Let A be any measurable set in the continuum product σ -algebra \mathcal{A} on Λ . Then A is determined on a countable index set C in the sense that for any α and β in Λ , if $\alpha(i) = \beta(i)$ for all $i \in C$, then $\alpha \in A$ if and only if $\beta \in A$.

Now suppose that A contains \mathcal{M}_h . Take any $\omega \in \Lambda$. Define ω' so that it agrees with ω on C and with h on $I - C$. Since $\omega'(i) = h(i)$ except for countably many $i \in I$, we have $\omega' \in \mathcal{M}_h$, and hence $\omega' \in A$. On the other hand, since ω and ω' agree on C , the property of C implies $\omega \in A$ also. This means that $A = \Lambda$, and hence the outer measure $\mu^*(\mathcal{M}_h)$ of \mathcal{M}_h is one. ■

It was noted in [12] that if one takes h to be non-Lebesgue measurable, then so is every function in \mathcal{M}_h . Thus, the set \mathcal{N} of all non-Lebesgue measurable samples must have μ -outer measure one, and hence its complement \mathcal{N}^c of all Lebesgue measurable samples has μ -inner measure zero. On the other hand, if one takes a Lebesgue measurable function h , then $\mu^*(\mathcal{M}_h) = 1$ implies that the μ -outer measure of the set \mathcal{N}^c to be one ([23]).

Using the method in [13] (see p. 69), one can extend μ to a measure $\bar{\mu}$ on the σ -algebra generated by the set \mathcal{M}_h and the sets in \mathcal{A} with $\bar{\mu}(\mathcal{M}_h) = 1$. Thus, one obtains the absurd claim that almost all sample functions are essentially equal to an arbitrarily given function h . In particular, if h is Lebesgue (or non-Lebesgue) measurable, then one can say that almost all (or almost no) sample functions are Lebesgue measurable. In fact, the set \mathcal{N}^c can be assigned a completely arbitrary new probability r for any $r \in [0, 1]$ (see [23]). This means that the size of the set of Lebesgue measurable samples is completely indeterminate.

For any given number c , one can take h to be the constant function with value c , and then claim that $\bar{\mu}$ -almost all sample functions are essentially constant c . When the common mean $m = \int_{\mathbb{R}} x d\tau$ exists, let $c = m$. One can thus claim that almost all sample means equal the theoretical mean m . This appears to be an exact law of large numbers. However, it is actually based on an absurd underlying statement, namely, almost all sample functions take the constant value m .²⁶ For example, when τ concentrates at the points 0 and 1 with equal probability 1/2 (the theoretical mean is thus 1/2), this so-called exact law of large numbers is actually based on the claim that almost all sample functions equal the constant function 1/2 essentially. Since each random variable f_i takes either 0 or 1 as its values essentially, there is absolutely no valid reason to believe that almost all sample functions take the constant value 1/2. Hence, it is illegitimate to claim the validity of the exact law of large numbers in this way.

The following simple result shows that $(\Lambda, \mathcal{A}, \mu)$ cannot be a marginal probability space of a rich product probability space in which f is measurable.

Proposition 6.2 *Let $I = [0, 1]$. There is no atomless probability space $(I, \mathcal{I}, \lambda)$ with the following properties: (i) $(I \times \Lambda, \mathcal{I} \boxtimes \mathcal{A}, \lambda \boxtimes \mu)$ is a rich product probability space with marginals $(I, \mathcal{I}, \lambda)$ and $(\Lambda, \mathcal{A}, \mu)$; (ii) the process f of coordinate functions is $\mathcal{I} \boxtimes \mathcal{A}$ -measurable.*

Proof: Suppose that there is an atomless probability space $(I, \mathcal{I}, \lambda)$ with properties (i) and (ii). Take an interval B in \mathbb{R} such that $\tau(B) < 1$. Let $c \in B$, h be the constant function on I with value c , and $g = 1_B \circ f$. Then, g is a bounded $\mathcal{I} \boxtimes \mathcal{A}$ -measurable process with mutually independent random variables. Corollary 3.6 shows that for μ -almost all $\omega \in \Lambda$, $\lambda(f_\omega^{-1}(B)) = Eg_\omega = Eg = \tau(B) < 1$. This implies that for μ -almost all $\omega \in \Lambda$, $f_\omega = \omega \notin \mathcal{M}_h$.²⁷ This means that $\mu(\mathcal{M}_h) = 0$, which contradicts Proposition 6.1. ■

Remark 6.3 (1) *From the proof of Proposition 6.1, one can observe that it is still valid for any probability measure μ on Λ which is not necessarily a continuum product measure as stated*

²⁶Judd ([23], p. 24) did point out that such kind of claim may appear to be weak straw to clutch.

²⁷If $f_\omega \in \mathcal{M}_h$, then f_ω is essentially the constant c . Since $c \in B$, $\lambda(f_\omega^{-1}(B)) = 1$.

earlier ([12]). Thus, one can still claim (by measure extension) that almost all sample functions differ from an arbitrarily given function h at only countably many points, which also essentially equal h when $I = [0, 1]$ is endowed with any given atomless probability measure ν' .

(2) The requirement that I is the unit interval in Proposition 6.1 is not essential. It can be any space endowed with any atomless measure. In that case, one needs to work with the general space \mathbb{R}^I with \mathcal{A} the usual product σ -algebra constructed from I copies of the Borel algebra on \mathbb{R} (see [6], p. 230). Hence, (Λ, \mathcal{A}) is always a bad sample space for any process with any atomless index space.

In the literature, individual risks are usually modeled by a nontrivial iid process whose index space is the unit Lebesgue interval $([0, 1], \mathcal{L}, \nu)$ and sample space the continuum product space $(\Lambda, \mathcal{A}, \mu)$ based on the Kolmogorov construction. This mathematical model does not fit the economic assumption of individual risks in three aspects.

First, as noted in Remark 6.3, $(\Lambda, \mathcal{A}, \mu)$ is always a bad sample space for any process with any atomless index space. One can draw an analogy here. The algebra \mathcal{F}_0 of cylinders in Section 6.2 below only depends on finitely many indexes. It is not sufficient for studying an infinite sequence of independent random variables. While events in \mathcal{A} only depend on countably many indexes, the collection \mathcal{A} of such events is again inadequate for the study of a continuum of independent random variables. Proposition 6.2 also shows that it cannot be a marginal probability space of any rich product probability space in which the relevant iid process is measurable.

Second, $([0, 1], \mathcal{L}, \nu)$ is not an appropriate index space for an independent process since Corollary 5.3 actually shows that almost all sample functions cannot be Lebesgue measurable in a richer framework. The fact that almost no sample functions are essentially measurable with respect to any fixed countably generated σ -algebra implies that the sample functions generate a “big” σ -algebra on the index space I . When they are viewed based on a comparatively much smaller σ -algebra (say the Lebesgue σ -algebra), too much information about them is lost.

Third, the process is not jointly measurable in the usual product measure-theoretic framework (Proposition 2.1), and thus it is not meaningful to integrate the process itself or take its distribution.

6.2 A sequence of iid random variables on a finitely additive measure space

Section 6.1 shows how absurd results could be obtained when one attempts to study a continuum of independent random variables in an inadequate mathematical model. The purpose of this section is to present an analog for a sequence of independent random variables in a finitely

additive measure-theoretic framework.

Let τ be a probability distribution on \mathbb{R} , $\Lambda_0 = \mathbb{R}^\infty$ the space of all sequences of real numbers, and \mathcal{F}_0 the class of cylinders in the form $B_1 \times \dots \times B_n \times \mathbb{R}^\infty$ for Borel sets B_1, \dots, B_n on \mathbb{R} . Define a set function μ_0 on \mathcal{F}_0 such that $\mu_0(B_1 \times \dots \times B_n \times \mathbb{R}^\infty) = \tau(B_1) \times \dots \times \tau(B_n)$. Then \mathcal{F}_0 is an algebra of sets on Λ_0 and μ_0 a finitely additive measure on $(\Lambda_0, \mathcal{F}_0)$. Define a sequence of iid random variables $\{g_n\}_{n=1}^\infty$ on $(\Lambda_0, \mathcal{F}_0, \mu_0)$ by letting $g_n(\omega) = \omega_n$ for $\omega = \{\omega_n\}_{n=1}^\infty \in \Lambda_0$. The following is an analog of Proposition 6.1.

Proposition 6.4 *For any sequence $\alpha = \{a_n\}_{n=1}^\infty$ of real numbers, let \mathcal{M}_α be the set of all sequences which differ from α at only finitely many terms. Then, \mathcal{M}_α is not measurable in \mathcal{F}_0 and has μ_0 -outer measure one in the setting of finitely additive measures. Moreover, there is a finitely additive measure space $(\Lambda_0, \bar{\mathcal{F}}_0, \bar{\mu}_0)$ extending $(\Lambda_0, \mathcal{F}_0, \mu_0)$ such that $\mathcal{M}_\alpha \in \bar{\mathcal{F}}_0$ and $\bar{\mu}_0(\mathcal{M}_\alpha) = 1$.*

Proof: Pick any $F \in \mathcal{F}_0$ that contains \mathcal{M}_α . Then, there are Borel sets B_1, \dots, B_k in \mathbb{R} such that $F = B_1 \times \dots \times B_k \times \mathbb{R}^\infty$. Suppose $F \neq \Lambda_0$. Then, there is an index $1 \leq i_0 \leq k$ and a real number x_{i_0} not in B_{i_0} . Let $x_n = a_n$ for any $n \neq i_0$. The sequence $\beta = \{x_n\}_{n=1}^\infty$ is in \mathcal{M}_α and hence in F , which implies $x_{i_0} \in B_{i_0}$. This is a contradiction. Thus, $F = \Lambda_0$. Hence, \mathcal{M}_α has μ_0 -outer measure one in the setting of finite additive measures.

Let $\bar{\mathcal{F}}_0$ be the collection of sets in the form $F = (F_1 \cap \mathcal{M}_\alpha) \cup (F_2 \cap \mathcal{M}_\alpha^c)$ with $F_1, F_2 \in \mathcal{F}_0$, and for such a set F , define $\bar{\mu}_0(F) = \mu_0(F_1)$ (see [13], p. 69). The above result implies that $\bar{\mu}_0$ is a uniquely defined finitely additive measure on the algebra $\bar{\mathcal{F}}_0$ with $\bar{\mu}_0(\mathcal{M}_\alpha) = 1$. ■

Thus, in the finitely additive measure-theoretic framework, one can claim that almost all sample sequences differ from any given sequence α of real numbers at only finitely many terms. This is obviously absurd. By taking α such that $(a_1 + \dots + a_n)/n$ diverges (or converges to an arbitrary constant c) as n goes to infinity, one can say that the arithmetic averages of almost all sample sequences diverge (or converge to c).²⁸ This also means that results stated on finitely additive measures may be arbitrary and completely unreliable. In the context of finitely additive measures, the outer and inner measures of the set of all samples ω such that $(g_1(\omega) + \dots + g_n(\omega))/n$ converges as n goes to infinity are also respectively one and zero, and thus can be assigned an arbitrary value in $[0, 1]$ by extension. This is analogous to the indeterminacy problem mentioned in Section 6.1.

²⁸If the common mean is m , one can claim that for almost all ω , the sequence $\{g_n(\omega)\}_{n=1}^\infty$ differs from the constant sequences with value m at only finitely many terms, and hence $\lim_{n \rightarrow \infty} (g_1(\omega) + \dots + g_n(\omega))/n = m$. One certainly cannot regard this statement as a law of large numbers.

As already shown by Kolmogorov, all these anomalies can be removed by the axiomatic condition of a countably additive measure on a σ -algebra. In particular, for any sequence $\{f_n\}_{n=1}^\infty$ of iid real-valued random variables on any countably additive measure space (Ω, \mathcal{F}, P) with a common mean m , the law of large numbers holds: for P -almost all $\omega \in \Omega$, $\lim_{n \rightarrow \infty} (f_1(\omega) + \dots + f_n(\omega))/n = m$. In addition, assume that the sequence $\{f_n\}_{n=1}^\infty$ is bounded. Then for those samples ω such that $\lim_{n \rightarrow \infty} (f_1(\omega) + \dots + f_n(\omega))/n = m$, some Banach limits corresponding to the density charges for the sequence $\{f_n(\omega)\}_{n=1}^\infty$ are also m .²⁹ Since such a Banach limit can be viewed as an integral of a bounded sequence of real numbers with respect to a density charge ν_0 (which is a purely finitely additive measure), Kolmogorov's strong law of large numbers can thus be *weakened* to the statement that the ν_0 -integral of almost all sample functions is m . This weakened statement was claimed in [14] as an exact version of the law of large numbers.

There are a number of problems with this weakened statement on the law of large numbers in terms of the purely finitely additive measure ν_0 on the index set \mathbb{Z}_+ . First, it is well known that one cannot conduct any limit operations with such a measure and thus mathematical analysis is not applicable.³⁰ Second, an integral with respect to a density charge may not have any asymptotic meaning. For a bounded sequence $\alpha = \{a_n\}_{n=1}^\infty$ of real numbers, if $\lim_{n \rightarrow \infty} (a_1 + \dots + a_n)/n$ does not exist, then it does not have a uniquely defined integral with respect to different density charges, and thus its integral on a given density charge may not have any useful asymptotic interpretations. Third, as shown in the following proposition, if one works with the index space \mathbb{Z}_+ endowed with a density charge ν_0 , then one can derive the result that every sample distribution (which is finitely additive)³¹ cannot be the countably additive theoretical distribution τ when τ has a nontrivial continuous part.³² This shows further that results on finitely additive measure spaces may be totally unreliable. In one case, it may say something that shares some similarity with what one wants; in another case, it may lead to something completely meaningless.

Proposition 6.5 *Assume that the probability distribution τ has a nontrivial continuous part. Then, for every sample realization $\omega \in \Omega$, the sample distribution τ_ω is not equal to the theoretical distribution τ .*

Proof: Note that for every $\omega \in \Omega$, τ_ω only concentrates at the countable set $C_\omega = \{f_n(\omega) : n \in \mathbb{Z}_+\}$, and hence $\tau_\omega(\mathbb{R} \setminus C_\omega) = 0$. Since τ has a nontrivial continuous part (denoted by τ_c),

²⁹Such a Banach limit is simply a bounded linear functional L on the space l_∞ of bounded sequences of real numbers such that for any sequence $\alpha = \{a_n\}_{n=1}^\infty \in l_\infty$ with $\lim_{n \rightarrow \infty} (a_1 + \dots + a_n)/n = a$, $L(\alpha) = a$; see [14].

³⁰See the argument against the use of finitely additive measures in this context in [4].

³¹For a sample realization $\omega \in \Omega$, the sample distribution τ_ω can be naturally defined by letting $\tau_\omega(B) = \nu_0(\{n : f_n(\omega) \in B\})$ for any Borel set B in \mathbb{R} .

³²For example, τ is the uniform distribution on $[0, 1]$.

we know that $0 < \tau_c(\mathbb{R}) = \tau_c(\mathbb{R} \setminus C_\omega) \leq \tau(\mathbb{R} \setminus C_\omega)$. Therefore, $\tau \neq \tau_\omega$ for every $\omega \in \Omega$. ■

7 Constructions of rich product probability spaces

This section presents two existence results for rich product probability spaces. One involves an extension of the usual continuum product based on the Kolmogorov construction, and another an extension of the product of a probability space with itself. Their proofs are given in Section 9.5.

7.1 Extending the continuum product

Let τ be the uniform distribution on $[0, 1]$, μ the continuum product measure on Λ based on the distribution τ on \mathbb{R} , and f the iid process defined by the coordinate functions. Section 6.1 shows that the continuum product measure space $(\Lambda, \mathcal{A}, \mu)$ is always a bad sample space with any atomless index space. However, the following proposition shows that some extension of $(\Lambda, \mathcal{A}, \mu)$ will have the desired properties.

Proposition 7.1 *There is an atomless probability space $(I, \mathcal{I}, \lambda)$ with $I = [0, 1]$ and a probability space $(\Lambda, \mathcal{F}, P)$ extending $(\Lambda, \mathcal{A}, \mu)$ such that (i) there is a rich product probability space $(I \times \Lambda, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$; (ii) the iid process f of coordinate functions is $\mathcal{I} \boxtimes \mathcal{F}$ -measurable.³³*

7.2 A rich product probability space on the unit square

For two atomless Loeb probability spaces, their Loeb product is shown to extend the usual product in [2], retains the Fubini property ([25]), and can be endowed with essentially independent processes with any variety of distributions in Theorem 6.2 of [34].³⁴ Thus, the Loeb product probability spaces are always rich product probability spaces. The following proposition which is a simplified version of that result shows that rich product probability spaces even exist on the unit square $[0, 1] \times [0, 1]$ with the same marginal probability spaces.

Proposition 7.2 *There is a rich product probability space $([0, 1] \times [0, 1], \mathcal{I} \boxtimes \mathcal{I}, \lambda \boxtimes \lambda)$ with identical atomless marginal probability space $([0, 1], \mathcal{I}, \lambda)$.*

³³Chiaki Hara asked the author the following question. Though the probability of \mathcal{M}_h is indeterminate under μ (whose μ -outer measure and μ -inner measure are respectively one and zero), what is its probability under P ? If the function h is not \mathcal{I} -measurable, then $P(\mathcal{M}_h) = 0$ since almost all sample functions are \mathcal{I} -measurable. If h is \mathcal{I} -measurable, then Theorem 5.2 says that almost no sample functions are essentially measurable with respect to the countably generated σ -algebra $\sigma(h)$; and hence $P(\mathcal{M}_h)$ is also 0 since each function in \mathcal{M}_h is essentially $\sigma(h)$ -measurable.

³⁴[27] shows that the Loeb product space is uniquely defined by the marginal Loeb spaces.

8 Concluding remarks

As a mathematical result, the law of large numbers is a conditional statement that involves two parts. The first is the condition of independence or its generalizations. The second is the conclusion on the stability of sample means or distributions. The result says that *whenever* the condition is satisfied, the sample means or distributions will not depend on particular sample realizations.³⁵ This conditional statement only holds in a suitable analytic framework. One needs to work with a countably additive measure on a σ -algebra in the discrete setting, and with a rich product measure-theoretic framework in the continuum setting. When one does not work within a suitable analytic framework,³⁶ the general conditional statement may fail badly as illustrated in [14], [23] and Section 6.

Note that the independence condition or its generalizations is a crucial component in the statement of the law of large numbers. These conditions have very important empirical and behavioral content. For example, when one models the real-world situation that some economic agents make decisions or receive signals independently of each other, one uses the mathematical condition of independence. If the independence condition is simply dropped, one can certainly produce many examples of processes that have stable sample means or distributions.³⁷ However, the processes in these examples may not have any useful economic content. For example, one may allow half of the random variables in a process to be a fixed real-valued random variable φ with mean zero and the other half $-\varphi$, then all sample means are exactly zero, which is the common mean of all the random variables.³⁸ No one would regard this example as a law of large numbers.

This paper points out that the mathematical framework of a rich product probability space is sufficient for a meaningful study of individual risks. In particular, the earlier economic literature that uses various versions of the exact law of large numbers in an incorrect way can be easily validated. One can simply replace the product of the unit Lebesgue interval

³⁵The classical law of large numbers says that for a sequence of independent and identically distributed random variables, the sample mean/distribution is close to the common theoretical mean/distribution. Using nonstandard analysis, one can simply transfer the same result to the nonstandard model to claim the *existence* of a continuum of independent random variables whose sample mean/distribution is constant (Footnote 41 is relevant here). However, one does not obtain an exact law of large numbers for a continuum of independent random variables in this way since the transferred version of independence (namely the $*$ -independence condition) is much stronger than the usual independence condition in the nonstandard model as used in [34]. Also the transferred law of large numbers has exactly the same mathematical content as the classical law of large numbers.

³⁶For example, when one only works with a finitely additive measure structure in the discrete case or with a continuum index process in a framework without the Fubini property.

³⁷On the other hand, if one requires the stability of sample means or distributions on all the non-negligible coalitions, the converse law of large numbers in Theorems 3.4 and 3.5 does show that the original process must be essentially uncorrelated or essentially pairwise independent.

³⁸See [14] for a similar example in terms of distributions, where the independence condition is completely dropped.

with a sample probability space used in that literature by a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ with $I = [0, 1]$ and then appeal to an appropriate version of the exact law of large numbers in Section 3. The other point of the paper is that the framework of a rich product probability space allows one to discover completely new phenomena that have no known discrete analogs, such as the converse law of large numbers in Theorems 3.4 and 3.5, and the characterization of insurable risks in Theorem 4.10.

9 Appendix

9.1 Proof of some results in Section 2

Proof of Proposition 2.1: We only prove the case that f is real-valued and bounded. The general case can be proven by using the composition of a Borel measurable one-to-one mapping from X to $[0, 1]$ with f .³⁹

Assume that f is real-valued and bounded. Let A be any measurable set in \mathcal{I} . The function $(f(s, \omega) - Ef_s)(f(i, \omega) - Ef_i)dP(\omega)$ for $(s, i, \omega) \in I \times I \times \Omega$ is integrable on $(I \times I \times \Omega, \mathcal{I} \otimes \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \lambda \otimes P)$. By the Fubini Theorem, it is easy to establish the following identities:

$$\begin{aligned} & \int_{\Omega} \left[\int_A (f(i, \omega) - Ef_i) d\lambda(i) \right]^2 dP(\omega) \\ &= \int_{\Omega} \int_A (f(s, \omega) - Ef_s) d\lambda(s) \int_A (f(i, \omega) - Ef_i) d\lambda(i) dP(\omega) \\ &= \int_{A \times A} \int_{\Omega} (f(s, \omega) - Ef_s)(f(i, \omega) - Ef_i) dP(\omega) d\lambda \otimes \lambda(s, i), \end{aligned}$$

which is zero by the condition of essential pairwise independence. Hence, for P -almost all $\omega \in \Omega$, $\int_A (f(i, \omega) - Ef_i) d\lambda(i) = 0$.

Thus, for any measurable set $B \in \mathcal{F}$, $\int_{A \times B} (f(i, \omega) - Ef_i) d\lambda \otimes P = 0$. This means that the signed measure ν defined on $(I \times \Omega, \mathcal{I} \otimes \mathcal{F})$ by integrating $f(i, \omega) - Ef_i$ on sets in $\mathcal{I} \otimes \mathcal{F}$ agrees with the zero measure on the rectangles. Note that the product algebra $\mathcal{I} \otimes \mathcal{F}$ is generated by all the rectangles $A \times B$, and the collection of rectangles is also closed under finite intersections, i.e., a π -system. By applying Dynkin's π - λ theorem (see [9], p. 44), we obtain that the signed measure ν is equal to the zero measure. Thus, both $f(i, \omega) - Ef_i$ and 0 are Radon-Nikodym derivatives of the same measure $\lambda \otimes P$. By the uniqueness of the Radon-Nikodym derivatives, we have $f(i, \omega) = Ef_i$ for $\lambda \otimes P$ -almost all $(i, \omega) \in I \times \Omega$. Therefore, for λ -almost all $i \in I$, f_i is the constant random variable Ef_i . ■

Proof of Proposition 2.4: (1) By the definition of a rich product probability space, there is a real-valued $\mathcal{I} \boxtimes \mathcal{F}$ -measurable function g on $I \times \Omega$ such that for λ -almost all $i \in I$, the

³⁹The existence of such a mapping follows from the Borel Isomorphism Theorem; see [9], p. 275.

random variable g_i on (Ω, \mathcal{F}, P) has the uniform distribution τ on $[0, 1]$ and is independent of the random variable g_s for λ -almost all $s \in I$. By Theorem 3.9, the sample functions f_ω have exactly the same property. The rest is clear.

(2) Let $g^{A \times \Omega}$ and $g^{A \times B}$ be the respective restrictions of g to $A \times \Omega$ and to $A \times B$. It is obvious that $(A \times \Omega, (\mathcal{I} \boxtimes \mathcal{F})^{A \times \Omega}, (\lambda \boxtimes P)^{A \times \Omega})$ is a rich product probability space since the random variables $g_i, i \in A$ are still essentially pairwise independent with uniform distribution τ on $[0, 1]$. By Theorem 3.9, the sample functions $g_\omega^{A \times \Omega}, \omega \in \Omega$ are essentially pairwise independent with distribution τ , which implies that the sample functions $g_\omega^{A \times B}, \omega \in B$ have the same property. By Theorem 3.9 again, the random variables $g_i^{A \times B}, i \in A$ are also essentially pairwise independent with distribution τ . The Fubini property for $(A \times B, (\mathcal{I} \boxtimes \mathcal{F})^{A \times B}, (\lambda \boxtimes P)^{A \times B})$ is clear; and hence, it is a rich product probability space with marginals $(A, \mathcal{I}^A, \lambda^A)$ and (B, \mathcal{F}^B, P^B) .

■

We reproduce Lemma 2.22 on page 34 in [24] using our notation for the convenience of the reader.

Lemma 9.1 *Let τ be the Lebesgue measure on the Borel algebra \mathcal{B} on $[0, 1]$, and ϕ a measurable mapping from a measurable space (S, \mathcal{S}) to the space $\mathcal{M}(X)$ of Borel probability measures on a Polish space X , where $\mathcal{M}(X)$ is endowed with the topology of weak convergence of measures.⁴⁰ Then there exists some $\mathcal{S} \otimes \mathcal{B}$ -measurable function $f : S \times [0, 1] \rightarrow X$ such that for any $s \in S$, the distribution τf_s^{-1} is $\phi(s)$.*

Proof of Proposition 2.6: By the richness assumption, there is a $\mathcal{I} \boxtimes \mathcal{F}$ -measurable process g from $I \times \Omega$ to $[0, 1]$ such that the random variables g_i are essentially pairwise independent with the uniform distribution τ on $[0, 1]$. By Lemma 9.1, there is a $\mathcal{I} \otimes \mathcal{B}$ -measurable function $f : I \times [0, 1] \rightarrow X$ such that for any $i \in I$, the distribution τf_i^{-1} is $\varphi(i)$. Define a mapping G from $I \times \Omega$ to X by letting $G(i, \omega) = f(i, g(i, \omega))$. Then, for λ -almost all $i \in I$, the distribution

$$PG_i^{-1} = (Pg_i^{-1})f_i^{-1} = \tau f_i^{-1} = \varphi(i).$$

It is also obvious that for λ -almost all $i \in I$, $G_i = f_i(g_i)$ and $G_s = f_s(g_s)$ are independent for λ -almost all $s \in I$.

What remains is to check the $\mathcal{I} \boxtimes \mathcal{F}$ -measurability of G . Define a measurable mapping H from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F})$ to $(I \times [0, 1], \mathcal{I} \otimes \mathcal{B})$ by letting $H(i, \omega) = (i, g(i, \omega))$. For any Borel set B in X , $f^{-1}(B)$ is in $\mathcal{I} \otimes \mathcal{B}$; and hence $H^{-1}(f^{-1}(B))$ is in $\mathcal{I} \boxtimes \mathcal{F}$. Since $G^{-1}(B) = H^{-1}(f^{-1}(B))$, we thus obtain the $\mathcal{I} \boxtimes \mathcal{F}$ -measurability of G . ■

⁴⁰The second measurability condition mentioned in Footnote 17 above is used in [24]. That is equivalent to the measurability condition used here.

9.2 Proof of some results in Section 3

The following lemma is a generalized version of Lemma 3.1.

Lemma 9.2 *Let g and h be real-valued square integrable processes on a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, and a and b real-valued square integrable functions on $(I, \mathcal{I}, \lambda)$. Define a real-valued function G on $I \times I \times \Omega$ by letting $G(s, i, \omega) = a(s)g(s, \omega)h(i, \omega)b(i)$ for $(s, i, \omega) \in I \times I \times \Omega$. Then*

(i) *For λ -almost all $s \in I$, λ -almost all $i \in I$, the function $G(s, i, \cdot)$ on Ω is P -integrable.*

(ii) *For λ -almost all $s \in I$, the function $\int_{\Omega} G(s, i, \omega) dP(\omega)$ in terms of $i \in I$ is λ -integrable.*

(iii) *The function $\int_I \int_{\Omega} G(s, i, \omega) dP(\omega) d\lambda(i)$ in terms of $s \in I$ is λ -integrable.*

(iv) *The product function $\int_I a(s)g(s, \omega) d\lambda(s) \int_I b(i)h(i, \omega) d\lambda(i)$ in terms of $\omega \in \Omega$ is P -integrable, and*

$$\begin{aligned} & \int_I \int_I \int_{\Omega} a(s)g(s, \omega)h(i, \omega)b(i) dP(\omega) d\lambda(i) d\lambda(s) \\ &= \int_{\Omega} \left[\int_I a(s)g(s, \omega) d\lambda(s) \int_I b(i)h(i, \omega) d\lambda(i) \right] dP(\omega). \end{aligned}$$

Proof: (i) The Fubini property implies that for λ -almost all $s \in I$, λ -almost all $i \in I$, both the functions $g_s(\omega)$ and $h_i(\omega)$ in terms of $\omega \in \Omega$ are P -square integrable; and hence the product $G(s, i, \omega) = a(s)g(s, \omega)h(i, \omega)b(i)$ is P -integrable for $\omega \in \Omega$.

(ii) The product of any real-valued functions φ and ψ that are respectively P -square integrable on Ω and λ -square integrable on I is still square integrable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Thus the product $\varphi(\omega)h(i, \omega)\psi(i)$ is integrable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

For λ -almost all $s \in I$, g_s is P -square integrable; and hence the product $g_s(\omega)h(i, \omega)b(i)$ is integrable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. It also follows easily from the Cauchy-Schwartz inequality that both $\int_I b(i)h(i, \omega) d\lambda(i)$ and $\int_I a(s)g(s, \omega) d\lambda(s)$ are P -square integrable. The Fubini property implies that for λ -almost all $s \in I$, the function $\int_{\Omega} G(s, i, \omega) dP(\omega)$ in terms of $i \in I$ is λ -integrable, and also

$$\begin{aligned} \int_I \int_{\Omega} G(s, i, \omega) dP(\omega) d\lambda(i) &= \int_I \int_{\Omega} a(s)g(s, \omega)h(i, \omega)b(i) dP(\omega) d\lambda(i) \\ &= \int_{\Omega} a(s)g(s, \omega) \int_I b(i)h(i, \omega) d\lambda(i) dP(\omega). \end{aligned} \quad (6)$$

(iii) Since $\int_I b(i)h(i, \omega) d\lambda(i)$ is P -square integrable on Ω , the product function (in terms of (s, ω)) $a(s)g(s, \omega) \int_I b(i)h(i, \omega) d\lambda(i)$ is integrable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. The Fubini property implies that the function

$$\int_{\Omega} a(s)g(s, \omega) \int_I b(i)h(i, \omega) d\lambda(i) dP(\omega)$$

in terms of $s \in I$ is λ -integrable, and also

$$\begin{aligned} & \int_I \int_{\Omega} a(s)g(s, \omega) \int_I b(i)h(i, \omega) d\lambda(i) dP(\omega) d\lambda(s) \\ &= \int_{\Omega} \int_I a(s)g(s, \omega) d\lambda(s) \int_I b(i)h(i, \omega) d\lambda(i) dP(\omega). \end{aligned} \quad (7)$$

(iv) Since both $\int_I b(i)h(i, \omega) d\lambda(i)$ and $\int_I a(s)g(s, \omega) d\lambda(s)$ are P -square integrable on Ω , their product is P -integrable on Ω . By the identities in Equations (6) and (7), we have

$$\begin{aligned} & \int_I \int_I \int_{\Omega} a(s)g(s, \omega) h(i, \omega) b(i) dP(\omega) d\lambda(i) d\lambda(s) \\ &= \int_{\Omega} \left[\int_I a(s)g(s, \omega) d\lambda(s) \int_I b(i)h(i, \omega) d\lambda(i) \right] dP(\omega), \end{aligned}$$

and we are done. ■

Lemma 9.3 *Let f be a real-valued square integrable process on a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. If the random variables f_i are essentially uncorrelated and have a common mean m , then any given λ -square integrable function on $(I, \mathcal{I}, \lambda)$ is uncorrelated with the sample function f_{ω} for P -almost all $\omega \in \Omega$.*

Proof: Let a be any real-valued square integrable function on $(I, \mathcal{I}, \lambda)$, and $b = a$, $g = h = f - m$ as in Lemma 9.2. Then we have

$$\begin{aligned} & \int_I \int_I \int_{\Omega} a(s)(f(s, \omega) - m)(f(i, \omega) - m)a(i) dP(\omega) d\lambda(i) d\lambda(s) \\ &= \int_{\Omega} \left(\int_I a(i)(f(i, \omega) - m) d\lambda(i) \right)^2 dP(\omega), \end{aligned} \quad (8)$$

which is zero by the assumption of uncorrelatedness, and hence $\int_I a(i)(f(i, \omega) - m) d\lambda(i) = 0$ for P -almost all $\omega \in \Omega$. On the other hand, the exact law of large numbers in Proposition 3.3 implies that $Ef_{\omega} = m$ for P -almost all $\omega \in \Omega$. Therefore, $a(\cdot)$ and $f_{\omega}(\cdot)$ are uncorrelated for P -almost all $\omega \in \Omega$. ■

Proof of Theorem 3.9: We only consider (i) \implies (ii). By Corollary 3.6, the sample function f_{ω} has distribution μ for P -almost all $\omega \in \Omega$.

As in the proof of Theorem 3.5, fix a countable open base $\{O_n\}_{n=1}^{\infty}$ for X so that it is closed under finite intersections. Take any $m, n \geq 1$. Then the process $1_{O_n}(f)$ has essentially uncorrelated random variables. Lemma 9.3 implies that for P -almost all $q \in \Omega$, the functions $1_{O_m}(f_q)$ and $1_{O_n}(f_{\omega})$ on I are uncorrelated for P -almost all $\omega \in \Omega$, which means that Equation (5) holds here by interchanging the variables i and ω . The proof for essential pairwise independence of the f_i 's after Equation (5) still applies here; and hence the sample functions f_{ω} are essentially pairwise independent. ■

Proof of Lemma 3.10: When $T = \mathbb{Z}_+$, the result is clear. For the case of continuous parameters, let

$$\mathcal{E} = \{D \in (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}_T : D_i \in \mathcal{F} \otimes \mathcal{B}_T \text{ for } \lambda\text{-almost all } i \in I\}.$$

Then it is clear that for any $C \in \mathcal{I} \boxtimes \mathcal{F}$, $B \in \mathcal{B}_T$, $C \times B \in \mathcal{E}$. It can also be checked that \mathcal{E} is a λ -system. By applying Dynkin's π - λ theorem (see [9], p. 44), we obtain that $\mathcal{E} = (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}_T$. Thus there is a λ -null set \mathcal{N} such that for any $i \notin \mathcal{N}$, $\{(\omega, t) \in \Omega \times T : F_i(\omega, t) < r\} \in \mathcal{F} \otimes \mathcal{B}_T$ holds for all rational numbers $r \in \mathbb{Q}$. The rest is clear. ■

Proof of Theorem 3.13: Define a process G from $I \times \Omega$ into \mathbb{R}^∞ by letting $G(i, \omega) = \{F(i, \omega, t_n)\}_{n=1}^\infty$. Then, for $\lambda \boxtimes P$ -almost all $(i, \omega) \in I \times \Omega$,

$$F(i, \omega, t) = \psi(G(i, \omega), t) \tag{9}$$

for all $t \in T$. The fact that the stochastic processes F_i are essentially pairwise independent implies that the random variables G_i are essentially pairwise independent. Here we use the fact that the Borel algebra on \mathbb{R}^∞ is generated by the cylinders of a finite product of Borel sets in \mathbb{R} with infinitely many copies of \mathbb{R} .

By Theorem 3.5, we have for P -almost all $\omega \in \Omega$, the distribution μ_ω on \mathbb{R}^∞ induced by the random variable G_ω on I is equal to the distribution μ on \mathbb{R}^∞ induced by G on $I \times \Omega$. Also, the Fubini property implies that for P -almost all $\omega \in \Omega$, Equation (9) holds for λ -almost $i \in I$ and for all $t \in T$. Take such an ω with both properties, and choose any t_1, \dots, t_n from T . For any bounded continuous functions ϕ on \mathbb{R}^n , we have

$$\begin{aligned} \int_{I \times \Omega} \phi(F_{t_1}, \dots, F_{t_n}) d\lambda \boxtimes P &= \int_{I \times \Omega} \phi(\psi(G(\cdot, \cdot), t_1), \dots, \psi(G(\cdot, \cdot), t_n)) d\lambda \boxtimes P \\ &= \int_{y \in \mathbb{R}^\infty} \phi(\psi(y, t_1), \dots, \psi(y, t_n)) d\mu(y) = \int_{y \in \mathbb{R}^\infty} \phi(\psi(y, t_1), \dots, \psi(y, t_n)) d\mu_\omega(y) \\ &= \int_I \phi(\psi(G_\omega(\cdot), t_1), \dots, \psi(G_\omega(\cdot), t_n)) d\lambda = \int_I \phi(F_{\omega t_1}, \dots, F_{\omega t_n}) d\lambda \end{aligned}$$

Hence the stochastic processes F_ω and F have the the same finite dimensional distributions, and we are done. ■

Proof of Theorem 3.14: We first show that F (viewed as a stochastic process on $I \times \Omega$) and H have the same finite dimensional distributions. Note that essential pairwise independence is not needed for this point. Choose t_1, \dots, t_n from T . For any bounded continuous functions ϕ on \mathbb{R}^n , we have

$$\int_{I \times \Omega} \phi(F_{t_1}, \dots, F_{t_n}) d\lambda \boxtimes P = \int_I \int_\Omega \phi(F_{t_1}, \dots, F_{t_n}) dP d\lambda$$

$$= \int_I \int_{\Lambda} \phi(H_{t_1}, \dots, H_{t_n}) d\nu d\lambda = \int_{\Lambda} \phi(H_{t_1}, \dots, H_{t_n}) d\nu.$$

Hence F and H have the same finite dimensional distributions.

Next, by the essential pairwise independence assumption on the stochastic processes F_i , Theorem 3.13 implies that for P -almost all $\omega \in \Omega$, the empirical process F_{ω} has the same finite dimensional distributions as F , and hence H .

It remains to show that the empirical processes F_{ω} are essentially pairwise independent. As in the proof of Theorem 3.13, we use the expression in Equation (9). It is easy to see that the random variables G_i are essentially pairwise independent and identically distributed. By Theorem 3.9, the sample functions G_{ω} are also essentially pairwise independent and identically distributed. Then, for P -almost all $\omega \in \Omega$, P -almost all $\omega' \in \Omega$,

$$F_{\omega}(i, t) = \psi(G_{\omega}(i), t), \quad F_{\omega'}(i, t) = \psi(G_{\omega'}(i), t) \quad (10)$$

for λ -almost $i \in I$ and for all $t \in T$; and G_{ω} and $G_{\omega'}$ are independent. Thus, the empirical processes F_{ω} and $F_{\omega'}$ are independent. The rest follows from symmetry. ■

9.3 Proof of some results in Section 4

Proof of Lemma 4.2: (1) Let Δ^k be the set of all probability measures q on S^k with $q(\{s\}) > 0$ for all $s \in S^k$. For a given complete continuous preordering \preceq in \mathcal{P}^k and a probability measure $q \in \Delta^k$, define a mapping E^q from $(\mathbb{R}_+^m)^{S^k}$ to \mathbb{R}_+^m by letting $E^q(x) = \sum_{s \in S^k} q(\{s\})x(s)$ for each $x \in (\mathbb{R}_+^m)^{S^k}$. For any given $a \in \mathbb{R}_+^m$, the set $(E^q)^{-1}(\{a\}) = \{z \in (\mathbb{R}_+^m)^{S^k} : E^q z = a\}$ is compact since $q(\{s\}) > 0$ for all $s \in S^k$. In fact, it is easy to check that $E^q(\cdot)$ is a closed mapping. It is also an open mapping since it is linear (see [32], p. 230). Thus, Theorem 14.7 on page 463 of [1] implies that $(E^q)^{-1}(\cdot)$ defines a continuous compact-valued correspondence from \mathbb{R}_+^m to $(\mathbb{R}_+^m)^{S^k}$.

Define a correspondence φ^k from $\mathcal{P}^k \times \Delta^k \times \mathbb{R}_+^m$ to $(\mathbb{R}_+^m)^{S^k}$ by letting $\varphi^k(\preceq, q, a)$ be the nonempty set of all maximal elements in the compact set $(E^q)^{-1}(\{a\})$ under the complete continuous preordering \preceq . It is easy to check that φ^k a compact valued correspondence with a closed graph. It is obvious that elements in $\varphi^k(\preceq, q, a)$ are indifferent to each other under \preceq . Take another $b \in \mathbb{R}_+^m$. There is no confusion to use $\varphi^k(\preceq, q, a) \preceq \varphi^k(\preceq, q, b)$ to mean that there are $x \in \varphi^k(\preceq, q, a)$ and $y \in \varphi^k(\preceq, q, b)$ with $x \preceq y$. Thus, we can define a binary relation \preceq^q on \mathbb{R}_+^m such that for any $a, b \in \mathbb{R}_+^m$, $a \preceq^q b$ holds if and only if $\varphi^k(\preceq, q, a) \preceq \varphi^k(\preceq, q, b)$ holds.

Then it is obvious that \preceq^q is a reflexive, transitive and complete binary relation (thus a complete preordering). We shall show that \preceq^q is also a continuous preordering. Suppose that $a_n, b_n \in \mathbb{R}_+^m$ for $n = 0, 1, 2, \dots$ with $a_n \preceq^q b_n$ for $n \geq 1$ and $\lim_{n \rightarrow \infty} (a_n, b_n) = (a_0, b_0)$. Take

$x_n \in \varphi^k(\preceq, q, a_n)$ and $y_n \in \varphi^k(\preceq, q, b_n)$ for $n \geq 1$. Since the set $\cup_{n=0}^{\infty} \{a_n, b_n\}$ is compact and since $q(\{s\}) > 0$ for all $s \in S^k$, one can find convergent subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{y_{n_k}\}_{k=1}^{\infty}$ with respective limits x_0 and y_0 . Since $\varphi^k(\preceq, q, \cdot)$ also has a closed graph, one obtains that $x_0 \in \varphi^k(\preceq, q, a_0)$ and $y_0 \in \varphi^k(\preceq, q, b_0)$. By the definition of \preceq^q , one also has $x_{n_k} \preceq y_{n_k}$ for all $k \geq 1$. The continuity of \preceq implies $x_0 \preceq y_0$. This means that $a_0 \preceq^q b_0$. Thus the preordering \preceq^q is continuous.

Next, take any $1 \leq k \leq K$. For $i \in I^k$, it is clear that the binary relation \preceq_i^- on \mathbb{R}_+^m in the statement of Lemma 4.2 is simply $\preceq_i^{\pi_i}$, corresponding to the preordering \preceq_i and probability measure π_i . Then, \preceq_i^- is indeed a continuous complete preordering as shown above.

(2) for any \mathcal{I} -measurable mappings $a(\cdot)$ and $b(\cdot)$ from I to \mathbb{R}_+^m , we have

$$\{i \in I : a(i) \preceq_i^- b(i)\} = \cup_{k=1}^K \{i \in I^k : \varphi^k(\preceq_i, \pi_i, a(i)) \preceq_i \varphi^k(\preceq_i, \pi_i, b(i))\}.$$

Let $\Phi_a^k(i) = \varphi^k(\preceq_i, \pi_i, a(i))$ and $\Phi_b^k(i) = \varphi^k(\preceq_i, \pi_i, b(i))$. Then $\Phi_a^k(\cdot)$ and $\Phi_b^k(\cdot)$ define two compact-valued measurable correspondences on I^k . Let $\phi_a^k(\cdot)$ and $\phi_b^k(\cdot)$ be measurable selections of $\Phi_a^k(\cdot)$ and $\Phi_b^k(\cdot)$ respectively. By the measurability of α_p on I^k , we know that the set $\{i \in I^k : \phi_a^k(i) \preceq_i \phi_b^k(i)\}$ is measurable. Since $\varphi^k(\preceq_i, \pi_i, a(i)) \preceq_i \varphi^k(\preceq_i, \pi_i, b(i))$ actually means $\phi_a^k(i) \preceq_i \phi_b^k(i)$, we know that the set $\{i \in I : a(i) \preceq_i^- b(i)\}$ is measurable. ■

Proof of Lemma 4.3: First, assume that (x, p) is an equilibrium in \mathcal{E} . Take any $z \in E_i^{-1}(\{E_i x_i\})$; then $pE_i z = pE_i x_i$. Since x_i is a maximal element in agent i 's budget set for λ -almost all $i \in I$, we have $z \preceq_i x_i$, i.e., $x_i \in \Phi_a(i)$.

Next, consider that x is a Pareto optimal allocation in \mathcal{E} . Let $B = \{i \in I^k : x_i \prec_i \phi_a^k(i)\}$ and suppose $\lambda(B) > 0$. Then define an allocation y such that $y(i) = \phi_a^k(i)$ for $i \in B$, and $y(i) = x(i)$ for $i \in I - B$. It is clear that $E_i y_i = E_i x_i$ for all $i \in I$. Since x is feasible, y is also a feasible allocation in \mathcal{E} , which is strictly better than x for agents in B and the same as x for other agents. This contradicts the Pareto optimality of x . ■

Before proving the results on Pareto optimality, we present a useful simple lemma.

Lemma 9.4 *For any $1 \leq k \leq K$, if $y \in (\mathbb{R}_+^m)^{S^k}$, then $y \preceq_i z$ for any $z \in \Psi^k(i, E_i y)$, and $i \in I^k$.*

Proof: Note that y belongs to the set $(E_i)^{-1}(E_i y)$. Since z is a maximal element in $(E_i)^{-1}(E_i y)$ under \preceq_i . The rest is clear. ■

Proof of Proposition 4.4: (1) \implies (2): The second part of (2) follows from Lemma 4.3. Suppose that there exists a feasible allocation b in $\bar{\mathcal{E}}$ such that $a_i \preceq_i^- b_i$ for all $i \in I$ with

$\lambda(\{i \in I^k : a_i \prec_i^- b_i\}) > 0$ for some k . Define an allocation y in \mathcal{E} such that $y(i) = \phi_b^k(i)$ for $i \in I^k$, $k = 1, \dots, K$. Then it is clear that y is still feasible. By the definition of \preceq_i^- , we have $x_i \preceq_i^- y_i$ for all $i \in I$, and $\lambda(\{i \in I^k : x_i \prec_i^- y_i\}) > 0$ for some k . This contradicts the Pareto optimality of x .

(2) \implies (1): Suppose that there exists a feasible allocation y such that $x_i \preceq_i y_i$ for all $i \in I$ with $\lambda(\{i \in I^k : x_i \prec_i^- y_i\}) > 0$ for some k . Let $b(i) = E_i y_i$. Lemma 9.4 says that for any k , $y_i \preceq \phi_b^k(i)$ for all $i \in I^k$. Hence, ϕ_b is a feasible allocation in \mathcal{E} , and $x_i \preceq_i^- \phi_b(i)$ for all $i \in I$, and $\lambda(\{i \in I^k : x_i \prec_i^- \phi_b^k(i)\}) > 0$ for some k . By the definition of \preceq_i^- and the second part of (2), we know that $a_i \preceq_i^- b_i$ for all $i \in I$ with $\lambda(\{i \in I^k : a_i \prec_i^- b_i\}) > 0$ for some k . This contradicts the Pareto optimality of a . ■

Proof Proposition 4.5: (1) For any $d \in \mathbb{R}_+^m$ with $a_i \preceq_i^- d$, we have $\phi_a(i) \preceq_i z$ for some $z \in \Psi(i, d)$. Lemma 9.4 implies that $x_i \preceq_i \phi_a(i)$, and hence $x_i \preceq_i z$. Since (x, p) be a pseudo-equilibrium, we have, $pa_i \leq pE_i z$, i.e., $pa_i \leq pd$. Thus, (a, p) is a pseudo-equilibrium in the economy $\bar{\mathcal{E}}$.

(2) Take any $z \in (\mathbb{R}_+^m)^{S_i}$ with $y_i \preceq_i z$. Then, Lemma 9.4 implies that $z \preceq_i v$ for some $v \in \Psi(i, E_i z)$, and hence, $y_i \preceq_i v$. Thus, $b_i \preceq_i^- E_i z$. Since (b, p) be a pseudo-equilibrium, we have, $pb_i \leq pE_i z$, i.e., $pE_i y_i \leq pE_i z$. Thus, (y, p) is a pseudo-equilibrium in the economy \mathcal{E} . ■

Proof Proposition 4.6: (1) For any $d \in \mathbb{R}_+^m$ with $pd \leq pa_i$, we have $pE_i z \leq pE_i x_i$ for some $z \in \Psi(i, d)$. Since (x, p) be an equilibrium, we have $z \preceq_i x_i$. Lemma 9.4 implies that $x_i \preceq_i \phi_a(i)$, and hence $z \preceq_i \phi_a(i)$. Thus, $d = E_i z \preceq_i^- a_i$. This means that (a, p) is an equilibrium in the economy $\bar{\mathcal{E}}$.

(2) Take any $z \in (\mathbb{R}_+^m)^{S_i}$ with $pE_i z \leq pE_i y_i = pb_i$. Since (b, p) be an equilibrium, we have $E_i z \preceq_i^- b_i$. By Lemma 9.4 and the definition of \preceq_i^- , we obtain $z \preceq_i y_i$. Hence, (y, p) is an equilibrium in the economy \mathcal{E} . ■

Proof Proposition 4.7: The monotonicity of \preceq_i^- and Lemma 4.2 imply that Conditions (ii) and (iii) on page 366 of [18] are satisfied. Thus, it follows from the theorem on page 367 of [18] (the other conditions of the theorem are trivially satisfied) that there is a price system p such that (a, p) is a pseudo-equilibrium in $\bar{\mathcal{E}}$. Based on some standard method (see, for example, [4]), the monotonicity of \preceq_i^- and the strict positivity of the average expected endowment imply that p is strictly positive. Hence (1) \implies (2) follows.

Next, we consider (2) \implies (3). Assume that (a, p) is a pseudo-equilibrium in $\bar{\mathcal{E}}$ with

a strictly positive price system p . Take any $b \in \mathbb{R}_+^m$ with $pb \leq pa_i$. If $pb = 0$, then $b = 0$, and $b \preceq_i^- a_i$ by monotonicity. If $pb > 0$, one can find a sequence $b_n \in \mathbb{R}_+^m, n = 1, 2, \dots$ with $pb_n < pb$ such that the sequence converges to b . Thus $pb_n < pa_i$, which implies that $b_n \preceq_i^- a_i$. By continuity of the preference, $b \preceq_i^- a_i$.

The thin preference condition as defined in [18] is obviously satisfied. Thus, (3) \implies (1) follows from the argument on page 366 of [18]. ■

Proof Corollary 4.8: (1) \implies (2) follows from Lemma 4.3, Propositions 4.4, 4.5 and 4.7. The same proof for (2) \implies (3) in Proposition 4.7 also applies here. (3) \implies (1) follows from Lemma 4.3, Propositions 4.4, 4.6 and 4.7. ■

9.4 Proof of Theorem 5.2

Before proving Theorem 5.2, we need to prove the following lemma.

Lemma 9.5 *Let g be a square integrable process on a rich product probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that the random variables g_i are essentially uncorrelated, and \mathcal{C} any fixed countably generated sub- σ -algebra of \mathcal{I} . If there is a subset B_0 of Ω with outer measure $P^*(B_0) > 0$ such that for all $\omega \in B_0$, g_ω is essentially \mathcal{C} -measurable, then there is a P -null set S_0 such that for all $\omega \in B_0 - S_0$, $g_\omega(i) = Eg_i$ for λ -almost all $i \in I$.*

Proof: For any given measurable set $A \in \mathcal{I}$, Theorem 3.4 implies that

$$\int_A g_\omega(i) d\lambda(i) = \int_A Eg_i d\lambda(i) \quad (11)$$

holds for P -almost all $\omega \in \Omega$.

Let \mathcal{C}_0 be a countable collection of sets in \mathcal{C} that includes I , generates \mathcal{C} and is closed under finite intersections. Let h be the conditional expectation of the mean function Eg_i with respect to \mathcal{C} . Then, for any $C \in \mathcal{C}$, $\int_C h(i) d\lambda(i) = \int_C Eg_i d\lambda(i)$; and hence Equation (11) implies that

$$\int_C g_\omega(i) d\lambda(i) = \int_C h(i) d\lambda(i) \quad (12)$$

holds for P -almost all $\omega \in \Omega$. By grouping countably many P -null sets together, one can find a P -null set S_0 such that for all $\omega \notin S_0$, g_ω is λ -square integrable, and Equation (12) holds for all $C \in \mathcal{C}_0$, which means that the conditional expectation $E(g_\omega | \mathcal{C})$ is simply h (see Theorem 1 on page 212 of [8]).

Let $E_0 = B_0 - S_0$. Fix $\omega \in E_0$. There is a function \bar{g}_ω which equals g_ω essentially on $(I, \mathcal{I}, \lambda)$ such that \bar{g}_ω is \mathcal{C} -measurable. Thus, $E(g_\omega | \mathcal{C}) = E(\bar{g}_\omega | \mathcal{C}) = \bar{g}_\omega = h$ hold essentially on $(I, \mathcal{I}, \lambda)$. Hence $g_\omega(i) = h(i)$ for λ -almost all $i \in I$.

For any fixed $A \in \mathcal{I}$, since $P^*(E_0) > 0$, we can find $\omega \in E_0$ such that $g_\omega(i) = h(i)$ for λ -almost all $i \in I$ and Equation (11) also holds for such a ω . Hence $\int_A h(i)d\lambda(i) = \int_A Eg_i d\lambda(i)$. Since A is arbitrary, we obtain that $Eg_i = h(i)$ for λ -almost all $i \in I$. Therefore, for all $\omega \in E_0$, $g_\omega(i) = Eg_i$ for λ -almost all $i \in I$. ■

Proof of Theorem 5.2: Suppose that it is not true that almost all the random variables f_i are constant. Let B_0 be the set of all $\omega \in \Omega$ such that f_ω is essentially \mathcal{C} -measurable. Suppose $P^*(B_0) > 0$.

By the usual Borel isomorphism theorem ([9], p. 275), there is a Borel measurable mapping ϕ from X to $[0, 1]$ such that ϕ is one-to-one. Let $p = \phi \circ f$. Then, the random variables p_i are still essentially pairwise independent. Take any rational number r . Let $g = 1_{(-\infty, r]} \circ p$. Then g satisfies the condition of Lemma 9.5. Hence there is a P -null set S_0 such that for all $\omega \in B_0 - S_0$, $g_\omega(i) = Eg_i$ holds for λ -almost all $i \in I$. Since p itself also satisfies the condition of Lemma 9.5, we know that there is a P -null set \bar{S}_0 such that for all $\omega \in B_0 - \bar{S}_0$, $p_\omega(i) = Ep_i$ holds for λ -almost all $i \in I$. Since $P^*(B_0) > 0$, $B_0 - (S_0 \cup \bar{S}_0)$ is nonempty. Take $\omega \in B_0 - (S_0 \cup \bar{S}_0)$. Then, for λ -almost all $i \in I$,

$$g_\omega(i) = 1_{(-\infty, r]}(p_\omega(i)) = 1_{(-\infty, r]}(Ep_i) = Eg_i = E(1_{(-\infty, r]} \circ p_i) = F_i(r), \quad (13)$$

where F_i is the distribution function of p_i . It is easy to see that $1_{(-\infty, r]}(Ep_i) = 1_{[Ep_i, \infty)}(r)$, and hence $F_i(r) = 1_{[Ep_i, \infty)}(r)$ for λ -almost all $i \in I$. By grouping countably many λ -null sets together, we obtain that for λ -almost all $i \in I$, $F_i(r) = 1_{[Ep_i, \infty)}(r)$ for all rational numbers r , which implies that p_i is the constant random variable taking value Ep_i . Since ϕ is one-to-one, we derive that for λ -almost all $i \in I$, the random variables f_i are constant, which contradicts the hypothesis. Therefore, $P^*(B_0) = 0$, which implies that $P(B_0) = 0$. ■

9.5 Proof of Propositions 7.1 and 7.2

Proof of Proposition 7.1: Take an atomless hyperfinite Loeb space $(I_0, \mathcal{I}_0, \lambda_0)$ and another atomless Loeb space $(\Omega_0, \mathcal{F}_0, P_0)$ constructed using the usual ultrapower construction (see [21]) such that there is a measurable real-valued process \bar{g} on the Loeb product space $(T_0 \times \Omega_0, \mathcal{I}_0 \boxtimes \mathcal{F}_0, \lambda_0 \boxtimes P_0)$ whose random variables \bar{g}_i are mutually independent with uniform distribution τ on $[0, 1]$.⁴¹

⁴¹ The existence of such a process \bar{g} is easy. One can simply take the transfer $\{\beta_n\}_{n \in \mathbb{Z}_+^*}$ of a sequence $\{\beta_n\}_{n \in \mathbb{Z}_+}$ of iid random variables with uniform distribution τ . For an unlimited hyperfinite integer ρ , let $I_0 = \{1, 2, \dots, \rho\}$ be endowed with the counting probability measure on its internal subsets. Its corresponding Loeb space is denoted by $(I_0, \mathcal{I}_0, \lambda_0)$. For $i_0 \in I_0$, let \bar{g}_{i_0} be ${}^\circ\beta_{i_0}$. Then this process \bar{g} satisfies the property. See also [3] for another construction.

Since a hyperfinite set in the ultrapower construction on \mathbb{N} only involves sequences of finite sets, the usual external cardinality of I_0 is the cardinality of the continuum. Let φ be a bijection from the unit interval $I = [0, 1]$ to I_0 . Then, there is an atomless probability space $(I, \mathcal{I}, \lambda)$ such that φ is an isomorphism between the probability spaces $(I_0, \mathcal{I}_0, \lambda_0)$ and $(I, \mathcal{I}, \lambda)$. Similarly, there is a probability space $(I \times \Omega_0, \mathcal{I} \boxtimes \mathcal{F}_0, \lambda \boxtimes P_0)$ such that the mapping (φ, id_{Ω_0}) is an isomorphism between the probability spaces $(I_0 \times \Omega_0, \mathcal{I}_0 \boxtimes \mathcal{F}_0, \lambda_0 \boxtimes P_0)$ and $(I \times \Omega_0, \mathcal{I} \boxtimes \mathcal{F}_0, \lambda \boxtimes P_0)$, where id_{Ω_0} is the identity mapping on Ω_0 . For $(i, \omega) \in I \times \Omega_0$, let $g(i, \omega) = \bar{g}(\varphi(i), \omega)$. Then g is a $\mathcal{I} \boxtimes \mathcal{F}_0$ -measurable process whose random variables are mutually independent with uniform distribution τ . It is clear that $(I \times \Omega_0, \mathcal{I} \boxtimes \mathcal{F}_0, \lambda \boxtimes P_0)$ extends the usual product and retains the Fubini property.

Define a mapping ψ from $(\Omega_0, \mathcal{F}_0, P_0)$ to Λ such that $\psi(\omega)$ is the sample function g_ω on I . Note that ψ may not be one-to-one. For any i_1, \dots, i_n in I and for any Borel sets B_1, \dots, B_n on \mathbb{R} , let H be the cylinder of all functions $y \in \Lambda$ such that $y(i_k) \in B_k$ for $1 \leq k \leq n$. Then, $\psi^{-1}(H) = \{\omega \in \Omega_0 : \forall k = 1, \dots, n, g_{i_k}(\omega) \in B_k\}$, which is \mathcal{F}_0 -measurable. By the iid assumption on g , $P_0(\psi^{-1}(H)) = \prod_{k=1}^n \tau(B_k) = \mu(H)$. Define $\mathcal{F} = \{D \subseteq \Lambda : \psi^{-1}(D) \in \mathcal{F}_0\}$. Then \mathcal{F} is a σ -algebra that contains all the cylinders H and hence the product σ -algebra \mathcal{A} . Define a measure P on \mathcal{F} such that $P(D) = P_0(\psi^{-1}(D))$ for $D \in \mathcal{F}$. Since P and μ agree on all the cylinders (which form a π -system), P is an extension of μ by the uniqueness of measures (two probability measures that agree on a π -system \mathcal{D} must agree on $\sigma(\mathcal{D})$; see [9] p. 44).

Now let Ψ be the mapping from $I \times \Omega_0$ to $I \times \Lambda$ by letting $\Psi(i, \omega) = (i, \psi(\omega))$. Define $\mathcal{I} \boxtimes \mathcal{F} = \{E \subseteq I \times \Lambda : \Psi^{-1}(E) \in \mathcal{I} \boxtimes \mathcal{F}_0\}$, and a set function $\lambda \boxtimes P$ on $\mathcal{I} \boxtimes \mathcal{F}$ such that $\lambda \boxtimes P(E) = \lambda \boxtimes P_0(\Psi^{-1}(E))$ for $E \in \mathcal{I} \boxtimes \mathcal{F}$. Then $\mathcal{I} \boxtimes \mathcal{F}$ is a σ -algebra that contains the rectangles $A \times D$ for $A \in \mathcal{I}$ and $D \in \mathcal{F}$ and hence the product σ -algebra $\mathcal{I} \otimes \mathcal{F}$; since $\lambda \boxtimes P(A \times D) = \lambda(A)P_0(\psi^{-1}(D)) = \lambda \otimes P(A \times D)$, $\lambda \boxtimes P$ is thus an extension of $\lambda \otimes P$ on $\mathcal{I} \otimes \mathcal{F}$.

Since the Fubini property holds on $(I \times \Omega_0, \mathcal{I} \boxtimes \mathcal{F}_0, \lambda \boxtimes P_0)$ with marginal measures λ and P_0 , it also holds on $(I \times \Lambda, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ with marginal measures λ and P .

By the definition of $(I \times \Lambda, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, any real-valued function α on $I \times \Lambda$ is $\mathcal{I} \boxtimes \mathcal{F}$ -measurable if and only if $\alpha \circ \Psi$ is $\mathcal{I} \boxtimes \mathcal{F}_0$ -measurable. For the process f defined by the coordinate functions on Λ , it is obvious that $f \circ \Psi(i, \omega) = f(i, \psi(\omega)) = \psi(\omega)(i) = g_\omega(i) = g$ for all $(i, \omega) \in I \times \Lambda$. The fact that g is $\mathcal{I} \boxtimes \mathcal{F}_0$ -measurable implies that f is $\mathcal{I} \boxtimes \mathcal{F}$ -measurable. Thus, the process f has mutually independent random variables with uniform distribution on $[0, 1]$; and $(I \times \Lambda, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a rich product probability space. ■

Proof of Proposition 7.2: Take an atomless hyperfinite Loeb space $(I_0, \mathcal{I}_0, \lambda_0)$ constructed

using the usual ultrapower construction on \mathbb{N} . Let $(I_0 \times I_0, \mathcal{I}_0 \boxtimes \mathcal{I}_0, \lambda_0 \boxtimes \lambda_0)$ be the Loeb product space of $(I_0, \mathcal{I}_0, \lambda_0)$ with itself, which extends the usual product space $(I_0 \times I_0, \mathcal{I}_0 \otimes \mathcal{I}_0, \lambda_0 \otimes \lambda_0)$.

As noted in the proof of Proposition 7.1, the usual cardinality of the hyperfinite set I_0 is the cardinality of the continuum. Thus, there is a bijection ϕ from I_0 to the unit interval $[0, 1]$. Let $([0, 1], \mathcal{I}, \lambda)$ be the probability space induced by ϕ from $(I_0, \mathcal{I}_0, \lambda_0)$.⁴²

Next, define a bijection Φ from $I_0 \times I_0$ to $[0, 1] \times [0, 1]$ by letting $\Phi(i, \omega) = (\phi(i), \phi(\omega))$ for $(i, \omega) \in I_0 \times I_0$. Let $([0, 1] \times [0, 1], \mathcal{I} \boxtimes \mathcal{I}, \lambda \boxtimes \lambda)$ be the probability space induced by Φ from the Loeb product space $(I_0 \times I_0, \mathcal{I}_0 \boxtimes \mathcal{I}_0, \lambda_0 \boxtimes \lambda_0)$.

For any $D_1, D_2 \in \mathcal{I}$, there are $A_1, A_2 \in \mathcal{I}_0$ with $\phi(A_i) = D_i$ for $i = 1, 2$. Since $A_1 \times A_2 \in \mathcal{I}_0 \boxtimes \mathcal{I}_0$, $\Phi(A_1 \times A_2) = D_1 \times D_2 \in \mathcal{I} \boxtimes \mathcal{I}$ and

$$\lambda \boxtimes \lambda(D_1 \times D_2) = \lambda \boxtimes \lambda(\Phi(A_1 \times A_2)) = \lambda_0 \boxtimes \lambda_0(A_1 \times A_2) = \lambda_0(A_1)\lambda_0(A_2) = \lambda(D_1)\lambda(D_2).$$

Since the class \mathcal{R} of all rectangles $D_1 \times D_2$ generates $\mathcal{I} \otimes \mathcal{I}$, the fact that the σ -algebra $\mathcal{I} \boxtimes \mathcal{I}$ contains the rectangles implies that $\mathcal{I} \boxtimes \mathcal{I}$ extends $\mathcal{I} \otimes \mathcal{I}$. On the other hand, since \mathcal{R} is also closed under finite intersections, i.e., a π -system and the restrictions of $\lambda \boxtimes \lambda$ and $\lambda \otimes \lambda$ to \mathcal{R} are the same, the usual result on the uniqueness of measures (see [9] p. 44) implies that the restriction of $\lambda \boxtimes \lambda$ to $\mathcal{I} \otimes \mathcal{I}$ must equal $\lambda \otimes \lambda$ on $\mathcal{I} \otimes \mathcal{I}$.

The Fubini property for the Loeb product space $(I_0 \times I_0, \mathcal{I}_0 \boxtimes \mathcal{I}_0, \lambda_0 \boxtimes \lambda_0)$ was shown by Keisler (see Section IV.5 in [21] and [25]). Theorem 6.2 in [34] implies that there exists a process g from $(I_0 \times I_0, \mathcal{I}_0 \boxtimes \mathcal{I}_0, \lambda_0 \boxtimes \lambda_0)$ to \mathbb{R} such that the random variables $g_i(\cdot)$ are essentially pairwise independent with uniform distribution on $[0, 1]$. Let $f = g \circ \Phi^{-1}$, i.e., for $(i, \omega) \in [0, 1] \times [0, 1]$, $f(i, \omega) = g(\phi^{-1}(i), \phi^{-1}(\omega))$. Since both ϕ and Φ are isomorphisms between the corresponding measure spaces, the rest is clear. ■

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⁴²This means that $\mathcal{I} = \{\phi(A) : A \in \mathcal{I}_0\}$ and $\lambda(\phi(A)) = \lambda_0(A)$.

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