

# Moderate Deviations and Cluster Measures in Geometric Probability

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## Abstract

Functionals in geometric probability are often expressed as sums of bounded functions exhibiting exponential stabilization. Methods based on cumulant expansions and cluster measures show that such functionals satisfy moderate deviation principles. This leads to moderate deviation principles and laws of the iterated logarithm for random packing models, the process of maximal points, as well as for statistics associated with graphs in computational geometry, including the Euclidean  $k$  nearest neighbors graph, the Voronoi graph, and the sphere of influence graph.

## 1 Introduction, main results

### 1.1 Introduction

Functionals and measures induced by binomial and Poisson point processes in  $d$ -dimensional Euclidean space often satisfy a weak spatial dependence structure termed stabilization [25, 26, 5], which, roughly speaking, quantifies the degree to which functionals are determined by the local configuration of points. Stabilization has been used to establish thermodynamic limits for many functionals [26, 27] and it has also been employed in a general setting to establish Gaussian limits for re-normalized functionals as well as re-normalized spatial point measures [5, 24]. Such general

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results can be applied to deduce limit laws for a variety of functionals and measures, including those defined by percolation models [24], random graphs in computational geometry [5, 25], random packing models [4, 26], germ grain models [5], and the process of maximal points [7].

Here we show that stabilization, coupled with cumulant expansion techniques, also yields moderate deviation principles and laws of the iterated logarithm for many of these same functionals. By explicitly identifying rate functions we relate the large scale limit behavior of stabilizing functionals to the local behavior of the underlying density of points.

When the stabilization is of exponential type, which roughly translates into exponential decay of correlations, then Gaussian limit behavior can be proved using either the Stein method [28] or the method of cumulants [5]. The latter method, which yields precise asymptotics for correlation functions, depends heavily on cumulant expansion methods and exponential clustering of measures [5]. While technically more involved, the method of cumulant expansions allows one to express cumulant measures in terms of *cluster measures* exhibiting exponential decay of correlations. This information gives growth rates for cumulants in terms of the scale parameter.

In this paper we refine the cumulant expansion method in order to establish more precise rates of growth on the cumulants, in both their scale parameter and their order. This yields asymptotics for random measures and functionals on scales intermediate between those appearing in Gaussian and thermodynamic limit behavior. In other words, by appealing to Gärtner-Ellis and Dawson-Gärtner theory, we may establish moderate deviation principles, and conclude almost sure laws of the iterated logarithm. This leads to moderate deviation principles and laws of the iterated logarithm for functionals of random sequential packing models, the process of maximal points, as well as for statistics associated with graphs in computational geometry, including the  $k$  nearest neighbors graph, the Voronoi graph, and the sphere of influence graph.

## 1.2 Terminology

As in [5, 27] we let  $\xi(x; \mathcal{X})$  be a bounded measurable  $\mathbb{R}$ -valued function defined for all pairs  $(x, \mathcal{X})$ , where  $\mathcal{X} \subset \mathbb{R}^d$  is finite and where  $x \in \mathcal{X}$ . Assume that  $\xi$  is translation invariant, that is  $\xi(x; \mathcal{X}) = \xi(x - y; \mathcal{X} - y)$  for all  $y \in \mathbb{R}^d$ . When  $x \notin \mathcal{X}$ , we abbreviate notation and write  $\xi(x; \mathcal{X})$  instead of  $\xi(x; \mathcal{X} \cup x)$ . For all  $\lambda > 0$  let  $\xi_\lambda(x; \mathcal{X}) := \xi(\lambda^{1/d}x; \lambda^{1/d}\mathcal{X})$ , where given  $a > 0$ , we let  $a\mathcal{X} := \{ax : x \in \mathcal{X}\}$ .  $\xi(x; \mathcal{X})$  should be thought of as a measure of the interaction between  $x$  and the point set  $\mathcal{X}$ .

Given a continuous probability density  $\kappa$  with support  $[0, 1]^d$ , for all  $\lambda > 0$  we let  $\mathcal{P}_{\lambda\kappa}$  denote a

Poisson point process with intensity  $\lambda\kappa$  on  $[0, 1]^d$ . The density  $\kappa$  and the weight  $\xi$  generate scaled *random point measures* on  $[0, 1]^d$ :

$$\mu_{\lambda\kappa}^\xi := \sum_{x \in \mathcal{P}_{\lambda\kappa}} \xi_\lambda(x; \mathcal{P}_{\lambda\kappa}) \delta_x$$

as well as the corresponding measures generated by fixed-size binomial samples

$$\rho_{n,\kappa}^\xi := \sum_{i=1}^n \xi_n(X_i; \{X_j\}_{j=1}^n) \delta_{X_i} \quad (1.1)$$

where  $X_i$  are i.i.d. with density  $\kappa$ . We will establish moderate deviation principles for the centered version of  $\mu_{\lambda\kappa}^\xi$ , namely for  $\bar{\mu}_{\lambda\kappa}^\xi := \mu_{\lambda\kappa}^\xi - \mathbb{E} \mu_{\lambda\kappa}^\xi$  and laws of the iterated logarithm for  $\bar{\mu}_{\lambda\kappa}^\xi$  and  $\bar{\rho}_{n,\kappa}^\xi := \rho_{n,\kappa}^\xi - \mathbb{E} \rho_{n,\kappa}^\xi$ . We first introduce some terminology [5].

Exponential stabilization, used heavily in [5, 25], plays a central role in all that follows. For  $x \in \mathbb{R}^d$  and  $r > 0$   $B_r(x)$  denotes the Euclidean ball centered at  $x$  of radius  $r$ ,  $\mathbf{0}$  denotes the origin of  $\mathbb{R}^d$ , and for all  $\tau > 0$ ,  $\mathcal{P}_\tau$  denotes a homogeneous Poisson point process on  $\mathbb{R}^d$  of intensity  $\tau$ .

For all  $0 \leq a < b < \infty$ ,  $\mathcal{F}(a, b)$  consists of  $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$  having support either  $[0, 1]^d$  or  $\mathbb{R}^d$  and such that the range of  $f$  is a subset of  $[a, b]$ . For  $f \in \mathcal{F}(a, b)$ , let  $\mathcal{P}_f$  denote a Poisson point process with intensity  $f$ .

**Definition 1.1** ([5]) *The functional  $\xi$  is exponentially stabilizing if for all  $0 \leq a < b < \infty, 0 < \lambda \leq \infty, f \in \mathcal{F}(a, b)$  and  $x \in \lambda[0, 1]^d$ , there exists a random variable  $R := R_{\lambda;f}^\xi(x)$  (a radius of stabilization for  $\xi$  at  $x$  with intensity  $f$  and under scaling  $\lambda$ ) such that for all  $f \in \mathcal{F}(a, b)$  with  $\text{Supp } f = \lambda[0, 1]^d$ ,*

$$\xi(x; (\lambda^{1/d} \mathcal{P}_f \cap B_R(x)) \cup \mathcal{X})$$

*is independent of  $\mathcal{X}$  for all finite  $\mathcal{X} \subset \lambda A \setminus B_R(x)$  and there exists finite constants  $L := L(a, b) > 0$ ,  $\alpha := \alpha(a, b) > 0$  such that for all  $t > 0$*

$$\sup_{x \in \mathbb{R}^d, 0 < \lambda \leq \infty} P[R_{\lambda;f}^\xi(x) > t] \leq L \exp(-\alpha t). \quad (1.2)$$

*When  $\xi$  stabilizes then for all  $\tau > 0$  we let  $\xi(\mathbf{0}; \mathcal{P}_\tau) := \lim_{l \rightarrow \infty} \xi(\mathbf{0}; \mathcal{P}_\tau \cap B_l(\mathbf{0}))$ .*

Thus  $R := R_{\lambda;f}^\xi(x)$  is a radius of stabilization if the value of  $\xi(x; \mathcal{P}_f)$ ,  $f \in \mathcal{F}(a, b)$ , is unaffected by changes outside  $B_R(x)$ . Say that  $\xi$  is *strongly exponentially stabilizing* if for all  $z \in \mathbb{R}^d$ , Definition 1.1 also holds with  $\mathcal{P}_f$  replaced by  $\mathcal{P}_f \cup z$ . Note that in the sequel we usually take  $f := \kappa$ . Throughout we assume that  $\xi$  is strongly exponentially stabilizing and bounded, that is  $|\xi| \leq C_\xi$  for some finite constant  $C_\xi$ . See [5] for examples of exponentially stabilizing functionals.

For all  $\mathcal{X}$ , let  $H(\mathcal{X}) := H^\xi(\mathcal{X}) := \sum_{x \in \mathcal{X}} \xi(x; \mathcal{X})$ . If  $\xi$  is strongly exponentially stabilizing, then (see Proposition 2.1 of [5]) for all  $\tau > 0$  there is a random variable  $\Delta^\xi(\tau)$  such that almost surely

$$\lim_{l \rightarrow \infty} [H(\mathcal{P}_\tau \cap B_l(\mathbf{0}) \cup \{\mathbf{0}\}) - H(\mathcal{P}_\tau \cap B_l(\mathbf{0}))] = \Delta^\xi(\tau). \quad (1.3)$$

Let  $D^\xi(\tau) := \mathbb{E}[\Delta^\xi(\tau)]$  for all  $\tau > 0$ .

We let  $\langle f, \mu \rangle$  denote the integral with respect to a signed finite variation Borel measure  $\mu$  of a  $\mu$ -integrable function  $f$ . Let  $\mathcal{C}([0, 1]^d)$  be the collection of continuous  $f : [0, 1]^d \rightarrow \mathbb{R}$ .

### 1.3 Moderate deviation principles

From [5, 27], we know that if  $\xi$  is exponentially stabilizing, then the one and two point correlation functions for  $\xi_\lambda(x; \mathcal{P}_{\lambda\kappa})$  converge in the large  $\lambda$  limit, which establishes volume order asymptotics for  $\mathbb{E}[\mu_{\lambda\kappa}^\xi([0, 1]^d)]$  and  $\text{Var}[\mu_{\lambda\kappa}^\xi([0, 1]^d)]$  as  $\lambda \rightarrow \infty$ . Moreover, under strong exponential stabilization, the limit of the re-normalized measures  $(\lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi)_\lambda$  is a generalized mean zero Gaussian field in the sense that the finite dimensional distributions of  $(\lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi)_\lambda$  over test functions  $f \in \mathcal{C}([0, 1]^d)$  converge to those of a Gaussian field. In formal terms, we have:

**Theorem 1.1** (Theorem 2.1 of [5], Theorem 2.1 of [27]) *If  $\xi$  is exponentially stabilizing then*

$$\lim_{\lambda \rightarrow \infty} \frac{\mathbb{E}[\mu_{\lambda\kappa}^\xi([0, 1]^d)]}{\lambda} = \int_{[0, 1]^d} \mathbb{E}[\xi(\mathbf{0}; \mathcal{P}_{\kappa(x)})] \kappa(x) dx.$$

*If  $\xi$  is strongly exponentially stabilizing then for all  $\tau > 0$  there is a constant  $V^\xi(\tau)$  such that*

$$\lim_{\lambda \rightarrow \infty} \frac{\text{Var}[\mu_{\lambda\tau}^\xi([0, 1]^d)]}{\lambda} = \tau V^\xi(\tau)$$

*and the finite-dimensional distributions of  $\lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi$  converge in distribution as  $\lambda \rightarrow \infty$  to those of a generalized mean-zero Gaussian field with covariance kernel*

$$(f_1, f_2) \mapsto \int_{[0, 1]^d} f_1(x) f_2(x) V^\xi(\kappa(x)) \kappa(x) dx.$$

*If the distribution of  $\Delta^\xi(\kappa(X))$  is non-degenerate (not identically zero) then the limiting Gaussian field is non-degenerate.*

Theorem 1.1 captures the weak law of large numbers and the Gaussian limit behavior of the re-normalized measures  $\lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi$ . Similar results hold for the binomial measures (1.1) (Theorem 2.2. of [5]). It is natural to investigate the asymptotics of  $(\bar{\mu}_{\lambda\kappa}^\xi)_\lambda$  on intermediate scales. This leads us to moderate deviations principles (MDP). We say that a family of probability measures

$(\mu_\varepsilon)_{\varepsilon>0}$ , on some topological space  $\mathcal{T}$  obeys a large deviation principle (LDP) with speed  $\varepsilon$  and good rate function  $I(\cdot) : \mathcal{T} \rightarrow \mathbb{R}_0^+ \cup \{+\infty\}$  if

- $I$  is lower semi-continuous and has compact level sets  $N_L := \{x \in \mathcal{T} : I(x) \leq L\}$ , for every  $L \in [0, \infty)$ .
- For every open set  $G \subseteq \mathcal{T}$  it holds

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mu_\varepsilon(G) \geq - \inf_{x \in G} I(x). \quad (1.4)$$

- For every closed set  $A \subseteq \mathcal{T}$  it holds

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mu_\varepsilon(A) \leq - \inf_{x \in A} I(x). \quad (1.5)$$

Similarly we will say that a family of random variables  $(Y_\varepsilon)_{\varepsilon>0}$  with topological state space  $\mathcal{T}$  obeys a large deviation principle with speed  $\varepsilon$  and good rate function  $I(\cdot) : \mathcal{T} \rightarrow \mathbb{R}_0^+ \cup \{+\infty\}$  if the sequence of their distributions does. Formally a moderate deviation principle is nothing but an LDP. However, we will speak about a moderate deviation principle (MDP) for a sequence of random variables, whenever the scaling of the corresponding random variables is between that of an ordinary Law of Large Numbers and that of a Central Limit Theorem.

Now let  $(\alpha_\lambda)_{\lambda>0}$  be such that

$$\lim_{\lambda \rightarrow \infty} \alpha_\lambda = \infty \text{ and } \lim_{\lambda \rightarrow \infty} \alpha_\lambda \lambda^{-1/2} = 0.$$

Under these assumptions first we obtain the following MDP:

**Theorem 1.2** (*MDP on Poisson samples*) *Assume that the distribution of  $\Delta^\xi(\kappa(X))$  is non-degenerate. For each  $f \in \mathcal{C}([0, 1]^d)$  the family of random variables  $(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle)_\lambda$  satisfies on  $\mathbb{R}$  the moderate deviation principle with speed  $\alpha_\lambda^2$  and good rate function*

$$K_{\kappa;f}^\xi(t) := \frac{t^2}{2} \left( \int_{[0,1]^d} f^2(x) V^\xi(\kappa(x)) \kappa(x) dx \right)^{-1}. \quad (1.6)$$

*Remarks.* (i) Theorem 1.2 interpolates between the law of large numbers and CLT statements in Theorem 1.1. Through the rate function (1.6), Theorem 1.2 relates the large scale behavior of the family  $(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle)_\lambda$  to the local behavior of the underlying Poisson point process. In view of [5, 25], the non-degeneracy assumption on  $\Delta^\xi(\kappa(X))$  insures that  $V^\xi$  is non-zero and thus the rate function is finite.

(ii) Specific applications of Theorem 1.2, including one example with an explicit identification of the rate function (subsection 2.2), are given in section two.

(iii) Our method of proof can be modified to show that Theorem 1.2 also holds whenever the support of  $\kappa$  is a compact convex subset of  $\mathbb{R}^d$  with non-empty interior. By following the approach of [5], the results could be modified to yield an MDP for non-translation invariant functionals.

(iv) We do not know how to prove the analog of Theorem 1.2 for de-Poissonized measures (1.1) nor do we know if, in the general setting of geometric probability, it is possible to remove the boundedness assumption on  $\xi$ . In fact the MDP often breaks down for unbounded  $\xi$ 's: as a natural example, one can take  $\xi(x, \mathcal{X})$  to be the squared volume of the largest ball centered at  $x$  which does not contain any other sample points. This functional is clearly exponentially stabilizing, yet it is easy to see that the MDP is not valid for rates above  $\lambda^{1/6}$ . Note that in addition to being unbounded, the functional  $\xi$  has an infinite Laplace transform for all positive arguments, which puts us in a situation where, in general, one does not expect the MDP to hold.

The next result is a MDP on the level of *measures*. To formulate the result we still need some more terminology. Let us denote by  $\mathcal{M}([0, 1]^d)$  the real vector space of finite signed measures on  $[0, 1]^d$ . Equip  $\mathcal{M}([0, 1]^d)$  with the *weak topology* generated by the sets  $\{U_{f,x,\delta}, f \in \mathcal{C}([0, 1]^d), x \in \mathbb{R}, \delta > 0\}$ , where

$$U_{f,x,\delta} := \{\nu \in \mathcal{M}([0, 1]^d) : |\langle f, \nu \rangle - x| < \delta\}.$$

The Borel- $\sigma$ -field generated by the weak topology is denoted by  $\mathcal{B}$ . It is well known, that since the collection of linear functionals  $\{\nu \mapsto \langle f, \nu \rangle : f \in \mathcal{C}([0, 1]^d)\}$  is separating in  $\mathcal{M}([0, 1]^d)$ , this topology makes  $\mathcal{M}([0, 1]^d)$  into a locally convex, Hausdorff topological vector space, whose topological dual is the preceding collection, hereafter identified with  $\mathcal{C}([0, 1]^d)$ .

**Theorem 1.3** (*measure level MDP*) *If  $\xi$  is exponentially stabilizing and the distribution of  $\Delta^\xi(\kappa(X))$  is non-degenerate, then the family*

$$(\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda, \kappa}^\xi)_\lambda$$

*satisfies a MDP on  $\mathcal{M}([0, 1]^d)$ , endowed with the weak topology, with speed  $\alpha_\lambda^2$  and a convex, good rate function*

$$I_\kappa^\xi(\nu) := \frac{1}{2} \int_{[0,1]^d} \left( \frac{d\nu}{V^\xi(\kappa(x)) \kappa(x) dx} \right)^2 V^\xi(\kappa(x)) \kappa(x) dx, \quad (1.7)$$

*if  $\nu \in \mathcal{M}([0, 1]^d)$  is absolutely continuous with respect to  $V^\xi(\kappa(x)) \kappa(x) dx$ , and  $+\infty$  otherwise.*

It is an easy observation to obtain a multi-dimensional version of Theorem 1.3.

**Theorem 1.4** *If  $\xi$  is exponentially stabilizing and the distribution of  $\Delta^\xi(\kappa(X))$  is non-degenerate, then for each linearly independent collection of continuous functions  $f_1, \dots, f_l : [0, 1]^d \rightarrow \mathbb{R}$ ,  $l \in \mathbb{N}$ , the family*

$$(\alpha_\lambda^{-1} \lambda^{-1/2} (\langle f_1, \bar{\mu}_{\lambda\kappa}^\xi \rangle, \dots, \langle f_l, \bar{\mu}_{\lambda\kappa}^\xi \rangle))_\lambda$$

*satisfies a MDP on  $\mathbb{R}^l$  with speed  $\alpha_\lambda^2$  and a good rate function*

$$I_{\kappa, f_1, \dots, f_l}^\xi(t) := \frac{1}{2} \langle t, C^{-1}(\xi, \kappa, f_1, \dots, f_l)t \rangle, \quad t \in \mathbb{R}^l, \quad (1.8)$$

*where  $C(\xi, \kappa, f_1, \dots, f_l)$  denotes the covariance matrix with entries*

$$C_{ij}(\xi, \kappa, f_1, \dots, f_l) := \int_{[0,1]^d} f_i(x) f_j(x) V^\xi(\kappa(x)) \kappa(x) dx.$$

Note that the linear independence of  $f_1, \dots, f_l$  guarantees that the matrix  $C(\xi, \kappa, f_1, \dots, f_l)$  above is invertible so that  $C^{-1}(\xi, \kappa, f_1, \dots, f_l)$  is well defined.

*Remarks.* (i) Starting with Theorem 1.4, we can alternatively apply Theorem 3.3 in [1] to get the measure-valued result, Theorem 1.3. See also [18] and [19], where this approach is applied to prove large and moderate deviations for empirical measures.

(ii) We expect that the measure-valued MDP Theorem 1.3 holds with respect to the strong topology on  $\mathcal{M}([0, 1]^d)$ . Proving this would necessitate showing that the upcoming Proposition 3.1 holds for all bounded functions on  $[0, 1]^d$ .

## 1.4 General laws of the iterated logarithm

For all  $\lambda > e$ , put  $\alpha_\lambda := \sqrt{\log \log \lambda}$  and

$$\zeta_{\lambda\kappa}^\xi := \alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi. \quad (1.9)$$

Further, denote by  $\mathcal{K}_\kappa^\xi$  the ‘unit ball’ for  $I_\kappa^\xi$ , given by

$$\mathcal{K}_\kappa^\xi := \{\theta \in \mathcal{M}([0, 1]^d) \mid I_\kappa^\xi(\theta) \leq 1\}. \quad (1.10)$$

We will show that Theorem 1.3 yields the following general Strassen-type law of the iterated logarithm (LIL).

**Theorem 1.5** *For any possibly coupling of the family of random measures  $(\mu_{\lambda\kappa}^\xi)_\lambda$  on a common probability space  $(\Omega, \mathcal{F}, \mathcal{P})$  and for any countable sequence  $\lambda \rightarrow \infty$  the family of random measures  $(\zeta_{\lambda\kappa}^\xi)_\lambda \subseteq \mathcal{M}([0, 1]^d)$  is a.s. relatively compact and all its accumulation points a.s. fall into  $\mathcal{K}_\kappa^\xi$ . Moreover, there exists a coupling of  $(\mu_{\lambda\kappa}^\xi)_\lambda$  on a common probability space such that the set of accumulation points of  $(\zeta_{\lambda\kappa}^\xi)_\lambda$  a.s. coincides with  $\mathcal{K}_\kappa^\xi$ .*

It should be emphasized that we consider the families of random measures  $(\zeta_{\lambda\kappa}^\xi)_\lambda$  along countable sequences  $\lambda \rightarrow 0$  rather than over all of  $\mathbb{R}^+$  in order to avoid technicalities due to the presence of accumulation points arising along subsequences of  $\lambda$  converging to a finite limit in  $\mathbb{R}^+$ . We do so in all of our LIL results below, without further mention.

The following scalar LIL is an immediate consequence of Theorem 1.5 above.

**Theorem 1.6** (*LIL on Poisson samples*) *For any possible coupling of the family of random measures  $(\mu_{\lambda\kappa}^\xi)_\lambda$  on a common probability space  $(\Omega, \mathcal{F}, \mathcal{P})$  for any  $f \in \mathcal{C}([0, 1]^d)$  we have almost surely*

$$\limsup_{\lambda \rightarrow \infty} \langle f, \zeta_{\lambda\kappa}^\xi \rangle \leq \sqrt{2 \int_{[0,1]^d} f^2(x) V^\xi(\kappa(x)) \kappa(x) dx} \quad (1.11)$$

and

$$\liminf_{\lambda \rightarrow \infty} \langle f, \zeta_{\lambda\kappa}^\xi \rangle \geq -\sqrt{2 \int_{[0,1]^d} f^2(x) V^\xi(\kappa(x)) \kappa(x) dx}. \quad (1.12)$$

Moreover, there exists a coupling of  $(\mu_{\lambda\kappa}^\xi)_\lambda$  on  $(\Omega, \mathcal{F}, \mathcal{P})$  such that the above bounds are attained.

‘De-Poissonization’ techniques for stabilizing functionals, as developed in [5], yield a corresponding LIL for the binomial measures (1.1). Note that we are only able to state this result in the scalar setting. For notational convenience put

$$\theta_{n,\kappa}^\xi := \alpha_n^{-1} n^{-1/2} \bar{\rho}_{n,\kappa}^\xi. \quad (1.13)$$

Before stating a LIL for the random variables  $(\langle f, \theta_{n,\kappa}^\xi \rangle)_n$  we need some additional terminology. For all  $x \in \mathcal{X}$ ,  $\mathcal{X}$  a finite point set in  $\mathbb{R}^d$ , let  $\Delta_x(\mathcal{X}) := H(\mathcal{X} \cup x) - H(\mathcal{X})$ . Let  $\mathcal{X}_{m,n}$  be a point process consisting of  $m$  i.i.d. random variables  $n^{1/d}X$  on  $n^{1/d}[0, 1]^d$ , where  $X$  has density  $\kappa$ . Say that  $H$  satisfies the *bounded moments condition for  $\kappa$*  (cf. [25]) if

$$\sup_n \sup_{x \in n^{1/d}[0,1]^d} \sup_{m \in [n/2, 3n/2]} \mathbb{E} [\Delta_x^4(\mathcal{X}_{m,n})] < \infty. \quad (1.14)$$

For any  $f \in \mathcal{C}([0, 1]^d)$  let

$$\sigma(\xi, \kappa, f) := \sqrt{2 \int_{[0,1]^d} f^2(x) V^\xi(\kappa(x)) \kappa(x) dx - 2 \left( \int_{[0,1]^d} f(x) D^\xi(\kappa(x)) \kappa(x) dx \right)^2}.$$

**Theorem 1.7** (*LIL on binomial samples*) *Assume that  $H^\xi$  satisfies the bounded moments condition (1.14). For the family of random measures  $(\rho_{\lambda\kappa}^\xi)_\lambda$  and for any  $f \in \mathcal{C}([0, 1]^d)$  we have almost surely*

$$\limsup_{n \rightarrow \infty} \langle f, \theta_{n,\kappa}^\xi \rangle \leq \sigma(\xi, \kappa, f) \quad (1.15)$$

and

$$\liminf_{n \rightarrow \infty} \langle f, \theta_{n, \kappa}^\xi \rangle \geq -\sigma(\xi, \kappa, f). \quad (1.16)$$

*Remarks.* (i) As evident from the expression for  $\sigma(\xi, \kappa, f)$ , Poissonization contributes extra randomness. As noted in Theorem 2.1 of [25] and Theorem 2.2 of [5],  $\sigma(\xi, \kappa, f)$  is strictly positive whenever  $\Delta^\xi$  is non-degenerate.

(ii) It should be noted, as further discussed in the proof of Theorem 1.6 (see (4.4) there), that the coupling under which the bounds (1.11) and (1.12) are attained, in the special case of the uniform density  $\kappa$ , coincides with a certain natural coupling often appearing in applications.

(iii) The bounded moments condition (1.14) is needed to facilitate de-Poissonization.

## 1.5 Further limit theory

### 1.5.1 Surface order growth

Theorem 1.1 shows that the growth rates for  $\mathbb{E}[\mu_{\lambda\kappa}^\xi[0, 1]^d]$  are of volume order. In some instances the functional  $\xi(x; \mathcal{X})$  defined for  $x \in S$ ,  $\mathcal{X} \subseteq S$ , with  $S$  standing for a sufficiently smooth subset of  $\mathbb{R}^d$ , exhibit for typical configurations  $\mathcal{X}$  a rapid (usually exponential) decay in the distance  $d(x, \partial S)$  between  $x$  and  $\partial S$ . In formal terms it means that

$$P[\xi(x; \mathcal{P}_1 \cap S) > 0] \leq \exp(-Cd(x, \partial S)), \quad x \in \mathbb{R}^d. \quad (1.17)$$

We postpone a representative example of such a situation – maximal point counting functionals – to Subsection 2.2, while dedicating the current section to a general discussion. Choosing a projection operator  $\pi : S \rightarrow \partial S$  specific for a particular setting it is natural to consider the measures

$$\mu_{\lambda\kappa}^{\xi; \pi} := \mu_{\lambda\kappa} := \sum_{x \in \mathcal{P}_{\lambda\kappa}} \xi_\lambda(x; \mathcal{P}_{\lambda\kappa}) \delta_{\pi(x)},$$

where the density  $\kappa$  is concentrated on  $S$ . The measures  $\mu_{\lambda\kappa}$  typically exhibit surface order growth rates and an analog of Theorem 1.1 holds with the volume order term  $\lambda$  replaced by the surface order term  $\lambda^{(d-1)/d}$ . See [7] for details. In other words, as shown for a representative example in [7], under the standard exponential stabilization condition, appropriate moment boundedness conditions and the surface decay condition (1.17) it can be concluded that there is a constant  $C(\xi, \kappa)$  depending on both  $\xi$  and the restriction of  $\kappa$  to  $\partial S$  such that

$$\lim_{\lambda \rightarrow \infty} \frac{\mathbb{E}[\mu_{\lambda\kappa}^\xi(S)]}{\lambda^{(d-1)/d}} = C(\xi, \kappa)$$

and for all  $\tau > 0$  there is a constant  $V^\xi(\tau)$  such that

$$\lim_{\lambda \rightarrow \infty} \frac{\text{Var}[\mu_{\lambda\kappa}^\xi(S)]}{\lambda^{(d-1)/d}} = \int_{\partial S} V^\xi(\kappa(x)) \kappa^{(d-1)/d}(x) \sigma(dx)$$

with  $\sigma$  standing for the surface measure. Moreover, the finite dimensional distributions of  $\lambda^{-(d-1)/2d} \bar{\mu}_{\lambda\kappa}^\xi$  converge as  $\lambda \rightarrow \infty$  to those of a generalized Gaussian field with covariance kernel

$$(f_1, f_2) \mapsto \int_{\partial S} f_1(x) f_2(x) V^\xi(\kappa(x)) \kappa^{(d-1)/d}(x) \sigma(dx).$$

Additionally, using the methods developed in the present paper and letting  $f$  range over the bounded continuous functions defined on  $\partial S$ , we can obtain a full moderate deviation principle with speed  $\alpha_\lambda^2$  for the family of random variables  $(\alpha_\lambda^{-1} \lambda^{-(d-1)/2d} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle)_\lambda$ ,  $\alpha_\lambda \ll \lambda^{-(d-1)/2d}$ , that is to say an analog of Theorem 1.2 holds. Likewise, the law of the iterated logarithm given by Theorem 1.6 holds with  $\lambda^{1/2}$  replaced by  $\lambda^{(d-1)/2d}$ . Moreover, one can define the measures

$$\rho_n^{\xi; \pi} = \rho_n := \sum_{i=1}^n \xi(X_i; \{X_i\}_{i=1}^n) \delta_{\pi(X_i)}$$

where  $X_i$  are i.i.d. copies of an  $S$ -valued random element distributed according to the density  $\kappa$  (we require that  $\int_S \kappa = 1$  in this case). It can be argued following the lines of [7] that, unlike in the volume order setting, in the surface setting the asymptotic CLT and LIL behavior of  $\rho_n$  coincides with that of  $\mu_{n\kappa}$  so that analogs of the corresponding theorems hold also for  $\rho_n$ . We believe it is so for the MDP as well, but we were not able to prove it so far.

### 1.5.2 Normal approximation

As explained at the outset, our methods rely upon establishing precise growth rates on the cumulants of  $\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle$  in terms of the scale parameter  $\lambda$  and the order  $k$ . For  $k \geq 3$ , we show that the cumulants are bounded by  $AB^k k! \lambda^{(2-k)/2}$  where  $A$  and  $B$  are finite constants (see Lemma 3.2). Such bounds give very good control of the moment generating function of  $\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle$  near the origin and consequently yield approximation, as  $\lambda \rightarrow \infty$ , of the distribution function of  $\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle$  by the normal distribution. This yields Berry-Esseen bounds (see e.g. Lemmas 1.2 and 1.3 of [30]) as well as Cramér-Petrov large deviation relations of the type

$$\frac{P \left[ \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \geq x \right]}{P \left[ N(0, 1) \geq x \right]} = \exp(L(x))(1 + o(1))$$

holding uniformly in  $x$  over bounded intervals as  $\lambda \rightarrow \infty$ , see e.g. Lemma 1.5 of [30]. Here, the function  $L(x)$  admits an asymptotic expansion into Cramér-Petrov series [30].

## 2 Applications

We provide here applications of our moderate deviation principle and laws of the iterated logarithm to functionals in geometric probability, including functionals of random sequential packing models, the process of maximal points, and functionals of graphs in computational geometry. The following examples have been considered in detail in the context of central limit theorems [25, 5, 28] and thermodynamic limits [26, 27] and it is only natural to consider them here as well. We envision applications to germ-grain models as well.

### 2.1 Packing

#### 2.1.1 Random sequential packing

The following prototypical random sequential packing model arises in diverse disciplines, including physical, chemical, and biological processes. See [26] for a discussion of the many applications, the many references, and also a discussion of previous mathematical analysis. In one dimension, this model is often referred to as the Rényi car parking model [29].

With  $N(\lambda)$  standing for a Poisson random variable with parameter  $\lambda$ , let  $B_{\lambda,1}, B_{\lambda,2}, \dots, B_{\lambda,N(\lambda)}$  be a sequence of  $d$ -dimensional balls of volume  $\lambda^{-1}$  whose centers are i.i.d. random  $d$ -vectors  $X_1, \dots, X_{N(\lambda)}$  with probability density function  $\kappa : [0, 1]^d \rightarrow [0, \infty)$ . Without loss of generality, assume that the balls are sequenced in the order determined by marks (time coordinates) in  $[0, 1]$ . Let the first ball  $B_{\lambda,1}$  be *packed*, and recursively for  $i = 2, 3, \dots$ , let the  $i$ -th ball  $B_{\lambda,i}$  be packed iff  $B_{\lambda,i}$  does not overlap any ball in  $B_{\lambda,1}, \dots, B_{\lambda,i-1}$  which has already been packed. If not packed, the  $i$ -th ball is discarded. The collection of centers of accepted balls induces a point measure on  $[0, 1]^d$ , denoted  $\mu_{\lambda^{-1}}$ . We call this the *random sequential packing measure* induced by balls (of volume  $\lambda^{-1}$ ) with centers arising from  $\kappa$ .

For any finite point set  $\mathcal{X} \subset \mathbb{R}^d$ , assume the points  $x \in \mathcal{X}$  have time coordinates which are independent and uniformly distributed over the interval  $[0, 1]$ . Assume unit volume balls centered at the points of  $\mathcal{X}$  arrive sequentially in an order determined by the time coordinates, and assume as before that each ball is packed or discarded according to whether or not it overlaps a previously packed ball. Let  $\xi(x; \mathcal{X})$  be either 1 or 0 depending on whether the ball centered at  $x$  is packed or discarded.  $\xi$  is strongly exponentially stabilizing,  $H^\xi$  satisfies the bounded moments condition, and  $\Delta^\xi$  is non-degenerate [26]. Let  $\xi_\lambda(x; \mathcal{X}) = \xi(\lambda^{1/d}x; \lambda^{1/d}\mathcal{X})$ , where  $\lambda^{1/d}x$  denotes

scalar multiplication of  $x$  and *not* the mark associated with  $x$ . The random measure

$$\mu_{\lambda\kappa}^\xi := \sum_{i=1}^{N(\lambda)} \xi_\lambda(X_i; \{X_i\}_{i=1}^{N(\lambda)}) \delta_{X_i}$$

coincides with  $\mu_{\lambda^{-1}}$ .

The following theorem provides a MDP and LIL for the random packing measures. Recall that  $\zeta_{\lambda\kappa}^\xi := (\lambda \log \log \lambda)^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi$ , and  $\theta_{n,\kappa}^\xi := (n \log \log n)^{-1/2} \bar{\rho}_{n,\kappa}^\xi$  as in (1.9) and (1.13), respectively.

**Theorem 2.1** (*MDP and LIL*) *For each  $f \in \mathcal{C}([0, 1]^d)$ , the family of random variables  $(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \mu_{\lambda\kappa}^\xi \rangle)_\lambda$  satisfies the moderate deviation principle as in Theorem 1.2 whereas  $\langle f, \zeta_{\lambda\kappa}^\xi \rangle_\lambda$  and  $\langle f, \theta_{n,\kappa}^\xi \rangle_n$  satisfy the law of the iterated logarithm as in Theorems 1.6 and 1.7 respectively. Moreover the family of measures  $(\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi)_\lambda$  satisfies a MDP on  $\mathcal{M}([0, 1]^d)$  with respect to the weak topology, as in Theorem 1.3, and the corresponding LIL, as in Theorem 1.5.*

*Remarks.* (i) By taking  $f = 1$ , Theorem 2.1 provides an MDP and LIL for the total number of balls accepted in the packing model with finite input.

(ii) Theorem 2.1 adds to existing central limit theorems [13, 4, 5, 26] and weak laws of large numbers [12, 26, 27] for random packing functionals.

(iii) We do not know how to establish a MDP or LIL in the infinite input setting in dimensions greater than  $d = 1$ , that is to say where the time coordinates arise as the realization of a homogeneous Poisson point process over  $[0, \infty)$ . Central limit theorems in this context are not known either (cf. Theorem 1.2 of [26]).

### 2.1.2 Spatial Birth-Growth Models

Consider the following spatial birth-growth model in  $\mathbb{R}^d$ . Seeds are born at random locations  $X_i \in \mathbb{R}^d$  at times  $T_i$ ,  $i = 1, 2, \dots$  according to a unit intensity homogeneous spatial temporal Poisson point process  $\Psi := \{(X_i, T_i) \in \mathbb{R}^d \times [0, 1]\}$ . When a seed is born, it forms a cell by growing radially in all directions with a constant speed  $v > 0$ . Whenever one growing cell touches another, it stops growing in that direction. Initially the seed takes the form of a ball of radius  $\rho_i \geq 0$  centered at  $X_i$ . If a seed appears at  $X_i$  and if the ball centered at  $X_i$  with radius  $\rho_i$  overlaps any of the existing cells, then the seed is discarded.

If seeds are born at random locations  $X_i \in [0, 1]^d$ , it is natural to study the *spatial distribution* of accepted seeds. [5] establishes the convergence of the random measure induced by the locations of the accepted seeds.

For any finite point set  $\mathcal{X} \subset [0, 1]^d$ , assume the points  $x \in \mathcal{X}$  have i.i.d. marks over  $[0, 1]$ . A mark at  $x \in \mathcal{X}$  represents the arrival time of a seed at  $x$ . Assume that the seeds are centered at the points of  $\mathcal{X}$  and that they arrive sequentially in an order determined by the associated marks, and assume that each seed is accepted or rejected according to the rules above. Let  $\xi(x; \mathcal{X})$  be either 1 or 0 according to whether the seed centered at  $x$  is accepted or not.  $H(\mathcal{X}) := \sum_{x \in \mathcal{X}} \xi(x; \mathcal{X})$  is the total number of seeds accepted.  $\xi$  is strongly exponentially stabilizing,  $H^\xi$  satisfies the bounded moments condition (1.14), and  $\Delta^\xi$  is non-degenerate [26].

As with the random sequential packing, let  $X_1, \dots, X_{N(\lambda)}$  be i.i.d. random variables with density  $\kappa$  on  $[0, 1]^d$  and with marks in  $[0, 1]$ . The random measure

$$\mu_{\lambda\kappa}^\xi := \sum_{i=1}^{N(\lambda)} \xi_\lambda(X_i; \{X_i\}_{i=1}^{N(\lambda)}) \delta_{X_i},$$

is the scaled spatial-birth growth measure on  $[0, 1]^d$  induced by  $X_1, \dots, X_{N(\lambda)}$ .

Put  $\zeta_{\lambda\kappa}^\xi := (\lambda \log \log \lambda)^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi$  and  $\theta_{n,\kappa}^\xi := (n \log \log n)^{-1/2} \bar{\rho}_{n,\kappa}^\xi$ .

**Theorem 2.2** *For each  $f \in \mathcal{C}([0, 1]^d)$ , the family of random variables  $(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \mu_{\lambda\kappa}^\xi \rangle)_\lambda$  satisfies the moderate deviation principle as in Theorem 1.2 whereas  $\langle f, \zeta_{\lambda\kappa}^\xi \rangle_\lambda$  and  $\langle f, \theta_{n,\kappa}^\xi \rangle_n$  satisfy the law of the iterated logarithm as in Theorems 1.6 and 1.7 respectively. Moreover the family of measures  $(\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi)_\lambda$  satisfies a MDP on  $\mathcal{M}([0, 1]^d)$  with respect to the weak topology, as in Theorem 1.3 and the corresponding LIL, as in Theorem 1.5.*

*Remarks.* (i) Theorem 2.2 adds to the CLT results of Chiu and Quine [9], Penrose and Yukich [26], and Baryshnikov and Yukich [5], who prove asymptotic normality for the number of accepted seeds.

(ii) Theorem 2.2 extends to more general versions of the prototypical packing model. The stabilization analysis of [26] yields MDPs and LILs in the finite input setting for the number of packed balls in the following general models: (a) models with balls replaced by particles of random size/shape/charge, (b) cooperative sequential adsorption models, and (c) ballistic deposition models (see [26] for a complete description of these models). In each case, our general MDP and LIL apply to the random packing measures associated with the centers of the packed balls, whenever the balls have a continuous density  $\kappa : [0, 1]^d \rightarrow [0, \infty)$ .

## 2.2 Maximal points, record values, Johnson-Mehl growth process

Let  $K \subset (\mathbb{R}^+)^d, d \geq 2$ , be a cone with apex at the origin. Given a point set  $\mathcal{X} \subset \mathbb{R}^d$ , a point  $w \in \mathcal{X}$  is called *maximal* (or Pareto extremal) if  $(K + w) \cap \mathcal{X} = \emptyset$ . The cone  $K$  defines a partial order (or dominance relation) on  $\mathbb{R}^d$ :  $z > w$  if and only if  $z - w \in K$ . Thus  $z$  is maximal with respect to  $\mathcal{X}$  if and only if  $z > y$  for all  $y \in \mathcal{X}$ . When  $K = (\mathbb{R}^+)^d$ ,  $>$  denotes the usual partial order on  $\mathbb{R}^d$ . Clearly  $z \in \mathcal{X}$  is maximal if the cone  $z + K$  contains no other points in  $\mathcal{X}$ .

Let  $M(\mathcal{X}) := M_K(\mathcal{X})$  denote the collection of points  $x \in \mathcal{X}$  which are maximal;  $M(\mathcal{X})$  is called the *maximal layer* for the point set  $\mathcal{X}$ . Maximal layers and maximal points have been widely used in a variety of scientific disciplines and are of broad interest in computational geometry. Maximal points appear in Pareto optimality, multi-criteria decision analysis, networks, data mining, analysis of linear programming, and statistical decision theory, among other areas. For a discussion and for references to the vast scientific literature, we refer to Chen et al. [8].

The probabilistic analysis of  $M_K(\mathcal{X})$ ,  $\mathcal{X}$  random, has received considerable attention in the following two special cases.

(i) *Record Values.* ( $K$  is positive orthant.) Let  $X_i, i \geq 1$ , be i.i.d. random variables distributed in the planar set  $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq f(x)\}$ , where  $f$  is a non-decreasing non-negative function on  $[0, 1]$ . Say that  $(X_i, Y_i)$  corresponds to a *record* if  $Y_i = \max\{Y_j, X_j \leq X_i\}$ . The number of records in the sequence  $\{(X_i, Y_i)\}_{i=1}^n$  has the same distribution as the cardinality of  $M_K(\{(X'_i, Y'_i)\}_{i=1}^n)$ , where  $(X'_i, Y'_i)$  represents a counterclockwise rotation of  $(X_i, Y_i)$  by ninety degrees. The number of records is a well studied problem and has received considerable attention since the pioneering papers of Rényi [29] and Barndorff-Nielsen and Sobel [3].

(ii) *Johnson-Mehl Growth Process.* ( $K$  is right circular cone.) Consider the following version of a classical birth-growth model on  $\mathbb{R}^{d-1}$ : seeds appear at random locations  $X_i \in \mathbb{R}^{d-1}$  at times  $T_i, i = 1, 2, \dots$  according to a spatial-temporal Poisson point process  $\Psi := \{(X_i, T_i) \in \mathbb{R}^{d-1} \times [0, 1]\}$ . When a seed is born, it has initial radius zero and then forms a cell by growing radially in all directions with a constant speed  $v > 0$ . Whenever one growing cell touches another, it stops growing in that direction. If a seed appears at  $X_i$  and if  $X_i$  belongs to any of the existing cells then the seed is discarded. This is the Johnson-Mehl growth process, originally studied to model crystal growth, and is described in Stoyan, Kendall, and Mecke [31]. Let  $K' := K'(v)$  denote the right circular cone in  $\mathbb{R}^d$  with apex at the origin, with aperture  $v$ , and with altitude the vertical axis. Clearly a seed is born at  $X_i$  (and not discarded) iff the cone  $K' + X_i$  contains no other points from  $\Psi$ . Thus, the number of seeds in the Johnson-Mehl growth model coincides with the

cardinality of the maximal layer  $M_{K'}(\{X_i, T_i\})$ .

Given  $\mathcal{X} \subset \mathbb{R}^d$ , let  $\xi_K(x; \mathcal{X})$  be one or zero according to whether the cone  $K$  contains a point in  $\mathcal{X}$  or not. Then  $\xi$  is strongly exponentially stabilizing and satisfies the decay condition (1.17) [7]. Suppose that  $X_i$  are i.i.d. with values in a compact convex  $S \subset \mathbb{R}^d$  with continuous probability density  $\kappa$  supported on  $S$ . Notice that  $H_\lambda := \sum_{i=1}^{N_\lambda} \xi_K(X_i, \{X_j\}_{j=1}^{N(\lambda)})$ , with  $N(\lambda)$  standing for a mean  $\lambda$  Poisson random variable, equals the number of maximal points in  $S$  generated by a Poisson sample of intensity measure  $\lambda\kappa$ . Under smoothness conditions on the boundary  $\partial S$  of  $S$  [7], there is a function  $V : \partial S \rightarrow \mathbb{R}^+$  such that

$$\frac{H_\lambda - \mathbb{E}[H_\lambda]}{\lambda^{(d-1)/2d}} \xrightarrow{\mathcal{D}} N(0, \tau_\kappa^2)$$

where

$$\tau_\kappa^2 := \int_{\partial S} V(x) \kappa(x)^{(d-1)/d} d\sigma(x), \quad \tau_\kappa > 0,$$

where  $\sigma$  denotes the surface measure on  $\partial S$  and where  $\kappa(x)$  denotes the restriction of  $\kappa$  to  $\partial S$ .

As in the discussion in Section 1.5.1, the methods of the present paper combined with the techniques developed in [7] are easily checked to yield an MDP and LIL for the number  $H_\lambda$  of maximal points arising from a Poisson sample of intensity  $\lambda\kappa$  on  $S$ :

**Theorem 2.3** (MDP and LIL) *The family of random variables  $(\alpha_\lambda^{-1} \lambda^{-(d-1)/2d} (H_\lambda - \mathbb{E}[H_\lambda]))_\lambda$  with  $\alpha_\lambda$  such that  $\alpha_\lambda \lambda^{-(d-1)/2d} \rightarrow 0$  and  $\alpha_\lambda \rightarrow \infty$ , satisfies on  $\mathbb{R}$  the full MDP with speed  $\alpha_\lambda^2$  and good rate function  $K_\kappa^\xi(t) := \frac{1}{2}(\frac{t}{\tau_\kappa})^2$ . Additionally, we have a.s.*

$$\limsup_{\lambda \rightarrow \infty} \frac{H_\lambda - \mathbb{E}[H_\lambda]}{\lambda^{(d-1)/2d} \sqrt{2 \log \log \lambda}} \leq \tau_\kappa$$

and

$$\liminf_{\lambda \rightarrow \infty} \frac{H_\lambda - \mathbb{E}[H_\lambda]}{\lambda^{(d-1)/2d} \sqrt{2 \log \log \lambda}} \geq -\tau_\kappa.$$

*Remarks.*

(i) *Record values.* Devroye [17], following the pioneering work of Rényi [29] and Barndorff-Nielsen and Sobel [3], showed that if  $X_i$ ,  $i \geq 1$ , are i.i.d. with the uniform density on the planar set  $S := \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq F(x)\}$  and if  $F$  satisfies suitable regularity conditions, then  $\mathbb{E} H_\lambda / \lambda^{1/2} \rightarrow (\pi/2)^{1/2} \int_0^1 (|F'(x)|)^{1/2} dx$ , later improved to  $\mathbb{E} H_\lambda / \lambda^{1/2} \rightarrow (\pi/2)^{1/2} \int_0^1 (|F'(x)| \kappa(x, F(x)))^{1/2} dx$ , if  $X_i$  have a continuous density  $\kappa$  which is bounded away from zero and infinity [7]. The MDP and LIL given by Theorem 2.3 thus adds to these existing LLNs and CLTs; see Corollaries 2.1 and 2.2 of [7] for details concerning the value of  $\tau_\kappa^2$ .

(ii) *Johnson-Mehl growth processes.* Assume that  $X_i$  are i.i.d. with values in a compact convex  $S \subset \mathbb{R}^d$  with continuous probability density  $\kappa$  supported on  $S$ . In the special case that  $K$  is a right circular cone and  $\partial S$  is a subset of a hyperplane, as is case for the Johnson-Mehl growth process, we obtain an exact expression for the rate function [7]. Indeed,  $\tau_\kappa^2$  reduces to the constant

$$\int_{\partial S} [(\Gamma(1/d)\frac{1}{d}|K(1)|^{-1/d} + C(d)]\kappa(x)^{(d-1)/d} dx,$$

where  $|K(1)|$  is the volume of a right circular cone of height one and where  $C(d)$  is a constant. In  $d = 2$ , the integrand  $[(\Gamma(1/2)\frac{1}{2}|K(1)|^{-1/2} + C(2))]$  is numerically evaluated as 0.2471984828 [7]. Chiu and Quine [9, 10], Heinrich and Molchanov [20] and Penrose [22] provide central limit theorems for Johnson-Mehl growth processes, but without specifying limiting variances in terms of underlying point densities.

## 2.3 Random graphs

### 2.3.1 $k$ -nearest Neighbors Graphs

Let  $k$  be a positive integer. Given a locally finite point set  $\mathcal{X} \subset \mathbb{R}^d$ , the  $k$ -nearest neighbors (undirected) graph on  $\mathcal{X}$ , denoted  $\text{NG}(\mathcal{X})$ , is the graph with vertex set  $\mathcal{X}$  obtained by including  $\{x, y\}$  as an edge whenever  $y$  is one of the  $k$  nearest neighbors of  $x$  and/or  $x$  is one of the  $k$  nearest neighbors of  $y$ . The  $k$ -nearest neighbors (directed) graph on  $\mathcal{X}$ , denoted  $\text{NG}'(\mathcal{X})$ , is the graph with vertex set  $\mathcal{X}$  obtained by placing a directed edge between each point and its  $k$  nearest neighbors.  $k$ -nearest neighbors graphs are translation invariant.

Let  $\xi^t(x; \mathcal{X}) = 1$  if the length of the edge joining  $x$  to its nearest neighbor in  $\mathcal{X}$  is less than  $t$  and zero otherwise. Put  $H^t(\mathcal{X}) = \sum \xi^t(x, \mathcal{X})$ .  $\xi^t$  is strongly exponentially stabilizing,  $H^t$  satisfies the bounded moments condition, and  $\Delta^\xi$  is non-degenerate [26]. Put  $\mu_{\lambda\kappa}^\xi := \sum_{x \in \mathcal{P}_{\lambda\kappa}} \xi_\lambda^t(x; \mathcal{P}_{\lambda\kappa}) \delta_x$ ,  $\zeta_{\lambda\kappa}^\xi := (\lambda \log \log \lambda)^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi$ , and  $\theta_{n,\kappa}^\xi := (n \log \log n)^{-1/2} \bar{\rho}_{n,\kappa}^\xi$ .

**Theorem 2.4** (MDP and LIL) *For each  $f \in \mathcal{C}([0, 1]^d)$ , the family of random variables*

*$(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \mu_{\lambda\kappa}^\xi \rangle)_\lambda$  satisfies the moderate deviation principle as in Theorem 1.2 whereas  $\langle f, \zeta_{\lambda\kappa}^\xi \rangle_\lambda$  and  $\langle f, \theta_{n,\kappa}^\xi \rangle_n$  satisfy the law of the iterated logarithm as in Theorems 1.6 and 1.7 respectively. Moreover the family of measures  $(\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi)_\lambda$  satisfies a MDP on  $\mathcal{M}([0, 1]^d)$  with respect to the weak topology, as in Theorem 1.3 and the corresponding LIL, as in Theorem 1.5.*

Theorem 2.4 yields an MDP and LIL for the empirical distribution function of the rescaled lengths of the edges in the nearest neighbors graph on  $\mathcal{P}_{\lambda\kappa}$ . This gives a MDP and LIL for the

number of pairs of rescaled points distant at most  $t$  from each other, adding to central limit theorems of Penrose (Chapter four of [23]) and [5, 7].

Alternatively, given a positive integer  $m$ , we could let  $\xi_m^{NG}(x; \mathcal{X})$  (respectively,  $\xi_m^{NG'}(x; \mathcal{X})$ ) be one or zero according to whether the degree of  $x$  is equal to  $m$ . Then  $\xi_m^{NG}$  and  $\xi_m^{NG'}$  are strongly exponentially stabilizing and bounded.  $H^\xi$ , which represents the total number of vertices of degree  $m$ , satisfies the bounded moments condition, and  $\Delta^\xi$  is non-degenerate [26, 5]. In this way we obtain a MDP and LIL for the total number of vertices in the  $k$ -nearest neighbor graph of fixed degree.

### 2.3.2 Voronoi and Delaunay Graphs

Given a locally finite set  $\mathcal{X} \subset \mathbb{R}^d$  and  $x \in \mathcal{X}$ , the locus of points closer to  $x$  than to any other point in  $\mathcal{X}$  is called the *Voronoi cell* centered at  $x$ . The graph on vertex set  $\mathcal{X}$  in which each pair of adjacent cell centers is connected by an edge is called the *Delaunay graph* on  $\mathcal{X}$ ; if  $d = 2$ , then the planar dual graph consisting of all boundaries of Voronoi cells is called the *Voronoi graph* generated by  $\mathcal{X}$ . Edges of the Voronoi graph can be finite or infinite. Let  $\xi^t(x; \mathcal{X})$  be either 1 or 0 depending on whether the sum of the edge lengths in the Voronoi cell around  $x$  is less than a fixed cutoff  $t, t > 0$ .  $\xi^t$  is strongly exponentially stabilizing (cf. section 3.1.2 of [5]),  $H^{\xi^t}$  satisfies the bounded moments condition (1.14), and  $\Delta^\xi$  is non-degenerate [26]. We thus obtain a MDP and LIL for the empirical distribution of the number of Voronoi cells on Poisson point sets  $\mathcal{P}_{\lambda\kappa}$  with edge length less than  $t$ . This adds to existing laws of large numbers and central limit theorems (see Theorem 2.5 of [27] and section eight of [25], respectively). We may similarly obtain an MDP and LIL for the empirical distribution of the number of Voronoi cells (or Delaunay triangles) with area less than  $t$ .

### 2.3.3 Sphere of Influence Graph

Given a locally finite set  $\mathcal{X} \subset \mathbb{R}^d$ , the sphere of influence graph  $\text{SIG}(\mathcal{X})$  is a graph with vertex set  $\mathcal{X}$ , constructed as follows: for each  $x \in \mathcal{X}$  let  $B(x)$  be a ball around  $x$  with radius equal to  $\min_{y \in \mathcal{X} \setminus \{x\}} \{|y - x|\}$ . Then  $B(x)$  is called the sphere of influence of  $x$ . Draw an edge between  $x$  and  $y$  iff the balls  $B(x)$  and  $B(y)$  overlap. The collection of such edges is the sphere of influence graph (SIG) on  $\mathcal{X}$  and is denoted by  $\text{SIG}(\mathcal{X})$ .

In section seven of [25] (respectively section 3.1.3 of [5]), CLTs are proved for the total edge length, the number of components, and the number of vertices of fixed degree of the SIG on point

sets with a continuous density. Using the strong exponential stabilizing property of the SIG (section 7 of [25]), Theorems 1.2-1.4 yield an MDP and LIL for the number of vertices of fixed degree in the sphere of influence graph on Poisson point sets  $\mathcal{P}_{\lambda\kappa}$ .

### 3 Proof of the moderate deviations principles

#### 3.1 The method of cumulants

Given  $f \in \mathcal{C}([0, 1]^d)$ , consider the logarithmic Laplace transform of  $\alpha_\lambda \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle$  defined by

$$\Lambda_{\lambda\kappa, \alpha_\lambda}^\xi(f) := \frac{1}{\alpha_\lambda^2} \log \left[ \mathbb{E} \exp(\alpha_\lambda \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle) \right]. \quad (3.1)$$

To prove Theorem 1.2 it will suffice to establish the following result.

**Proposition 3.1** *The logarithmic Laplace transform  $\Lambda_{\lambda\kappa, \alpha_\lambda}^\xi(\cdot)$  satisfies*

$$\Lambda_\kappa^\xi(f) := \lim_{\lambda \rightarrow \infty} \Lambda_{\lambda\kappa, \alpha_\lambda}^\xi(f) = \frac{1}{2} \int_{[0, 1]^d} f^2(x) V^\xi(\kappa(x)) \kappa(x) dx.$$

*Proof of Theorem 1.2.* Indeed the proof of Theorem 1.2 now follows from standard arguments.

Choose  $f \in \mathcal{C}([0, 1]^d)$  and use Proposition 3.1 to get for all  $s \in \mathbb{R}$

$$\Lambda_\kappa^\xi(sf) := \lim_{\lambda \rightarrow \infty} \Lambda_{\lambda\kappa, \alpha_\lambda}^\xi(sf) = \frac{s^2}{2} \int_{[0, 1]^d} f^2(x) V^\xi(\kappa(x)) \kappa(x) dx. \quad (3.2)$$

In particular,  $\Lambda_\kappa^\xi(sf)$  is finite for all  $s \in \mathbb{R}$  and, moreover, it is everywhere differentiable. Therefore, by the standard Gärtner-Ellis result (cf. Theorem 2.3.6 in Dembo and Zeitouni [15]), the family  $(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle)_\lambda$  satisfies on  $\mathbb{R}$  the full moderate deviation principle with speed  $\alpha_\lambda^2$  and good rate function (Fenchel Legendre transform of  $\Lambda^\xi$ )

$$K_f^\xi(t) := \sup_{s \in \mathbb{R}} (ts - \Lambda_\kappa^\xi(sf)) = \frac{t^2}{2} \left( \int_{[0, 1]^d} f^2(x) V^\xi(\kappa(x)) \kappa(x) dx \right)^{-1}$$

as in (1.6). This yields Theorem 1.2.  $\square$

It remains therefore to prove Proposition 3.1. We will establish Proposition 3.1 by refining the method of cumulants and cluster measures as developed in [5] in the context of the central limit theorem. We recall the formal definition of cumulants in the context specified for our purposes. Let  $W := [0, 1]^d$  and take  $f \in \mathcal{C}(W)$ . Expand  $\mathbb{E} \exp\left(\alpha_\lambda \lambda^{-1/2} \langle -f, \bar{\mu}_{\lambda\kappa}^\xi \rangle\right)$  in a power series in  $f$  as follows:

$$\mathbb{E} \exp\left(\alpha_\lambda \lambda^{-1/2} \langle -f, \bar{\mu}_{\lambda\kappa}^\xi \rangle\right) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha_\lambda \lambda^{-1/2})^k \langle (-f)^k, M_\lambda^k \rangle}{k!}, \quad (3.3)$$

where  $f^k : \mathbb{R}^{dk} \rightarrow \mathbb{R}$ ,  $k = 1, 2, \dots$  is given by  $f^k(v_1, \dots, v_k) = f(v_1) \cdots f(v_k)$ , and  $v_i \in W, 1 \leq i \leq k$ .  $M_\lambda^k := M_\lambda^k(\kappa)$  is a measure on  $\mathbb{R}^{dk}$ , the  $k$ -th moment measure (p. 130 of [14]). Both the existence of the moment measures and the convergence of the series (3.3) are direct consequences of the boundedness of  $\xi$ .

We have

$$dM_\lambda^k = m_\lambda(v_1, \dots, v_k) \cdot \prod_{i=1}^k \kappa(v_i) d(\lambda v_i), \quad (3.4)$$

where the Radon-Nikodym derivative  $m_\lambda(v_1, \dots, v_k)$  is given by

$$m_\lambda(v_1, \dots, v_k) := \mathbb{E} \left[ \prod_{i=1}^k \bar{\xi}_\lambda(v_i; \mathcal{P}_{\lambda\kappa}) \right], \quad (3.5)$$

where for all  $i = 1, \dots, k$ , we abbreviate notation and write  $\bar{\xi}_\lambda(v_i; \mathcal{P}_{\lambda\kappa})$  for  $\xi_\lambda(v_i; \mathcal{P}_{\lambda\kappa}) - \mathbb{E} \xi_\lambda(v_i; \mathcal{P}_{\lambda\kappa})$  and  $\xi(v_i; \mathcal{P}_{\lambda\kappa})$  for  $\xi(v_i; \mathcal{P}_{\lambda\kappa} \cup \{v_j\}_{j=1}^k)$ . For each fixed  $k$ , the mixed moment on the right hand side of (3.5) is bounded in absolute value by  $C_\xi^k$  uniformly in  $\lambda$  since  $|\xi| \leq C_\xi$ .

Expanding the logarithm of the Laplace transform in a formal power series gives

$$\log \left[ 1 + \sum_{k=1}^{\infty} \frac{(\alpha_\lambda \lambda^{-1/2})^k \langle (-f)^k, M_\lambda^k \rangle}{k!} \right] = \sum_{l=1}^{\infty} \frac{(\alpha_\lambda \lambda^{-1/2})^l \langle (-f)^l, c_\lambda^l \rangle}{l!}; \quad (3.6)$$

the signed measures  $c_\lambda^l$  are *cumulant measures* [21]. To prove Proposition 3.1 it will be enough to establish growth bounds on  $\langle (-f)^l, c_\lambda^l \rangle$  in terms of both the scale parameter  $\lambda$  and the order  $l$ .

Note that we do not require the convergence of the formal series in (3.6) at this point, even though its convergence for large  $\lambda$  will eventually follow from our argument below. On the other hand, the existence of all cumulants  $c_\lambda^l$ ,  $l = 1, 2, \dots$  follows from the existence of all moments, in view of the representation

$$c_\lambda^l = \sum_{T_1, \dots, T_p} (-1)^{p-1} (p-1)! M_\lambda^{T_1} \cdots M_\lambda^{T_p}$$

with  $M_\lambda^{T_i}$  standing for a copy of the moment measure  $M_\lambda^{|T_i|}$  on the product space  $W^{T_i}$  (see below for formal details) and where  $T_1, \dots, T_p$  ranges over all unordered partitions of the set  $1, \dots, k$  (see p. 30 of [21]). The first cumulant measure coincides with the expectation measure and the second cumulant measure coincides with the covariance measure.

We will sometimes shorten notation and write  $M^k, m$  and  $c^l$  instead of  $M_\lambda^k, m_\lambda$  and  $c_\lambda^l$ .

### 3.2 Cluster measures and the proof of Proposition 3.1

Recall  $W = [0, 1]^d$  and for all  $i = 1, 2, \dots$  we let  $W_i$  denote the  $i$ th copy of  $W$ . For any subset  $T$  of the positive integers, we let

$$W^T := \prod_{i \in T} W_i.$$

If  $|T| = l$ , then by  $M^T$  we mean a copy of the  $l$ th moment measure on the  $l$ -fold product space  $W^T$ .  $M^T$  is equal to  $M^l$  as defined by (3.4).

A *cluster measure*  $U^{S,T}$  on  $W^S \times W^T$  for non-empty disjoint  $S, T \subseteq \{1, 2, \dots\}$ , is defined by

$$U^{S,T}(A \times B) = M^{S \cup T}(A \times B) - M^S(A)M^T(B)$$

for all Borel sets  $A$  and  $B$  in  $W^S$  and  $W^T$ , respectively.

For  $T$  a subset of  $\{1, 2, \dots\}$ , by  $c_\lambda^T$  we mean the cumulant

$$c_\lambda^T = \sum (-1)^{p-1} (p-1)! M_\lambda^{T_1} \cdots M_\lambda^{T_p}$$

where  $T_1, \dots, T_p$  ranges over all unordered partitions of  $T$ . When  $|T| = l$ ,  $c_\lambda^T$  is just a copy of  $c_\lambda^l$ . We use the notation  $c_\lambda^l$  and  $c_\lambda^{\{1, \dots, l\}}$  interchangeably.

Given a non-trivial partition  $(S, T)$  of  $\{1, 2, \dots, k\}$ ,  $k \geq 1$ , referred to as the *initial partition* in the sequel, a cluster  $U^{S_i, T_i}$  is called compatible with the initial partition if  $S_i \subset S$  and  $T_i \subset T$ . We claim that one can represent the  $k$ th cumulant  $c^k$ ,  $k = 1, 2, \dots$  as a linear combination of products of cluster measures, where each product involves at least one cluster compatible with  $(S, T)$ , i.e.,

$$c^k = \sum_{(S_1, T_1), \dots, (S_l, T_l)} a((S_1, T_1) \cdots (S_l, T_l)) U^{S_1, T_1} U^{S_2, T_2} \cdots U^{S_l, T_l}, \quad (3.7)$$

where the sum ranges over all partitions  $\pi$  of  $\{1, 2, \dots, k\}$  consisting of pairings  $(S_1, T_1), \dots, (S_l, T_l)$  with non-empty  $S_i$  and  $T_i$ , with at least one  $i \in \{1, 2, \dots, k\}$  such that  $U^{S_i, T_i}$  is compatible with  $(S, T)$ , where the prefactors  $a((S_1, T_1), \dots, (S_l, T_l))$  are integer numbers. Such a representation of  $c^k$  is called *compatible with the initial partition*  $(S, T)$ , and such partitions  $\pi$  showing up in this representation are called *compatible with*  $(S, T)$ . We may alternatively represent  $c^k$  in the form

$$\sum \hat{a}(\dots) U^{S_1, T_1} U^{S_2, T_2} \cdots U^{S_l, T_l}, \quad (3.8)$$

where  $\hat{a}(\dots) \in \{+1, -1\}$  and where we allow repetitions over the partitions  $\pi$ . Both representations (3.7) and (3.8) depend on the choice of the initial partition with which they are required to be compatible. However, we drop this dependence in our notation for the sake of readability.

We seek to estimate the sum

$$\sum_{(S_1, T_1), \dots, (S_l, T_l)} |a((S_1, T_1) \cdots (S_l, T_l))|$$

uniformly in the choice of initial partition  $(S, T)$ , in order to upper bound, uniformly in  $(S, T)$ , the number of terms in the finite sum (3.8). By refining the approach in section five of [5], we may prove the following two key lemmas; the proof of the first is deferred to section 3.3.

**Lemma 3.1** *Let  $(S, T)$  be a non-trivial (initial) partition of  $\{1, 2, \dots, k\}$ . The cumulant  $c^k$  admits the representation (3.7) and we have, uniformly in the choice of  $(S, T)$ ,*

$$\sum_{(S_1, T_1), \dots, (S_l, T_l)} |a((S_1, T_1), \dots, (S_l, T_l))| \leq k 3^{5k} k^k.$$

Let  $\Delta_k$  denote the diagonal in  $W^k$  and for all  $v := (v_1, \dots, v_k) \in W^k$ , let  $D_k(v)$  denote the distance to the diagonal.  $\Pi(k)$  denotes all initial partitions of  $\{1, 2, \dots, k\}$  into exactly two subsets  $S$  and  $T$ . For all such partitions consider the subset  $\sigma(S, T)$  of  $[0, 1]^S \times [0, 1]^T$  having the property that  $v \in \sigma(S, T)$  implies  $d(x(v), y(v)) \geq D_k(v)/k$ , where  $x(v)$  and  $y(v)$  denote the projections of  $v$  onto  $W^S$  and  $W^T$ , respectively, and where  $d(x(v), y(v))$  denotes the minimal distance between pairs of points from  $x$  and  $y$ .

Since for every  $v := (v_1, \dots, v_k) \in W^k$ , there is a splitting  $x := x(v)$  and  $y := y(v)$  of  $v$  such that  $d(x, y) \geq D_k(v)/k$ , it follows that  $W^k$  is the union of the sets  $\sigma(S, T)$ ,  $(S, T) \in \Pi(k)$ .

Given the initial partition  $(S, T)$  we evaluate the cumulant measure  $c_\lambda^k$  over the set  $\sigma(S, T)$ ; by the representations (3.7, 3.8) we know that the cumulant can be expressed as a linear combination of products of cluster measures, say  $\nu$ , compatible with the initial partition  $(S, T)$ . Lemma 5.3 of [5] implies there exists a finite constant  $A$  such that for each of the at most  $k 3^{5k} k^k$  cluster products  $\nu$ , all  $\lambda \geq 1$ , and all  $f \in \mathcal{C}(W)$ ,

$$\lambda^{-k/2} \int_{\sigma(S, T)} f^k d\nu \leq A 2^k C_\xi^k \|f\|_\infty^k \lambda^{(2-k)/2}.$$

Since there are at most  $2^k$  initial partitions  $(S, T)$ , we obtain the following growth bounds on the cumulants.

**Lemma 3.2** *There is a constant  $A < \infty$  such that for all  $f \in \mathcal{C}(W)$ ,  $k = 2, 3, \dots$  and all  $\lambda \geq 1$*

$$\lambda^{-k/2} \langle f^k, c_\lambda^k \rangle \leq A C_\xi^k \|f\|_\infty^k k 3^{5k} k^k 2^k \lambda^{(2-k)/2}.$$

We now deduce the proof of Proposition 3.1 as follows.

*Proof of Proposition 3.1.* We consider the terms in the power series on the right hand side of (3.6). The first cumulant coincides with the expectation measure and the second cumulant coincides with the covariance measure. Thus  $\langle f, c_\lambda^1 \rangle = 0$  for all  $f \in \mathcal{C}(W)$ . From section four of [5] we know that

$$\lambda^{-1} \langle f^2, c_\lambda^2 \rangle = \lambda^{-1} \text{Var}[\langle f, \bar{\mu}_\lambda^\xi \rangle] \rightarrow \int_{[0,1]^d} f^2(x) V^\xi(\kappa(x)) dx.$$

Thus the second term on the right hand side of (3.6) is:

$$\lim_{\lambda \rightarrow \infty} \frac{1}{2} \alpha_\lambda^{-2} (\alpha_\lambda \lambda^{-1/2})^2 \langle f^2, c_\lambda^2 \rangle = \frac{1}{2} \int_{[0,1]^d} f^2(x) V^\xi(\kappa(x)) dx.$$

Thus, to prove Proposition 3.1, we will need to show for all  $f \in \mathcal{C}(W)$  that the sum of the higher order terms in (3.6) goes to zero in the large  $\lambda$  limit, i.e., we need to show

$$\alpha_\lambda^{-2} \sum_{l \geq 3} \frac{1}{l!} (\alpha_\lambda \lambda^{-1/2})^l \langle f^l, c_\lambda^l \rangle \rightarrow 0 \quad (3.9)$$

as  $\lambda \rightarrow \infty$ .

Using Lemma 3.2 we now show (3.9). Indeed, by Lemma 3.2 it follows that

$$\begin{aligned} & (\alpha_\lambda)^{-2} \sum_{l=3}^{\infty} \frac{1}{l!} (\alpha_\lambda \lambda^{-1/2})^l \langle f^l, c_\lambda^l \rangle \\ & \leq (\alpha_\lambda)^{-2} \sum_{l=3}^{\infty} \frac{1}{l!} (\alpha_\lambda)^l A 2^l C_\xi^l \|f\|_\infty^l \cdot l \cdot 3^{5l} l! \lambda^{(2-l)/2} \\ & = A \sum_{l=3}^{\infty} \frac{1}{l!} 2^l C_\xi^l \|f\|_\infty^l \cdot l \cdot 3^{5l} l! (\alpha_\lambda \lambda^{-1/2})^{l-2}. \end{aligned}$$

Clearly, since  $\|f\|_\infty$  is finite and since  $l!/l!$  is bounded exponentially in  $l$ , there is  $M < \infty$  independent of  $l$  such that the above is bounded by

$$\leq A \sum_{l=3}^{\infty} M^l (\alpha_\lambda \lambda^{-1/2})^{l-2} = AM^2 \sum_{l=3}^{\infty} M^{l-2} (\alpha_\lambda \lambda^{-1/2})^{l-2}.$$

By our choice of  $\alpha_\lambda$ , for all  $\varepsilon > 0$  there is a  $\lambda(\varepsilon)$  such that for  $\lambda \geq \lambda(\varepsilon)$ ,  $\alpha_\lambda \lambda^{-1/2} \leq \varepsilon/M$ , showing that for  $\lambda \geq \lambda(\varepsilon)$  and with  $\varepsilon$  small enough, we have

$$AM^2 \sum_{l=3}^{\infty} M^{l-2} (\alpha_\lambda \lambda^{-1/2})^{l-2} \leq AM^2 \sum_{l=3}^{\infty} M^{l-2} (\varepsilon/M)^{l-2} \leq 2AM^2 \varepsilon.$$

Since  $\varepsilon$  is arbitrary, (3.9) follows as desired, proving Proposition 3.1.  $\square$

### 3.3 Trees and the proof of Lemma 3.1

*Proof of Lemma 3.1.* Our goal is the representation (3.8). We proceed in several steps. Given a subset  $T$  of the positive integers, recall that the moment measures  $M^T$  are expressed in terms of the cumulants via

$$M^T = \sum_{T_1, \dots, T_p} c^{T_1} \dots c^{T_p}, \quad (3.10)$$

where the sum is over all partitions of  $T$ , that is unordered collections  $T_1, T_2, \dots, T_p$  of mutually disjoint subsets of  $T$  whose union is  $T$  (p. 27 of [21]). Similarly,

$$M^{S \cup T} = \sum_{(S_1, T_1) \dots (S_p, T_p)} c^{S_1 \cup T_1} \dots c^{S_p \cup T_p}, \quad (3.11)$$

where the sum is over all partitions of  $S \cup T$ ,  $S_i$  denotes a subset of  $S$ , and  $T_i$  denotes a subset of  $T$ . A typical element  $(S_i, T_i)$  of a partition thus involves a pair of sets, one a subset of  $S$  and the other a subset of  $T$ . Some partitions of  $S \cup T$  are such that the empty set appears in each pair  $(S_i, T_i)$ ,  $1 \leq i \leq p$ . Call these the *degenerate partitions*.

We prove Lemma 3.1 by decomposing the  $k$ th cumulant  $c^k$  in steps. Choose an arbitrary initial partition  $(S, T)$  of  $\{1, 2, \dots, k\}$  and split (3.11) as

$$M^{S \cup T} = \sum_{\{\dots\}} c^{\{\dots\}} \dots c^{\{\dots\}} + \sum_{\{\dots\}^*} c^{\{\dots\}} \dots c^{\{\dots\}}, \quad (3.12)$$

where  $\{\dots\}^*$  denotes degenerate partitions whereas  $\{\dots\}$  stands for non-degenerate ones.

The first sum contains the  $k$ th cumulant  $c^k = c^{S \cup T}$  as well as products of *lower order cumulants* (by lower order cumulants we mean a cumulant of the form  $c^{S_i \cup T_i}$ , where  $S_i \cup T_i$  is a proper subset of  $\{1, \dots, k\}$ ). In view of (3.10), the second sum is just the product of  $M^S$  and  $M^T$ .

Thus the  $k$ th cumulant  $c^k$  is  $c^k = M^{S \cup T} - M^S M^T + l.c.$ , where l.c. denotes a linear combination of products of lower order cumulants, with integer pre-factors.

Summarizing and recalling the definition of  $U^{S, T}$ , the first step of our decomposition yields

$$c^k = c^{S \cup T} = U^{S, T} + \sum_{(S_1, T_1) \dots (S_p, T_p)} c^{S_1 \cup T_1} \dots c^{S_p \cup T_p}, \quad (3.13)$$

with the sum ranging over non-degenerate partitions and where  $S_i \cup T_i$  is a proper subset of  $\{1, \dots, k\}$  for all  $i = 1, 2, \dots, p$ . This representation holds for all non-trivial choices of the initial partition  $(S, T)$  of  $\{1, 2, \dots, k\}$ .

Next, on the second step, we represent each cumulant  $c^{S_i \cup T_i}$  appearing in (3.13) with both  $S_i$  and  $T_i$  non-empty as

$$c^{S_i \cup T_i} = U^{S_i, T_i} + \sum_{(S'_1, T'_1) \dots (S'_{p'}, T'_{p'})} c^{S'_1 \cup T'_1} \dots c^{S'_{p'} \cup T'_{p'}}, \quad (3.14)$$

where  $S'_j \cup T'_j$ ,  $1 \leq j \leq p'$ , is a proper subset of  $S_i \cup T_i$  and  $S'_j \subseteq S_i$ ,  $T'_j \subseteq T_i$ . Note that  $U^{S_i, T_i}$  is compatible with the initial partition  $(S, T)$ . Due to non-degeneracy of the partitions in (3.13) there always exists at least one  $i \in \{1, 2, \dots, p\}$  with  $c^{S_i \cup T_i}$  admitting the representation (3.14). Observe also that none of the products showing up in the sum on the right hand side of this representation is degenerate so that each of these contains at least one factor  $c^{S'_j \cup T'_j}$  with both  $S'_j$  and  $T'_j$  nonempty. In the particular situation where both  $T_i$  and  $S_i$  are singletons, the variance cumulant  $c^{S_i \cup T_i}$  coincides with  $U^{S_i, T_i}$  and the considered sum of products is empty.

For the remaining cumulants, namely for those for which either  $S_i$  or  $T_i$  is empty, we either have  $|T_i \cup S_i| = 1$ , in which case the corresponding cumulant reduces to 0, or we split  $S_i \cup T_i$  in an arbitrary way into a union  $S_i^+ \cup T_i^+$  of two non-empty sets and we expand it as in (3.14) with  $S_i, T_i$  replaced by  $S_i^+$  and  $T_i^+$  respectively. This yields the expansion

$$c^{S \cup T} = U^{S, T} + \left( U^{S_1^{(+)}, T_1^{(+)}} + \sum c^{S'_1 \cup T'_1} \dots c^{S'_{p'} \cup T'_{p'}} \right) \left( U^{S_2^{(+)}, T_2^{(+)}} + \sum c^{S''_1 \cup T''_1} \dots c^{S''_{p'} \cup T''_{p'}} \right) \dots \quad (3.15)$$

in which for at least one  $i \in \{1, \dots, p\}$  the measure  $U^{S_i, T_i}$  shows up, as in (3.14), with  $\emptyset \neq S_i^{(+)} = S_i \subseteq S$ ,  $\emptyset \neq T_i^{(+)} = T_i \subseteq T$  (here  $S_i^{(+)}$  denotes either  $S_i$  or  $S_i^+$ , likewise for  $T_i^{(+)}$ ). On the third step we similarly decompose each of the cumulants  $c^{S'_1 \cup T'_1}, \dots, c^{S'_{p'} \cup T'_{p'}}$ ,  $c^{S''_1 \cup T''_1}, \dots, c^{S''_{p'} \cup T''_{p'}}$ . We repeat this process, using as many steps as necessary to represent  $c^k$  as a sum of (terminal) products of cluster measures. The compatibility of these products with the initial partition  $(S, T)$  is implied by the discussion following (3.14). It should be emphasized, as already remarked before, that some of the terminal products showing up in the above expansion procedure involve not only cluster measures but also *singleton cumulants*  $c^{\{i\}}$ , which cannot be further expanded into clusters. However, such cumulants  $c^{\{i\}}$  are identically 0 for all  $i \in \{1, \dots, k\}$  and we will not keep track of them in either (3.15) or in the sequel.

It is convenient to represent each individual terminal product  $\Pi = U^{\hat{S}_1, \hat{T}_1} \dots U^{\hat{S}_m, \hat{T}_m}$  of cluster measures in the expansion (3.15) by a rooted tree  $\mathbb{T}_\Pi$ , keeping track of the *expansion history* of this product, i.e. the information about its factors and also about whence different factors come at subsequent stages of our expansion procedure. The advantage of *labelling* terminal products of cluster measures with their *expansion history* trees is that even though a given terminal product

may show up more than once in the sum (3.8), each *expansion history* can appear there only once. To put these ideas in formal terms, by an expansion history tree for  $c^k$ ,  $k \geq 1$ , we shall mean a finite rooted tree  $\mathbb{T}$ , with nodes labelled by nonempty subsets of  $\{1, 2, \dots, k\}$  and enjoying the following properties

- (T1) The root of  $\mathbb{T}$  is labelled with  $\{1, 2, \dots, k\}$ ,
- (T2) For any given node with label  $T \subseteq \{1, 2, \dots, k\}$ , the labels of the (ordered) offspring of the given node form a non-trivial (ordered) partition of  $T$ ,
- (T3) Each leaf (terminal node) has exactly one sibling (a node sharing common ancestor) and this sibling is also a leaf.

Observe that (T2) excludes nodes having only one offspring - a node is either terminal (a leaf) or it must have at least two outgoing branches. Another important observation is that the labels of nodes of any given generation (counting from the root) partition the index set  $\{1, 2, \dots, k\}$ . It should be also emphasized that we consider the offspring of any node in our tree *ordered*. Consider a tree  $\mathbb{T}$  satisfying the conditions (T1-3) and such that its *crop*, here *understood as the (ordered) collection of the pairs of sibling leaves*, is  $((L_1, R_1), (L_2, R_2), \dots, (L_m, R_m))$ . Then we say that  $\mathbb{T}$  describes an expansion history for the product of cluster measures  $U^{L_1, R_1} U^{L_2, R_2} \dots U^{L_m, R_m}$ . The remaining non-terminal branchings correspond to splittings performed at non-terminal stages of the expansion procedure.

Note that the conditions (T1-3) do not include the rather restrictive constraints due to the compatibility and non-degeneracy requirements (with respect to the initial partition) satisfied by the partitions showing up in our expansions (see (3.12) and below) and, moreover, the offspring of the tree nodes is considered ordered, while the partitions in (3.10) and (3.11) are unordered. All this means that the sum

$$\sum_{(S_1, T_1), \dots, (S_l, T_l)} |a((S_1, T_1), \dots, (S_l, T_l))|$$

in the expansion (3.11) for the  $k$ th cumulant  $c^k$ , is bounded above, uniformly in the initial partition  $(S, T)$ , by the total number of expansion history trees for  $c^k$ . Moreover, the above discussion suggests that this bound involves considerable over-counting and it could be possibly further improved, but this would just amount to gaining a factor exponential in  $k$ , which would be of no use for our further purposes.

To complete the proof of Lemma 3.1, we note first that the information stored in the non-terminal nodes of an expansion history tree is redundant as it can be easily recovered from the

labels attached to the leaves, by recursively labelling non-terminal nodes with the unions of labels held by their respective offspring. Therefore, below we consider trees with labels at the leaves only, and with non-terminal nodes unlabelled. A tree with a fixed number, say  $m$ , of leaves can be uniquely represented as a pair consisting of

- A *tree architecture*, which is a rooted tree with its leaves holding *label placeholders* numbered from 1 to  $m$ , rather than actual labels,
- A sequence of  $m$  data (more precisely, an ordered partition of  $\{1, 2, \dots, k\}$  with  $m$  groupings), to be matched against the above label placeholders.

**Lemma 3.3** *All tree architectures with  $m$  leaves can be described using sequences involving  $\circ, \uparrow$ , or  $\downarrow$  of length at most  $5m$ .*

*Proof.* Each tree architecture can be coded by a sequence of  $\circ, \uparrow$  and  $\downarrow$  as follows. Starting from the root, we ‘explore’ the tree, putting a  $\downarrow$  to go down the next branch, a  $\circ$  to mark the presence of a placeholder, and an  $\uparrow$  to go up to the direct ancestor of the current node. Thus each tree is encoded as a sequence of  $\circ, \uparrow$  and  $\downarrow$  but not every code sequence corresponds to a valid tree. The  $m$  leaves or terminal nodes each require a pair  $\uparrow, \downarrow$  and a  $\circ$  to describe them. In this way we see that terminal nodes require at most  $2m + m$  sequence entries. Since each terminal node has a parent which gives rise to at least two terminal nodes, the number of such parent nodes is bounded by  $m/2$  and so we require at most  $m$  sequence entries to describe the parents (the label placeholders are no more there, as they are attached to the leaves only). Similarly we need at most  $m/2$  sequence entries to describe the grandparents. Since  $(2m + m) + m + m/2 + \dots = 5m$ , Lemma 3.3 follows.  $\square$

Lemma 3.3 tells us that the number of tree architectures with  $m$  leaves is bounded by  $3^{5m}$ . The total number of trees with  $m$  leaves is bounded by the product of the number of tree architectures with  $m$  leaves and the number of partitions of  $\{1, 2, \dots, k\}$  involving  $m$  blocks. We deduce that the total number of trees with  $m$  leaves is bounded by  $3^{5m}m^k$  and thus the total number of trees is bounded by  $\sum_{m=1}^k 3^{5m}m^k \leq k3^{5k}k^k$ , which is precisely Lemma 3.1, as desired.

### 3.4 Proof of Theorems 1.3 and 1.4

*Proof of Theorem 1.3.* We will apply the theorems of Gärtner and Ellis combined with the theorem of Dawson and Gärtner dealing with large deviations for projective limits, see sections 4.5.3 and

4.6 in [15]. Actually we will apply Corollary 4.6.11, part (a), for the family  $(\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda, \kappa}^\xi)_\lambda$ , taking values in the topological vector space  $\mathcal{M}([0, 1]^d)$ . For each  $f \in \mathcal{C}([0, 1]^d)$  we define  $p_f(\nu) := \langle \nu, f \rangle : \mathcal{C}([0, 1]^d)' \rightarrow \mathbb{R}$ , where  $\mathcal{C}([0, 1]^d)'$  denotes the algebraic dual of  $\mathcal{C}([0, 1]^d)$  and  $\langle \cdot, \cdot \rangle$  denotes the duality brackets between these spaces. Taking the  $\sigma$ -algebra  $\mathcal{B}$  and the maps  $p_f$ ,  $f \in \mathcal{C}([0, 1]^d)$ , we note that  $\mathcal{C}([0, 1]^d)'$  satisfies Assumption 4.6.8 in [15]. Given the logarithmic moment generating function  $\Lambda_{\lambda, \kappa, \alpha_\lambda}^\xi(f)$  as in (3.1), we know by Proposition 3.1 that the limit

$$\Lambda_\kappa^\xi(f) := \lim_{\lambda \rightarrow \infty} \alpha_\lambda^{-2} \Lambda_{\lambda, \kappa, \alpha_\lambda}^\xi(f)$$

exists as an extended real number and moreover

$$\Lambda_\kappa^\xi(f) = \frac{1}{2} \int_{[0, 1]^d} f(x)^2 V^\xi(\kappa(x)) \kappa(x) dx, \quad f \in \mathcal{C}([0, 1]^d).$$

Next we have to show that  $\Lambda_\kappa^\xi(\sum_{i=1}^l t_i f_i) : \mathbb{R}^l \rightarrow (-\infty, +\infty]$  with  $f_1, \dots, f_l \in \mathcal{C}([0, 1]^d)$  and  $t_1, \dots, t_l \in \mathbb{R}$  is, for any  $l \in \mathbb{N}$ , essentially smooth (for a precise definition see [15, Definition 2.3.5]), lower semi-continuous and finite in some neighborhood of 0 (see also Corollary 4.6.14 in [15]). Since  $\Lambda_\kappa^\xi$  is everywhere finite, it suffices to show that for every  $f, g \in \mathcal{C}([0, 1]^d)$ , the function  $\Lambda_\kappa^\xi(f + tg) : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at  $t = 0$ , implying that  $\Lambda_\kappa^\xi(\cdot)$  is Gateaux differentiable. Using the boundedness of  $f$  and  $g$  we may interchange the order of differentiation and integration to obtain

$$\left. \frac{d}{dt} \Lambda_\kappa^\xi(f + tg) \right|_{t=0} = \int_{[0, 1]^d} f(x) g(x) V^\xi(\kappa(x)) \kappa(x) dx,$$

which is well defined. Consequently, all the conditions of part (a) of Corollary 4.6.11 in [15] are satisfied, and hence the family  $(\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda, \kappa}^\xi)_\lambda$  satisfies the LDP in  $\mathcal{C}([0, 1]^d)'$  with respect to the weak topology, with speed  $\alpha_\lambda^2$  and a convex, good rate function

$$\Lambda_{\xi, \kappa}^*(\nu) := \sup_{f \in \mathcal{C}([0, 1]^d)'} \{ \langle f, \nu \rangle - \Lambda_\kappa^\xi(f) \} \text{ for } \nu \in \mathcal{C}([0, 1]^d)'.$$

Now we restrict the LDP. We will show that  $\Lambda_{\xi, \kappa}^*(\nu) < \infty$  implies that  $\nu$  is a continuous linear form. Assume that  $\Lambda_{\xi, \kappa}^*(\nu) < \infty$ . Then for all  $f \in \mathcal{C}([0, 1]^d)$ ,  $f \neq 0$ , we obtain

$$\left\langle \frac{f}{\|f\|}, \nu \right\rangle \leq \Lambda_{\xi, \kappa}^*(\nu) + \Lambda_\kappa^\xi\left(\frac{f}{\|f\|}\right) \leq \Lambda_{\xi, \kappa}^*(\nu) + \Lambda_\kappa^\xi(1).$$

Now by assumption the right hand side is finite and hence  $\langle \nu, f \rangle \leq K \|f\|$ . Considering  $-f$ , we get  $|\langle \nu, f \rangle| \leq K \|f\|$ . Thus  $\nu$  is a continuous linear form. Since  $[0, 1]^d$  is compact, we can apply Riesz's representation theorem which implies that  $\nu$  can be represented as a  $\mathbb{R}$ -valued measure on  $[0, 1]^d$ , e.g.  $\nu \in \mathcal{M}([0, 1]^d)$ . We can apply Lemma 4.1.5 in [15] to obtain the LDP in the space  $\mathcal{M}([0, 1]^d)$ ,

endowed with the weak topology. The representation of the rate function is proved in Lemma 3.4, hence the proof of Theorem 1.3 is complete.  $\square$

**Lemma 3.4** *The identity*

$$\Lambda_{\xi, \kappa}^*(\cdot) = I_{\kappa}^{\xi}(\cdot)$$

holds over  $\mathcal{M}([0, 1]^d)$ . Moreover we obtain

$$\Lambda_{\xi, \kappa}^*(\nu) = \sup_{f \in B([0, 1]^d)} \{ \langle f, \nu \rangle - \Lambda_{\kappa}^{\xi}(f) \}, \quad \nu \in \mathcal{M}([0, 1]^d), \quad (3.16)$$

where  $B([0, 1]^d)$  denotes the collection of bounded measurable functions  $f : [0, 1]^d \rightarrow \mathbb{R}$ .

*Proof of Lemma 3.4.* Let us define the measure  $\mu(\xi, \kappa)$  by  $d\mu(\xi, \kappa) := V^{\xi}(\kappa(x)) \kappa(x) dx$ . For a fixed  $\nu \in \mathcal{M}([0, 1]^d)$  and  $f \in B([0, 1]^d)$  there exists a sequence  $f_n \in \mathcal{C}([0, 1]^d)$ ,  $n \in \mathbb{N}$ , such that  $\lim f_n = f$  both in  $L_1(\mu(\xi, \kappa))$  and in  $L_1(\nu)$  (truncating each  $f_n$  to the bounded range of  $f$ ). Consequently, there exists a sequence  $(f_n)_n \subset \mathcal{C}([0, 1]^d)$  such that

$$\lim_{n \rightarrow \infty} \left( \int_{[0, 1]^d} f_n d\nu - \Lambda_{\kappa}^{\xi}(f_n) \right) = \int_{[0, 1]^d} f d\nu - \Lambda_{\kappa}^{\xi}(f).$$

Since  $\nu \in \mathcal{M}([0, 1]^d)$  and  $f \in B([0, 1]^d)$  are arbitrary, the definition of  $\Lambda_{\xi, \kappa}^*$  agrees with (3.16) over  $\mathcal{M}([0, 1]^d)$ .

To proceed, let us assume that  $\nu \in \mathcal{M}([0, 1]^d)$  is chosen such that  $I_{\kappa}^{\xi}(\nu) < \infty$ . This is true by definition only for those  $\nu$  with a density  $\varrho$  with respect to the measure  $\mu(\xi, \kappa)$ . Hence  $I_{\kappa}^{\xi}(\nu) = \frac{1}{2} \int \varrho^2 d\mu(\xi, \kappa)$ . Since for every  $f \in \mathcal{C}([0, 1]^d)$

$$f \varrho \leq \frac{1}{2} f^2 + \frac{1}{2} \varrho^2,$$

we obtain

$$\int f d\nu - \frac{1}{2} \int f^2 d\mu(\xi, \kappa) = \int f \varrho d\mu(\xi, \kappa) - \frac{1}{2} \int f^2 d\mu(\xi, \kappa) \leq \frac{1}{2} \int \varrho^2 d\mu(\xi, \kappa) = I_{\kappa}^{\xi}(\nu).$$

and therefore  $\Lambda_{\xi, \kappa}^*(\cdot) \leq I_{\kappa}^{\xi}(\cdot)$  over the whole  $\mathcal{M}([0, 1]^d)$ . To get the converse inequality take  $\nu \in \mathcal{M}([0, 1]^d)$  such that

$$d\nu := f d\mu(\xi, \kappa), \quad f \in L_1(\mu(\xi, \kappa))$$

and define a sequence  $f_n \in B([0, 1]^d)$ ,  $n \in \mathbb{N}$ , by  $f_n(x) := \text{sign}(f(x)) \min(|f(x)|, n)$ . Then we

simply obtain the identities

$$\begin{aligned}
I_{\kappa}^{\xi}(\nu) &= \int_{[0,1]^d} \left( \frac{d\nu}{d\mu(\xi, \kappa)} \right)^2 d\mu(\xi, \kappa) - \frac{1}{2} \int_{[0,1]^d} \left( \frac{d\nu}{d\mu(\xi, \kappa)} \right)^2 d\mu(\xi, \kappa) \\
&= \int_{[0,1]^d} \frac{d\nu}{d\mu(\xi, \kappa)} d\nu - \frac{1}{2} \int_{[0,1]^d} \left( \frac{d\nu}{d\mu(\xi, \kappa)} \right)^2 d\mu(\xi, \kappa) \\
&= \int_{[0,1]^d} f d\nu - \frac{1}{2} \int_{[0,1]^d} f^2 d\mu(\xi, \kappa) \\
&= \lim_{n \rightarrow \infty} \int_{[0,1]^d} f_n d\nu - \frac{1}{2} \int_{[0,1]^d} f_n^2 d\mu(\xi, \kappa) \leq \Lambda_{\xi, \kappa}^*(\nu).
\end{aligned}$$

To complete the proof of Lemma 3.4 it remains to show that  $\Lambda_{\xi, \kappa}^*(\nu) = +\infty$  for  $\nu$  which are not absolutely continuous with respect to  $\mu(\xi, \kappa)$ . But this follows immediately from the fact that for such  $\nu$  we can find  $f \in B([0,1]^d)$  with  $\langle f, \nu \rangle$  arbitrarily large and  $\Lambda_{\kappa}^{\xi}(f) = \frac{1}{2} \int_{[0,1]^d} f^2 d\mu(\xi, \kappa)$  arbitrarily small. This finishes the proof of Lemma 3.4.  $\square$

*Proof of Theorem 1.4.* Denote by  $p_{f_1, \dots, f_l} : \mathcal{C}([0,1]^d)' \rightarrow \mathbb{R}^l$  the function

$$p_{f_1, \dots, f_l}(\nu) := (\langle f_1, \nu \rangle, \dots, \langle f_l, \nu \rangle).$$

Note that the limiting logarithmic moment generating function associated with the family

$$(\alpha_{\lambda}^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda \kappa}^{\xi} \circ p_{f_1, \dots, f_l}^{-1})_{\lambda}$$

is the function

$$h(t) := \Lambda_{\kappa}^{\xi} \left( \sum_{i=1}^l t_i f_i \right) : \mathbb{R}^l \rightarrow (-\infty, +\infty], \quad t = (t_1, \dots, t_l).$$

But we have checked in the proof of Theorem 1.3, that  $h$  is essentially smooth, lower semi-continuous and finite in some neighborhood of the origin. Hence the Gärtner-Ellis theorem, [15, Theorem 2.3.6], implies that these measures satisfy a LDP in  $\mathbb{R}^l$  with speed  $\alpha_{\lambda}^2$  and with the good rate function  $I_{f_1, \dots, f_l} : \mathbb{R}^l \rightarrow [0, \infty]$ , where for any  $\nu \in \mathcal{C}_b([0,1]^d)'$ , putting  $s_i := \langle f_i, \nu \rangle$  one gets

$$I_{f_1, \dots, f_l}((s_1, \dots, s_l)) = \sup_{t_1, \dots, t_l \in \mathbb{R}} \left( \sum_{i=1}^l t_i s_i - \Lambda_{\kappa}^{\xi} \left( \sum_{i=1}^l t_i f_i \right) \right). \quad (3.17)$$

It easy to see that

$$\Lambda_{\kappa}^{\xi} \left( \sum_{i=1}^l t_i f_i \right) = \Lambda_{\kappa}^{\xi}(\langle t, (f_1, \dots, f_l) \rangle) = \langle t, C(\xi, \kappa, f_1, \dots, f_l) t \rangle.$$

It follows that  $I_{f_1, \dots, f_l}$  in (3.17) coincides with  $I_{\kappa, f_1, \dots, f_l}^{\xi}$  in (1.8). This completes the proof.  $\square$

## 4 Proof of the LIL (Poisson case, Theorem 1.5)

The first step of the proof of Theorem 1.5 makes standard use of Theorem 1.3 in order to show that  $(\zeta_{\kappa\lambda}^\xi)_\lambda$  is a.s. compact and the set of its accumulation points is a.s. contained in  $\mathcal{K}_\kappa^\xi$ . Next, in the second part of the proof we use the rapid decay of dependencies, as concluded from the exponential stabilization in Lemma 4.2 below, to construct an appropriate coupling so that a.s. each point of  $\mathcal{K}_\kappa^\xi$  is attained as an accumulation point of  $(\zeta_{\lambda\kappa}^\xi)_\lambda$ .

### 4.1 Accumulation points

In view of the compactness of  $\mathcal{K}_\kappa^\xi$  (recall that  $I_\kappa^\xi$  is a good rate function) the relative compactness of  $(\zeta_{\lambda\kappa}^\xi)_\lambda$  and the fact that a.s. all accumulation points of  $(\zeta_{\lambda\kappa}^\xi)_\lambda$  fall into  $\mathcal{K}_\kappa^\xi$  will follow once we show that a.s.

$$\limsup_{\lambda \rightarrow \infty} \text{dist}_W(\zeta_{\lambda\kappa}^\xi, \mathcal{K}_\kappa^\xi) = 0. \quad (4.1)$$

We use the following lemma, which is a straightforward modification of Lemma 1.4.3 in Deuschel and Stroock [16].

**Lemma 4.1** *To establish (4.1) it is enough to show that for each  $s > 1$  we have almost surely*

$$\limsup_{k \rightarrow \infty} \text{dist}_W(\zeta_{s^k\kappa}^\xi, \mathcal{K}_\kappa^\xi) = 0. \quad (4.2)$$

To proceed with the proof of (4.1), fix  $s > 1$  and choose arbitrary  $\eta > 0$ . Clearly,

$$\inf\{I_\kappa^\xi(\theta) \mid \text{dist}_W(\theta, \mathcal{K}_\kappa^\xi) \geq \eta\} > 1.$$

In particular, in view of the moderate deviation principle upper bound in Theorem 1.3 we have for  $k$  large enough,

$$P \left[ \text{dist}_W(\zeta_{s^k\kappa}^\xi, \mathcal{K}_\kappa^\xi) \geq \eta \right] \leq \exp(-\alpha_{s^k}^2(1 + \delta)) = (k \log s)^{-(1+\delta)} \quad (4.3)$$

with some  $\delta > 0$ . Hence, in view of the Borel-Cantelli lemma, the event

$$\{\text{dist}_W(\zeta_{s^k\kappa}^\xi, \mathcal{K}_\kappa^\xi) \geq \eta\}$$

occurs almost surely at most a finite number of times. Consequently, almost surely each accumulation point  $\theta^*$  of  $(\zeta_{s^k\kappa}^\xi)_{k=1}^\infty$  in  $\mathcal{M}([0, 1]^d)$  has to satisfy  $\text{dist}_W(\theta^*, \mathcal{K}_\kappa^\xi) < \eta$ . As  $\eta$  was arbitrary, we conclude (4.2) and hence (4.1).

## 4.2 Conclusion of the proof of the LIL

The following lemma, stating rapid decay of dependencies (exponential  $\phi$ -mixing in fact) between  $\mu_{\lambda\kappa}^\xi(A)$  and  $\mu_{\lambda\kappa}^\xi(B)$  for separated  $A, B \subseteq [0, 1]^d$  will be an important tool in the sequel of our argument.

**Lemma 4.2** *Let  $A, B \subseteq [0, 1]^d$  be measurable sets satisfying  $\text{dist}(A, B) > r$  and denote by  $\mathcal{F}_A^\lambda$  and  $\mathcal{F}_B^\lambda$  the sigma fields generated by the restrictions of the random measure  $\mu_{\lambda\kappa}^\xi$  to  $A$  and  $B$  respectively. Then, for any two events  $\mathcal{E}_1 \in \mathcal{F}_A^\lambda$  and  $\mathcal{E}_2 \in \mathcal{F}_B^\lambda$  with  $P[\mathcal{E}_2] > 0$  we have*

$$|P[\mathcal{E}_1|\mathcal{E}_2] - P[\mathcal{E}_1]| \leq \lambda L \int_A \kappa(x) dx \exp(-\alpha\lambda^{1/d}r).$$

*Proof of Lemma 4.2.* It follows by the definition of the re-scaled measure  $\mu_{\lambda\kappa}^\xi$  and by exponential stabilization (1.2) that, almost surely,

$$\begin{aligned} |P[\mathcal{E}_1|\mathcal{F}_B^\lambda] - P[\mathcal{E}_1]| &\leq P\left[\exists_{x \in A \cap \mathcal{P}_{\lambda\kappa}} R_{\lambda;\kappa}^\xi(x) > \lambda^{1/d}r \mid \mathcal{F}_B^\lambda\right] \leq \\ \mathbb{E}\left[\sum_{x \in A \cap \mathcal{P}_{\lambda\kappa}} \mathbf{1}_{\{R_{\lambda;\kappa}^\xi(x) > \lambda^{1/d}r\}} \mid \mathcal{F}_B^\lambda\right] &\leq \lambda L \int_A \kappa(x) dx \exp(-\alpha\lambda^{1/d}r). \end{aligned}$$

Integrating the above inequality over the event  $\mathcal{E}_2$  yields the assertion of the lemma.  $\square$

To proceed, consider the following coupling of the Poisson point processes  $\mathcal{P}_{\lambda\kappa}$  on  $(\Omega, \mathcal{F}, \mathcal{P})$ . Let  $\Pi_1$  be a homogeneous Poisson point process on  $(\mathbb{R}^+)^d \times \mathbb{R}^+$  with intensity 1. Then  $\mathcal{P}_{\lambda\kappa}$  can be identified with the point process

$$\{\lambda^{-1/d}x \mid \exists_{t>0}(x, t) \in \Pi_1, x \in [0, \lambda^{1/d}], t \leq \kappa(\lambda^{-1/d}x)\}. \quad (4.4)$$

It is worth noting that for constant  $\kappa(x) \equiv \kappa$  the above coupling corresponds to observing always the same homogeneous Poisson point process of intensity  $\kappa$  on  $\mathbb{R}_+^d$ , while successively increasing the observation window  $[0, \lambda^{1/d}]^d$  and then re-scaling the observations onto  $[0, 1]^d$ , thus getting copies of  $\mathcal{P}_{\lambda\kappa}$ . Clearly, this is a natural coupling construction appearing in a number of applications.

We shall show that with the coupling (4.4), almost surely each  $\theta \in \mathcal{K}_\kappa^\xi$  is attained as an accumulation point of  $\zeta_{\lambda\kappa}^\xi$  for  $\lambda \rightarrow \infty$ , in other words, almost surely for each  $\theta \in \mathcal{K}_\kappa^\xi$  there exists a sequence  $\lambda_k^{(\theta)} \rightarrow \infty$  with  $\zeta_{\lambda_k^{(\theta)}\kappa}^\xi \rightarrow \theta$ . Clearly, in view of the separability of  $\mathcal{K}_\kappa^\xi$ , it is enough to prove, separately for each  $\theta \in \mathcal{K}_\kappa^\xi$ , that, almost surely,

$$\liminf_{\lambda \rightarrow \infty} \text{dist}_W(\zeta_{\lambda\kappa}^\xi, \theta) = 0. \quad (4.5)$$

To this end, fix  $\theta \in \mathcal{K}_{\lambda_\kappa}^\xi$ , choose arbitrary  $\eta > 0$  and let  $f \in \mathcal{C}([0, 1]^d)$  be a function with its support contained in  $[\delta, 1 - \delta]^d$  for some  $\delta > 0$ . Without loss of generality we assume that  $\delta < 1/2$  and we set

$$\rho := \frac{1 - \delta}{\delta} > 1.$$

We claim that in order to establish (4.5) it is enough to show that, almost surely,

$$\liminf_{k \rightarrow \infty} \left| \langle f, \zeta_{\rho^{kd}\kappa}^\xi \rangle - \langle f, \theta \rangle \right| = 0. \quad (4.6)$$

Indeed, since  $f$  was arbitrary, by a standard diagonal argument (4.6) yields almost sure existence of a (random) subsequence  $k_n$  with

$$\lim_{k_n \rightarrow \infty} \langle f_m, \zeta_{\rho^{k_n}\kappa}^\xi \rangle = \langle f_m, \theta \rangle \quad (4.7)$$

for all  $f_m$  running through a countable collection of continuous functions with compact supports bounded away from  $\partial[0, 1]^d$ , uniformly dense in the set of all continuous functions on  $[0, 1]^d$  assuming the value 0 at the boundary  $\partial[0, 1]^d$ . Since  $(\zeta_{\lambda_\kappa}^\xi)_\lambda$  is almost surely relatively compact in  $\mathcal{M}([0, 1]^d)$ , as argued in Subsection 4.1 above, we conclude using (4.7) that almost surely there exists a random subsequence  $k'_n$  with  $\zeta_{\rho^{k'_n}\kappa}^\xi$  converging weakly to some (random) measure  $\theta'$  satisfying almost surely

$$\langle f_m, \theta \rangle = \langle f_m, \theta' \rangle$$

for all  $f_m$  as above. Recalling that  $I_\kappa^\xi(\theta') \leq 1$  and, consequently,  $\theta'(\partial[0, 1]^d) = 0$ , we conclude that  $\theta' = \theta$  almost surely, which shows that  $\theta$  arises almost surely as an accumulation point of  $(\zeta_{\rho^k\kappa}^\xi)_k$ , as required.

It remains to establish (4.6). To this end, consider the sequence of disjoint cubes

$$Q_k := [\delta\rho^k, (1 - \delta)\rho^k]^d$$

and, fixing some  $\eta > 0$ , denote by  $E_k := E_k^{(f, \eta)}$  the event

$$E_k := \left\{ \left| \langle f, \zeta_{\rho^{kd}\kappa}^\xi \rangle - \langle f, \theta \rangle \right| \geq \eta \right\}.$$

Applying the moderate deviation principle as stated in Theorem 1.3 we see that, for  $k$  large enough,

$$P[E_k^c] = P \left[ \left| \langle f, \zeta_{\rho^{kd}\kappa}^\xi \rangle - \langle f, \theta \rangle \right| < \eta \right] \geq \exp(-\alpha_{\rho^{kd}}^2 [\inf\{I_\kappa^\xi(\gamma) \mid |\langle f, \gamma \rangle - \langle f, \theta \rangle| < \eta\} + \varepsilon]),$$

with  $\varepsilon > 0$  chosen so that

$$\beta := \inf\{I_\kappa^\xi(\gamma) \mid |\langle f, \gamma \rangle - \langle f, \theta \rangle| < \eta\} + \varepsilon < 1$$

(note that such a choice is possible as the rate function  $I_\kappa^\xi$  has the property that in an arbitrarily small open neighborhood of any  $\theta \in \mathcal{K}_\kappa^\xi$  one can always find  $\gamma$  with  $I_\kappa^\xi(s) < I_\kappa^\xi(\theta) \leq 1$ ). Consequently, we obtain

$$P[E_k^c] \geq (kd \log \rho)^{-\beta}. \quad (4.8)$$

To proceed, consider the event

$$A_m := \bigcap_{k \geq m} E_k.$$

It is clear that in order to establish (4.6) it is enough to show that

$$P[A_m] = 0 \quad (4.9)$$

for all  $m > 0$ . Note that for all  $k \geq m$

$$P[A_m] \leq P[E_m]P[E_{m+1}|E_m]P[E_{m+2}|E_{m+1} \cap E_m] \dots P[E_k|E_m \cap \dots \cap E_{k-1}]. \quad (4.10)$$

Taking into account that the distance between  $Q_k$  and  $\bigcup_{j=1}^k Q_j$  is larger than  $\delta \rho^{k-1}$ , which corresponds to  $\lambda^{-1/d} \delta \rho^{k-1}$  with  $\lambda = \rho^{kd}$  under the re-scaling in the definition of  $\mu_{\lambda \kappa}^\xi$ , we can apply Lemma 4.2 and use (4.8) to conclude that

$$\begin{aligned} P[E_k|E_m \cap \dots \cap E_{k-1}] &\leq 1 - (kd \log \rho)^{-\beta} + \lambda L \exp(-\alpha \lambda^{1/d} \lambda^{-1/d} \delta \rho^{k-1}) \int_{[0,1]^d} \kappa(x) dx = \\ &1 - (kd \log \rho)^{-\beta} + \rho^{kd} L \exp(-\alpha \delta \rho^{k-1}) \int_{[0,1]^d} \kappa(x) dx. \end{aligned} \quad (4.11)$$

Consequently, combining (4.11) and (4.10) we obtain

$$P[A_m] \leq \exp \left( - \sum_{k=m}^{\infty} [(kd \log \rho)^{-\beta} - \rho^{kd} L \exp(-\alpha \delta \rho^{k-1}) \int_{[0,1]^d} \kappa(x) dx] \right) = 0.$$

This yields (4.9) and, consequently, also the relations (4.6) and (4.5). The proof of Theorem 1.5 is hence complete.  $\square$

## 5 Proof of the LIL (binomial case, Theorem 1.7)

Put

$$\gamma := \int_{[0,1]^d} f(x) D^\xi(\kappa(x)) \kappa(x) dx.$$

By coupling arguments and the bounded moments condition (see e.g. (6.13) in [5]), there exists a coupling of the binomial measures  $\rho_n^\xi$  and the Poisson measures  $\mu_{n\kappa}^\xi$ ,  $n = 1, 2, \dots$  such that

$$\mathbb{E} \left[ n^{-1/2} \left( \langle f, \mu_{n\kappa}^\xi \rangle - \langle f, \rho_n^\xi \rangle - \gamma \left( \sum_{j=1}^n \eta_j - n \right) \right)^2 \right] \rightarrow 0, \quad (5.1)$$

where  $\eta_1, \eta_2, \dots$  is a sequence of i.i.d. Poisson(1) random variables with  $(\eta_j)_{j \geq 1}$  independent of  $(\rho_n^\xi)_{n \geq 1}$ . In particular, we get from (5.1) that

$$\mathbb{E} \left( \left( \langle f, \zeta_{n\kappa}^\xi \rangle - \langle f, \theta_n^\xi \rangle - \gamma \alpha_n^{-1} n^{-1/2} \left( \sum_{j=1}^n \eta_j - n \right) \right)^2 \right) \rightarrow 0. \quad (5.2)$$

The classical law of the iterated logarithm yields almost surely

$$\limsup_{n \rightarrow \infty} \alpha_n^{-1} n^{-1/2} \left( \sum_{j=1}^n \eta_j - n \right) = \sqrt{2} \quad (5.3)$$

and

$$\liminf_{n \rightarrow \infty} \alpha_n^{-1} n^{-1/2} \left( \sum_{j=1}^n \eta_j - n \right) = -\sqrt{2}. \quad (5.4)$$

Thus, in view of the independence of  $(\rho_n^\xi)_n$  and  $(\eta_j)_j$ , putting (5.2), (5.3) and (5.4) together we conclude that a violation with a positive probability of either of the inequalities (1.15) and (1.16) would lead to a violation of (1.11) or (1.12). This completes the proof of Theorem 1.7.  $\square$

*Acknowledgements.* Cumulant-based methods in a spirit similar to ours have been widely developed in the Russian probability literature, see e.g. [30] and the references therein. However, we have chosen the techniques in the proof of Proposition 3.1 as they are presented in a self-contained way and are directly specialized for the particular multidimensional continuum measure-valued setting considered in this paper, as opposed to traditional formulations of the afore-mentioned results, which would require considerable effort to make them directly applicable to our setting. The third author wishes to express his gratitude to A.V. Nagaev and Z.S. Szewczak for pointing out the above references and for valuable comments. The second and fourth authors greatly appreciate the hospitality of the Institute for Mathematical Sciences at the National University of Singapore, where parts of this work began.

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