LOWER BOUNDS FOR L-FUNCTIONS AT THE EDGE OF THE CRITICAL STRIP

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1. Introduction

In 1899, de la Vallée Poussin extended his method of proving the Prime Number Theorem to showing that the Riemann zeta function has a zero-free region of the form

$$\{\sigma + it : \sigma > 1 - \frac{c}{\log(|t| + 2)}\}$$

for c an absolute positive constant; equivalently, for $t \gg 0$,

$$|\zeta(1 + it)| \geq \frac{c}{\log t}. \tag{1}$$

(Under the Riemann Hypothesis the stronger bound $|\zeta(1 + it)| \geq \frac{c}{\log \log t}$ holds.) From a modern point of view, the method of de la Vallée Poussin is based on Rankin-Selberg convolutions and a positivity argument (an effective version of Landau's Lemma – see [HL94, Appendix]). It can be applied to any Rankin-Selberg L-function $L(s, \pi_1 \otimes \pi_2)$, *provided that one of the π_i's is self-dual* (cf. [Sar03], [Mor85]). Here π_i, $i = 1, 2$ are cuspidal automorphic representations of $GL_{n_i}(\mathbb{A})$, $\mathbb{A}$ is the ring of adèles of a number field F, and the central characters of π_i's are

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trivial on the positive reals, imbedded diagonally in the (archimedean) ideles. The zero-free region takes the form

$$\sigma > 1 - \frac{c}{\log(Q_{\pi_1} Q_{\pi_2}(|t|+2))} \quad |t| \geq \delta > 0$$

with c an explicit constant depending only on δ and the n_i's; here Q_{π_i} is the product of the conductor of π_i and the archimedean size of π_i, as defined for example in [Sar98]. In particular this holds for standard L-functions of cuspidal representations of $GL_n(\mathbb{A})$. Hence, by functoriality, this is expected to hold for most automorphic L-functions[1]. We mention that for $t = 0$ an approach for obtaining an effective zero-free region of standard type for $n \geq 2$ using functoriality is discussed in [HR95]. However, this method does not give any information about Landau-Siegel zeros for L-functions of quadratic Dirichlet characters.

If the π_i's are not assumed to be self-dual then Brumley recently established a coarse zero-free region

$$\sigma > 1 - \frac{c}{(Q_{\pi_1} Q_{\pi_2}(|t|+2))^N}$$

where again c, N depend (explicitly) only on n_1, n_2 ([Bru]). Brumley's method also uses Rankin-Selberg convolutions and a positivity argument. Note that here we do not have to assume $t \neq 0$. Brmuley's result has applications, among other things, to the absolute convergence of the spectral expansion of Jacquet's relative trace formula ([Lap]).

In [Sar03] Sarnak explains how to obtain a slightly weaker form of (1) by quite a different method, using Eisenstein series on GL_2. His ingenious argument exploits the Maass-Selberg relations and the computation of Fourier coefficients. Comparing the two by Bessel's inequality gives a coarse lower bound for zeta. This can be viewed as an effectuation (for $n = 1$) of the non-vanishing result of Jacquet-Shalika for the standard L-function of cuspidal representations of $GL_n(\mathbb{A})$ at $\mathrm{Re}(s) = 1$ ([JS77]). (To obtain a better bound in the spirit of (1), a deeper analysis using a sieve method is required.)

The advantage in Sarnak's method is that it potentially carries over to any L-function which appears in Langlands' formula for the constant term of Eisenstein series ([Lan71]). Specifically, let G be a reductive group over a number field F with Langlands dual ${}^LG$ and let M be the Levi part of a maximal parabolic P of G defined over G. Let ${}^LP$ be the corresponding parabolic subgroup of ${}^LG$ with Levi decomposition ${}^LM\,{}^LU$. The adjoint representation of the dual ${}^LM$ on the Lie algebra

[1] i.e., those pertaining to cuspidal π's for which one expects the Ramanujan conjecture to hold

of ${}^LU$ decomposes as $\oplus_{j=1}^m r_j$ where the irreducible constituents are indexed by the terms in the lower central series of ${}^LU$. Given a cuspidal automorphic representation π of $M(\mathbb{A})$ the main term in the constant term of the Eisenstein series induced from π is essentially given by

$$\prod_{j=1}^{r} \frac{\Lambda(js, \tilde{\pi}, r_j)}{\Lambda(js+1, \tilde{\pi}, r_j)}.$$

(Following the advice of Peter Sarnak, $L(s, \pi, r)$ will always denote the partial L-function, i.e. without the archimedean Γ-factors. The completed L-function will be denoted by $\Lambda(s, \pi, r)$. This deviates from the notation of [Lan70], [Sha78], [GS88].)

For the argument of [Sar03] to generalize it is important that π is generic, namely, it that admits a non-zero non-degenerate Fourier coefficient. (This implies in particular that G, or equivalently, M, is quasi-split over F.) The non-degenerate Fourier coefficient of the Eisenstein series is computed by the so-called Langlands-Shahidi method. The purpose of this paper is to extend the argument of [Sar03] in this generality. Namely, we prove the following Theorem.

Theorem 1. *Let G, M, r_j be as above and let π be a generic cuspidal representation of $M(\mathbb{A})$. Then there exist constants c, $n > 0$ such that*

$$|L(1+it, \pi, r_j)| \geq c(1+|t|)^{-n} \quad |t| \geq 1, \quad j = 1, \ldots, m.$$

This Theorem answers in a strong form a conjecture posed in [GS01]. As mentioned before, its proof is based on the theory of Eisenstein series and especially the Langlands-Shahidi method and the Maass-Selberg relations. Once again, it can be viewed as an effectuation of Shahidi's non-vanishing result ([Sha81]).

Now let π be a cuspidal representation of $GL_2(\mathbb{A})$. As in [KS02a], we consider the exceptional group $G = E_8$ and construct a representation Π of the Levi factor isogenous to $GL_4 \times GL_5$ using the symmetric cube and quadruple functorial lifts of π ([KS02b], [Kim03]). The ninth symmetric power L-function of π appears as a factor in $L(s, \Pi, r_1)$ and we obtain the following.

Corollary 1. *Let π be a cuspidal representation of $GL_2(\mathbb{A})$. Then there exists constants $c, n > 0$ such that for all $t \in \mathbb{R}$ with $|t| \geq 1$*

$$L^S(1+it, \pi, \mathrm{sym}^9) \geq \frac{c}{(1+|t|)^n}.$$

This lower bound seems to be out of reach of the method of de la Vallee Poussin (with our current knowledge on the functoriality conjectures). It is quite remarkable since even the holomorphy of the symmetric ninth power L-function for $\mathrm{Re}(s) > 1$ is not known!

Another consequence of Theorem 1 is uniform majorization of Eisenstein series.

Corollary 2. *Let G, M, π be as in Theorem 1. Then there exist constants c, n such that for all $g \in G(\mathbb{A})$ and $s \in i\mathbb{R}$*

$$|E(g, \varphi, s)| \leq c \cdot (1 + \|g\|)^n \cdot (1 + |s|)^n.$$

We mention that the constant c appearing in Theorem 1 (and hence, in its Corollaries) depends on (a lower bound for) the size of non-degenerate Fourier coefficients of π. This is subtle, because for general M not all cuspidal representations of $M(\mathbb{A})$ are generic, and it may be difficult to "effectuate" the notion of a generic representation. However, if M is isogenous to GL_n (or a product thereof, which is the case of Corollary 1) then every cuspidal representation is generic (with respect to some non-degenerate character) and moreover, it is possible to relate the L^2-norm with the Fourier coefficients by using the integral representation of the Rankin-Selberg L-function ([JS81]). Thus, it is possible to bound c inverse polynomially in the analytic conductor of π. We will not pursue this point in the paper.

The paper is organized as follows. In §2 we set up the notation and recall some standard facts about Eisenstein series and meromorphic functions of finite order. In §3 we show, following Sarnak, the finiteness of order, in the sense of meromorphic functions, of the L-functions appearing in Langlands' computation of the constant term of Eisenstein series. The main analytic input, as in [GS01], is the finiteness of order of Eisenstein series which was proved by Müller ([Mül00]). Using Shahidi's results on the L-function in the case where π is generic, we obtain coarse upper bounds for the partial L-functions and their derivatives by the Phragmen-Lindelof principle. This sharpens and considerably simplifies the proof of the main result of [GS01]. The main argument appears in §4 where we bound from above and below the L^2-norm of truncated Eisenstein series. For the upper bound we use the Maass-Selberg relation and the results of §3. The lower bound comes rather directly from elementary Fourier analysis and the properties of the truncation operator. Here it is crucial to use the fact that π is generic. Finally, we obtain the main result in §5 by an easy continuity argument.

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2. Preliminaries

2.1. **Notation.** Throughout this paper let G be a reductive group over a number field F. We will often denote $G(F)$ by G as well. We fix a minimal parabolic subgroup P_0 of G with a Levi decomposition $P_0 = M_0U_0$. Let $P = MU$ be a maximal parabolic of G defined over F with $M \supset M_0$. Let T_M be the intersection of the maximal split torus of the center of M with the derived group of G. Thus $T_M \simeq \mathbb{G}_m$ (the multiplicative group) and we let A_M be the subgroup $\mathbb{R}_+$ imbedded in $\mathbb{I}_F \simeq T_M(\mathbb{A})$ through the embedding $\mathbb{R} \hookrightarrow \mathbb{A}_{\mathbb{Q}} \hookrightarrow \mathbb{A}_F$. Similarly, let T_0 (resp. A_0) be the split part of the center of M_0 inside G^{der} (resp. the $\mathbb{R}$-vector space corresponding to it).

Let ϖ be the fundamental weight corresponding to P (in the vector space spanned by the rational characters of M). Let δ_P (resp. δ_0) be the modulus function of $P(\mathbb{A})$ (resp. $P_0(\mathbb{A})$). Finally, choose a maximal compact subgroup $\mathbf{K}$ of $G(\mathbb{A})$ which is in a "good position" with respect to M_0 (cf. [MW95]). In particular, $G(\mathbb{A}) = M(\mathbb{A})U(\mathbb{A})\mathbf{K}$ and $M(\mathbb{A}) \cap \mathbf{K}$ is a maximal compact of $M(\mathbb{A})$. Extend the function

$$|\varpi|(m) = \prod_{v \in \mathfrak{V}} |\varpi(m_v)|_v \qquad m = (m_v)_{v \in \mathfrak{V}} \in M(\mathbb{A})$$

to a left-$U(\mathbb{A})$ right-$\mathbf{K}$-invariant function on $G(\mathbb{A})$ (which we continue to denote by $|\varpi|$). Here $\mathfrak{V}$ is the set of valuations of F.

2.2. **Eisenstein series.** Let π be a cuspidal automorphic representation of $M(\mathbb{A})$. We will always assume that the central character of π is trivial on A_M. Let $\mathcal{A}_P^\pi$ denote the space of automorphic forms φ on $U(\mathbb{A})M\backslash G(\mathbb{A})$ such that for all $k \in \mathbf{K}$ the function $m \mapsto \delta_P(m)^{-\frac{1}{2}}\varphi(mk)$ belongs to the space of π. The automorphic realization of π gives rise to an identification of $\mathcal{A}_P^\pi$ with (the K_∞-finite part of) the induced space $I(\pi) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi$. Set $\varphi_s(g) = \varphi(g)\,|\varpi|^s(g)$ for any $\varphi \in \mathcal{A}_P^\pi$, $s \in \mathbb{C}$. The map $\varphi \mapsto \varphi_s$ identifies $I(\pi)$ with any $I(\pi, s) = I_P(\pi, s) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi\,|\varpi|^s$.

For any $\varphi \in \mathcal{A}_P^\pi$ we consider the Eisenstein series which is the meromorphic continuation of the series

$$E(g, \varphi, s) = \sum_{\gamma \in P \backslash G} \varphi_s(\gamma g)$$

which converges for $\mathrm{Re}(s) \gg 0$. Whenever regular, it defines an intertwining map from $I(\pi, s)$ into the space of automorphic forms on $G(\mathbb{A})$.

Let $\overline{P}$ be the parabolic opposite to P containing M and let $P' = M'U'$ be the standard parabolic conjugate to $\overline{P}$ with $M' \supset M_0$. Thus, if w_0 is the longest Weyl element, then $M' = w_0 M w_0^{-1}$. The fundamental weight ϖ' corresponding to P' is $\varpi' = (w_0 \varpi)^{-1}$. Let π' be the cuspidal representation of $M'(\mathbb{A})$ on the space $\{\varphi(w_0^{-1} \cdot) : \varphi \in V_\pi\}$.

Let $M(s) : \mathcal{A}_P^\pi \to \mathcal{A}_{P'}^{\pi'}$ be the operator given by

$$M(s)\varphi(g) = |\varpi'|^s (g) \int_{U'(\mathbb{A})} \varphi_s(w_0^{-1} u' g) \, du'.$$

It admits a meromorphic continuation and gives rise to a family of intertwining operators $I(\pi, s) \to I_{P'}(\pi', -s)$.

The constant term of $E(\cdot, \varphi, s)$ along P' is given by

$$\{\varphi_s(g) +\} \ |\varpi'|^{-s} (g) M(s)\varphi(g) \tag{2}$$

where the term in brackets appears only if $P' = P$. Write $\pi = \otimes_v \pi_v$. By a well-known formula of Langlands we have ([Lan71])

$$M(s)(\otimes_v \varphi_v) = \prod_{v \in S} M_v(\pi_v, s)\varphi_v \times \prod_{j=1}^m \frac{L^S(js, \tilde{\pi}, r_j)}{L^S(1 + js, \tilde{\pi}, r_j)} \tag{3}$$

where as in the introduction $\oplus_{j=1}^m r_j$ is the decomposition of the adjoint representation of ${}^L M$ on the Lie algebra of ${}^L U$, $M_v(\pi_v, s)$ are the local intertwining operators, L^S is the partial L-function (the product over $v \notin S$), and finally, S is a sufficiently large finite set of places containing all the archimedean ones such that for all $v \notin S$ both G_v and π_v are unramified and φ_v is the standard section. Moreover, on every K_v-type $M_v(\pi_v, s)$ is either a rational functional in q_v^s if v is non-archimedean or a rational function in s times a product of quotients of Γ-functions if v is archimedean ([Art89]).

2.3. **Meromorphic functions of finite order.** Let $\mathcal{F}$ be the ring of entire functions of finite order. The class of meromorphic functions of finite order $\mathcal{M}$ is, by definition, the field of fractions of $\mathcal{F}$. For example by the remark above

(4) on any K_v-type the matrix coefficients of $M_v(\pi_v, s)$ are in $\mathcal{M}$.

By the Hadamard factorization Theorem, any entire function in $\mathcal{M}$ lies in $\mathcal{F}$. More generally, we have the following.

Lemma 1. *Let $f(s) \in \mathcal{M}$. Then there exists C, N such that $|f(s)| \leq Ce^{|s|^N}$ outside the union of discs of radius 1 around the poles of f.*

Proof. By the Hadamard factorization theorem we can write

$$f(s) = e^{P(s)} s^{m_f(0)} \prod_{\eta \neq 0} E(s/\eta, p)^{m_f(\eta)} \tag{5}$$

for $p \gg 0$ and a certain polynomial $P(s)$ where $m_f(\eta)$ is the multiplicity of f at η (negative if η is a pole) and

$$E(z,p) = (1-z)e^{z+z^2/2+\cdots+z^p/p}$$

is the usual Weierstrass primary factor.

Since $\inf\{|\eta| : \eta \neq 0, m_f(\eta) \neq 0\} > 0$, each factor $E(s/\eta, p)$ satisfies $|E(s/\eta,p)| \leq C_2(1+|s|)e^{C_1|s|^p}$ for some C_1, C_2. The same will be true for $|E(s/\eta,p)|^{-1}$ provided that $|s-\eta| \geq 1$.

We separate the product in (5) for $|\eta| \leq 2|s|$ and $|\eta| > 2|s|$. The number of η's of the first type is bounded by $C_3(1+|s|)^l$ for some l, C_3. Using the previous bound the product over $|\eta| \leq 2|s|$ is bounded by $Ce^{|s|^N}$ for some C, N, provided that $|s-\eta| \geq 1$ for all η. On the other hand, the product over $|\eta| > 2|s|$ is bounded by $e^{2\sum_\eta |s/\eta|^{p+1}}$. Indeed, for $u = s/\eta$, $|u| < \frac{1}{2}$, we have $|\log(E(u,p))| \leq 2|u|^{p+1}$ ([Tit32, p. 246]). All in all we get the required bound. □

We also need the following Lemma. We take this opportunity to thank Ron Livné for helpful discussions leading to it.

Lemma 2. *Suppose that $g(s) \in \mathcal{M}$ and let $\sigma \in \mathbb{R}$. Then there exists $f(s) \in \mathcal{M}$ such that $g(s) = \frac{f(s)}{f(s+1)}$ and $f(s)$ has neither zeros nor poles in the strip*

$$\mathfrak{S}_\sigma = \{s \in \mathbb{C} : \sigma \leq \mathrm{Re}(s) < \sigma + 1\}.$$

Proof. Let $n(s)$ be the unique function on $\mathbb{C}$ such that $n(s) - n(s-1) = m_g(s)$ for all $s \in C$ and n is zero on $\mathfrak{S}_\sigma$. Thus,

$$n(s) = \begin{cases} \sum_{k=0}^{\lfloor \mathrm{Re}(s)-\sigma-1 \rfloor} m_g(s-k), & \text{if } \mathrm{Re}(s) \geq \sigma \\ -\sum_{k=1}^{\lceil \sigma - \mathrm{Re}(s) \rceil} m_g(s+k) & \text{if } \mathrm{Re}(s) < \sigma. \end{cases}$$

Clearly,

$$\sum_{|s|<R} |n(s)| < c(1+R)^n$$

for some c, n because the same is true for m_g. Thus, using a Weierstrass product we can find $h \in \mathcal{M}$ such that $m_h \equiv n$. It follows that the function $\phi(s) = \frac{h(s+1)g(s)}{h(s)} \in \mathcal{M}$ has neither zeros nor poles, and hence equals $e^{P(s)}$ for some polynomial $P(s)$. Clearly, we can find a polynomial $Q(s)$ such that $P(s) = Q(s) - Q(s+1)$ for all $s \in \mathbb{C}$. It remains to set $f(s) = e^{Q(s)}h(s)$ to obtain Lemma 2. □

3. Finiteness of order of L-functions

Theorem 2. *Let π be a (not necessarily generic) cuspidal automorphic representation of $M(\mathbb{A})$ and let r_j, $j = 1, \dots, m$ be as in the introduction. Then each $L(s, \pi, r_j)$ belongs to $\mathcal{M}$.*

Proof. We will prove it by induction on m (the nilpotence degree of ${}^LU$). For $j > 1$ the statement follows from the induction hypothesis and Proposition 4.1 of [Sha90] and the remark on p. 298 of the same paper (the F_4 case is treated in [Sha88, p. 572–573]).

By [Mül00, Theorem 0.2][2] $E(g, \varphi, s)$ is of finite order, and moreover, there exists $q(s) \in \mathcal{F}$ and $c, n > 0$ such that for any compact set $W \subset G(\mathbb{A})$ there exists $C > 0$ such that

$$|q(s)E(g, \varphi, s)| \leq Ce^{c|s|^n} \quad \forall g \in W, s \in \mathbb{C}.$$

It follows that the constant term of E along P' lies in $\mathcal{M}$. Using (2), (3) and (4) we obtain that $\prod_{j=1}^m \frac{L^S(js,\pi,r_j)}{L^S(1+js,\pi,r_j)}$ belongs to $\mathcal{M}$. We already know that $L^S(s, \pi, r_j)$, $j > 1$, belongs to $\mathcal{M}$. Thus $g(s) = \frac{L(s)}{L(s+1)} \in \mathcal{M}$ where we write for brevity $L(s) = L^S(s, \pi, r_1)$. Choose $\sigma \gg 0$. Using Lemma 2 we can find $f(s) \in \mathcal{M}$ holomorphic and non-zero on $\mathfrak{S}_\sigma$ such that $\phi(s) = \frac{f(s)}{L(s)}$ is periodic with period 1. It follows that $\phi(s)$ is holomorphic (and non-zero) on $\mathfrak{S}_\sigma$ and hence it is entire. Thus, $f(s)$ is holomorphic for $\mathrm{Re}(s) \gg 0$. We infer from Lemma 1 that $|f(s)| \leq Ce^{|s|^N}$ for $\mathrm{Re}(s) \gg 0$. We obtain a similar bound for $\phi(s)$ since $L(s)$ is bounded away from zero for $\mathrm{Re}(s) \gg 0$. Thus, $\phi \in \mathcal{F}$ by periodicity. We conclude that $L(s)$, hence also $L(s, \pi, r_1)$, belongs to $\mathcal{M}$. □

Assume from now on that G (or equivalently, M) is quasi-split. Let ψ be a non-degenerate character of $U_0 \backslash U_0(\mathbb{A})$, and denote by ψ_M its restriction to $U_0 \cap M$ (a maximal unipotent of M). For any automorphic

[2]In fact, since π is cuspidal, this is already proved, if not explicitly stated, in [Mül89].

function ϕ on $G\backslash G(\mathbb{A})$ we write

$$\phi^{\psi}(g) = \int_{U_0\backslash U_0(\mathbb{A})} \phi(ug)\psi(u)\, du$$

for the ψ-th Whittaker coefficient of $\phi(\cdot g)$. Similarly for $\phi^{\psi_M}(\cdot)$. As in [Sha81] we can write

$$E^{\psi}(g, \varphi, s) = \int_{U'(\mathbb{A})} \varphi_s^{\psi_M}(w^{-1}u'g)\psi(u')\, du$$

where w is the right-W_M-reduced representative of w_0. From now on we assume that π is generic with respect to ψ_M.

Suppose that $\phi = \otimes_v \phi_v$ is decomposable. Let S be a finite set of places containing all the archimedean ones such that for all $v \notin S$, π_v and ψ_v are unramified, φ_v is the standard section, and $g \in K_v = G(\mathcal{O}_v)$. We have

$$E^{\psi}(g, \varphi, s) = \lambda \cdot \overbrace{\prod_{v \in S} \mathcal{W}_v^{\psi}(\varphi_v, g, s)}^{\mathcal{W}_S^{\psi}(\varphi, g, s)} \left[\prod_{j=1}^{m} L^S(1 + js, \tilde{\pi}, r_j) \right]^{-1}. \tag{6}$$

where $\lambda \neq 0$ is a global constant (depending on π) and $\mathcal{W}_v^{\psi}(\varphi_v, g, s)$ is the local Jacquet integral which will be considered in §5 below.

We recall the following Lemma (cf. [Sha88], [Kim02]).

Lemma 3. *The function $L^S(s, \pi, r_j)$ has at most finitely many poles for $\mathrm{Re}(s) \geq \frac{1}{2}$ and at most finitely many zeros for $\mathrm{Re}(s) \geq 1$. They are all real.*

Proof. By induction on m, we can assume by Proposition 3.1 of [GS01] that the Lemma holds for $j > 1$. Consider the case $j = 1$. As in [Sha88] the poles of the two functions

$$\prod_{j=1}^{m} \frac{L^S(js, \tilde{\pi}, r_j)}{L^S(js + 1, \tilde{\pi}, r_j)} \quad \text{and} \quad \prod_{j=1}^{m} \frac{1}{L^S(js + 1, \tilde{\pi}, r_j)}$$

are included in those of the Eisenstein series. Hence there are at most finitely many of them for $\mathrm{Re}(s) \geq 0$, and they are all simple and real ([MW95, IV.1.11]). Thus, by the assumption for $j > 1$, the function $\frac{L^S(s, \tilde{\pi}, r_1)}{L^S(s+1, \tilde{\pi}, r_1)}$ (resp. $\frac{1}{L^S(s+1, \tilde{\pi}, r_1)}$) has only finitely many poles for $\mathrm{Re}(s) \geq \frac{1}{2}$ (resp. $\mathrm{Re}(s) \geq 0$), and they are all real. Since $L^S(s, \tilde{\pi}, r_1)$ is holomorphic for $\mathrm{Re}(s) \gg 0$ the Lemma follows immediately by passing to $\tilde{\pi}$. $\square$

With the Lemma we can conclude from Theorem 2 the following result (for π generic).

Proposition 1. *There exists constants* $c, n > 0$ *such that*

$$\left|L^S(s,\pi,r_j)\right| \le c(1+|s|)^n \tag{7}$$

on $R = \{s \in \mathbb{C} : \mathrm{Re}(s) \ge \frac{1}{2}, |\mathrm{Im}(s)| \ge 1\}$. *A similar bound holds for the derivative of* L^S.

Proof. The statement certainly holds for $\mathrm{Re}(s) > \sigma_1$ with $\sigma_1 \gg 0$. Let

$$L_S(s,\pi,r_j) = \prod_{v\in S} L_v(s,\pi_v,r_j)$$

where the local factors are defined in [Sha90, §7]. The functional equation for $L(s,\pi,r_j)$ ([Sha90, Theorem 7.7]) has the form

$$L^S(s,\pi,r_j) = \varepsilon_0 Q_j^{\frac{1}{2}-s} \frac{L_S(1-s,\tilde{\pi},r_j)}{L_S(s,\pi,r_j)} \overline{L^S(1-\bar{s},\pi,r_j)}$$

for some $Q_j > 0$ and $|\varepsilon_0| = 1$. Set $g(s) = L^S(s,\pi,r_j)/L_{S_{fin}}(1-s,\tilde{\pi},r_j)$ where S_{fin} is the set of finite places in S. By Lemma 3, $g(s)$ has at most finitely many poles for $\mathrm{Re}(s) \ge \frac{1}{2}$. We write the functional equation as

$$g(s) = \varepsilon_0 Q_j^{\frac{1}{2}-s} \frac{L_\infty(1-s,\tilde{\pi},r_j)}{L_\infty(s,\pi,r_j)} \overline{g(1-\bar{s})}.$$

Since $L_\infty(s,\pi,r_j)$ has only finitely many poles for $\mathrm{Re}(s) \ge \frac{1}{2}$ the function $\tilde{g}(s) = P(s)g(s)$ will be entire for a suitable polynomial $P(s)$. Moreover, $\tilde{g}(s)$ will be of finite order by Theorem 2. The usual argument invoking the Phragmen-Lindelof principle (e.g. [Lan94, XIII, §5]) yields the bound

$$|\tilde{g}(s)| \le c(1+|s|)^n$$

on any vertical strip. By changing $P(s)$ if necessary we may assume, by the previous lemma, that $P(s)L^S(s,\pi,r_j)$ is holomorphic for $\mathrm{Re}(s) \ge \frac{1}{2}$. Let z_n be the poles of $L_{S_{fin}}(1-s,\tilde{\pi},r_j)$. They form a finite union of arithmetic progressions with imaginary difference. Let C be the union of discs around z_n and around the roots of $P(s)$ with a small fixed radius r. Then $P(s)^{-1}L_{S_{fin}}(1-s,\tilde{\pi},r_j)$ is bounded in the complement of C in R. Thus (7) holds on $R\backslash C$. By the maximum modulus principle it will hold on R as well. In fact, the proof shows that (7) holds on the slightly bigger domain $\mathrm{Re}(s) \ge \sigma$, $|\mathrm{Im}(s)| \ge 1$, for an appropriate $\sigma < \frac{1}{2}$. The last part of the Proposition follows from this by Cauchy's formula. □

Remark 1. In many cases one knows that $L^S(s,\pi,r_j)$ (and in fact, $\Lambda(s,\pi,r_j)$) has only finitely many poles in the entire plane (cf. [GS01]). In that case, the proof shows that Proposition 1 holds on any right-half plane. This is a slight improvement of the main result of [GS01] (proved under the same assumption). In principle, Proposition 1 can be proved using the technique of [GS01] which, like ours, exploits Müller's result on the finiteness of order of Eisenstein series. The main difference between the approach taken here and that of [GS01] is that we prove Theorem 2 first, without appealing to the functional equation, and allowing the use of meromorphic functions. This avoids the use of Matsaev's theorem in [GS01].

Theorem 2 is new in this generality. It is proved along the lines of [Sar01]. The method of proof is robust, and applies to all L-functions which admit an integral representation (i.e. Rankin-Selberg integrals). Moreover, Proposition 1 was obtained from Theorem 2 by the usual Phragmen-Lindelof argument. To apply it, one "only" needs to know the functional equation, as well as finiteness of poles for the partial L-function. Again, these properties can be obtained, at least in principle, from integral representations of L-functions.

4. Estimation of the L^2-norm of truncated Eisenstein series

4.1. **Truncated Eisenstein series.** We consider the truncated Eisenstein series which is given in this case by

$$\Lambda^T E(g,\varphi,s) = \sum_{\gamma\in P\backslash G} \varphi_s(\gamma g)\chi_{\le T}(|\varpi|(\gamma g)) - \sum_{\gamma\in P'\backslash G} M(s)\varphi(\gamma g)\,|\varpi'|^{-s}(\gamma g)\chi_{\ge T}(|\varpi'|(\gamma g))$$

(cf. [Art80, §4]). Here, $\chi_{\le T}$ (resp. $\chi_{\ge T}$) is the characteristic function of the interval $(0,e^T)$ (resp. (e^T,∞)).

Lemma 4. *Fix $c>0$. Then for T sufficiently large and for any $g\in G(\mathbb{A})$ with $|\varpi|(g), |\varpi'|(g) > c$ and any non-degenerate character ψ of $U_0\backslash U_0(\mathbb{A})$ we have*

$$(\Lambda^T E)^{\psi}(g,\varphi,s) = E^{\psi}(g,\varphi,s). \tag{8}$$

Proof. The left-hand side of (8) can be written (for $\mathrm{Re}(s)\gg 0$) as the difference of

$$\sum_{w\in W/W_M} \int_{U_0\cap wPw^{-1}\backslash U_0(\mathbb{A})} \varphi_s(w^{-1}ug)\chi_{\le T}(|\varpi|(w^{-1}ug))\psi(u)\,du \tag{9}$$

and

$$\sum_{w\in W/W_{M'}} \int_{U_0\cap wP'w^{-1}\backslash U_0(\mathbb{A})} M(s)\varphi(w^{-1}ug)\,|\varpi'|^{-s}(w^{-1}ug)\chi_{\ge T}(|\varpi'|(w^{-1}ug))\psi(u)\,du. \tag{10}$$

As in [Sha78], in both summands only the longest w contributes since ψ is non-degenerate. For this w we have

$$|\varpi'|\,(w^{-1}ug) = |\varpi|^{-1}\,(g) \cdot |\varpi'|\,(w^{-1}u')$$

for some $u' \in U_0(\mathbb{A})$. It is well-known that $\sup_{U_0(\mathbb{A})} |\varpi'|\,(w^{-1}u) < \infty$. Thus, for T sufficiently large (10) vanishes. We remain with the contribution from w to (9) which by a similar reasoning is equal to

$$\int_{U_0^{M'}\backslash U_0(\mathbb{A})} \varphi_s(w^{-1}ug)\psi(u)\,du.$$

This is equal to the right-hand side of (8). $\square$

Our goal is to estimate from above and from below the L^2-norm of $\Lambda^T E(\cdot,\varphi,s)$.

4.2. We first deal with the upper bound. To that end we use the Maass-Selberg relations to write $\|\Lambda^T E(\cdot,\varphi,s)\|_2^2$ as

(11)

$$-(M(-s)M'(s)\varphi,\varphi) + T(\varphi,\varphi)\{+\frac{(e^{2sT}M(-s)\varphi - e^{-2sT}M(s)\varphi,\varphi)}{2s}\}$$

for $s \in i\mathbb{R}$ ([Art80, §4]) where the last term appears only if $P = P'$. Here $(\cdot,\cdot)$ denotes the usual inner product on $I(\pi)$ given by

$$(\varphi_1,\varphi_2) = \int_{\mathbf{K}} (\varphi_1(k),\varphi_2(k))_\pi\,dk.$$

Set

$$\|\varphi\|_\infty = \max_{k\in\mathbf{K}} \|\varphi(k)\|_\pi$$

for $\varphi \in I(\pi)$. Let $\mathfrak{k}_\mathbb{C}$ be the complexified Lie algebra of K_∞ and let $U(\mathfrak{k}_\mathbb{C})$ be the universal enveloping algebra of $\mathfrak{k}_\mathbb{C}$. For any $N \geq 0$ denote by $U(\mathfrak{k}_\mathbb{C})_{\leq N}$ the finite dimensional vector subspace of $U(\mathfrak{k}_\mathbb{C})$ spanned by monomials of degree $\leq N$. Choose a basis X_i of $U(\mathfrak{k}_\mathbb{C})_{\leq N}$ and let μ_N be the semi-norm on $I(\pi)$ defined by

$$\mu_N(\varphi) = \sum_i \|X_i\varphi\|_\infty.$$

(Different bases give rise to equivalent norms.) Of course, these norms exist also in the local archimedean setting.

Lemma 5. *Suppose that v is archimedean. Then for any $X \in U(\mathfrak{g}_{v,\mathbb{C}})$ there exist c, n and N such that*

$$\|I(X,\pi_v,s)\varphi\|_\infty \leq c(1+|s|)^n \mu_N(\varphi).$$

Here $I(X,\pi_v,s)$ denotes the action of $U(\mathfrak{g}_{v,\mathbb{C}})$ on $I(\pi,s)$ where $\mathfrak{g}_{v,\mathbb{C}}$ is the complexification of the Lie algebra of $G(F_v)$.

Proof. Except for the dependence on s this is precisely Lemma 10.1.1 of [Wal92]. In fact, the proof immediately gives the dependence on s. $\square$

Proposition 2. *There exist c, n, N such that*
(12)
$$\|\Lambda^T E(\cdot,\varphi,s)\|_2^2 \le c(1+|s|)^n \cdot \mu_N(\varphi)^2 \cdot (1+\sum_{j=1}^m \left|L^S(1+js,\tilde{\pi},r_j)\right|^{-1})$$
for all $s \in i\mathbb{R}$ with $|\mathrm{Im}\, s| \ge 1$.

Proof. Since T is fixed, the last two summands in (11) are bounded by constant multiple $\|\varphi\|_2^2$ for $|\mathrm{Im}(s)| \ge 1$. It remains to estimate the logarithmic derivative of $M(s)$. Following Shahidi, we write

$$M(s) = \prod_j \frac{\Lambda(js,\tilde{\pi},r_j)}{\varepsilon(js,\tilde{\pi},r_j)\Lambda(js+1,\tilde{\pi},r_j)} \times R(s) = \prod_j \frac{\overline{\Lambda(1-j\overline{s},\tilde{\pi},r_j)}}{\Lambda(1+js,\tilde{\pi},r_j)} \times R(s)$$

where $R(s) : I_P(\pi,s) \to I_{P'}(\pi',-s)$ is the normalized intertwining operator ([Sha90]). Thus, for $s \in i\mathbb{R}$

$$M(-s)M'(s) = R(-s)R'(s) - \mathrm{Re}\sum_j j\frac{\Lambda'(1+js,\tilde{\pi},r_j)}{\Lambda(1+js,\tilde{\pi},r_j)}$$
$$= R(-s)R'(s) - \mathrm{Re}\sum_j j\frac{L'(1+js,\tilde{\pi},r_j)}{L(1+js,\tilde{\pi},r_j)} - \mathrm{Re}\sum_j j\frac{L'_\infty(1+js,\tilde{\pi},r_j)}{L_\infty(1+js,\tilde{\pi},r_j)}.$$

Note that for any meromorphic function f the function $\mathrm{Re}\,\frac{f'(s)}{f(s)}$ is continuous for $s \in i\mathbb{R}$, regardless of the poles of f.

By Stirling's formula $\mathrm{Re}\,\frac{L'_\infty(1+js,\tilde{\pi},r_j)}{L_\infty(1+js,\tilde{\pi},r_j)}$ is bounded polynomially in $1+|s|$ on $i\mathbb{R}$. For $v \in S$ finite, the function $\mathrm{Re}\,\frac{L'_v(1+js,\tilde{\pi}_v,r_j)}{L_v(1+js,\tilde{\pi}_v,r_j)}$ is bounded on $i\mathbb{R}$, since it is continuous and periodic. Taking into account Proposition 1 it remains to bound $(R(-s)R'(s)\varphi,\varphi)$ for $s \in i\mathbb{R}$. The normalized intertwining operator admits a factorization into local normalized intertwining operators

$$R_v(\pi_v,s,\psi_v) : I_P(\pi_v,s) \to I_{P'}(\pi'_v,-s).$$

This factorization depends on a choice of a non-trivial character $\psi = \otimes_v \psi_v$ of $F\backslash\mathbb{A}_F$. We have $R_v(\pi_v,s,\psi_v)\varphi_v = \varphi_v$ if π_v is unramified, the conductor of ψ_v is $\mathcal{O}_v$ and φ_v is the standard section. Thus, we can write

$$(R(-s)R'(s)\varphi,\varphi) = \sum_{v\in S}(R_v(\pi_v,-s,\psi_v)R'_v(\pi_v,s,\psi_v)\varphi_v,\varphi_v).$$

If v is finite then $(R_v(\pi_v,-s,\psi_v)R'_v(\pi_v,s,\psi_v)\varphi_v,\varphi_v)$ is continuous and periodic, hence bounded on $i\mathbb{R}$. Suppose that v is an infinite place. Recall ([Wal92, §10]) that for some $\rho>0$ there exists a scalar polynomial $b(s)$ and a polynomial $D(s)$ with values in $U(\mathfrak{g}_{v,\mathbb{C}})$ (both depending on π_v) such that

$$b(s)M_v(\pi_v,s)=M_v(\pi_v,s+\rho)I(D(s),\pi_v,s+\rho)$$

for the unnormalized intertwining operators. A similar relation will hold for $R_v(\pi_v,s,\psi_v)$ (with a different $b(s)$) because of the nature of the normalizing factors. We can also assume that ρ is chosen so that the integral defining $M_v(\pi_v,s+\rho)$ is absolutely convergent for $\mathrm{Re}(s)=0$, and in fact, $M_v(\pi_v,s+\rho)$ is a bounded operator with respect to $\|\cdot\|_\infty$ ([Wal92, Lemma 10.1.2]). Once again, the same will be true for $R_v(\pi_v,s,\psi_v)$. Since $R_v(\pi_v,s,\psi_v)$ is holomorphic for $\mathrm{Re}(s)=0$, we can now infer from Lemma 5 that there exist c, n and N such that

$$|(R_v(\pi_v,s,\psi_v)\varphi_1,\varphi_2)|\le c(1+|s|)^n\ \mu_N(\varphi_1)\|\varphi_2\|$$

for $\mathrm{Re}(s)$ small. By Cauchy's formula, a similar bound holds for the derivative of $R_v(\pi_v,s,\psi_v)$. Moreover, since $R_v(\pi_v,s,\psi_v)$ is unitary for $s\in i\mathbb{R}$, we will get the required bound for the logarithmic derivative of $R_v(\pi_v,s,\psi_v)$. Altogether, we obtain (12). □

Remark 2. More generally, if $X\in U(\mathfrak{g}_{\mathbb{C}})=U(\oplus_{v|\infty}\mathfrak{g}_{v,\mathbb{C}})$ the same proof gives a similar upper bound for $\|\Lambda^T E(\cdot,I(X,\pi,s)\varphi,s)\|_2$.

4.3. **A Lower bound.** We now turn to bounding $\|\Lambda^T E(\cdot,\varphi,s)\|_2^2$ from below. Fix $t_0>0$ and let A_0^+ be the set of $a\in A_0$ such that $\alpha(a)>t_0$ for any simple root of T_0 in U_0. Consider the set $\mathfrak{S}$ consisting of $utak$, where $k\in\mathbf{K}$, u (resp. t) is in a fundamental domain for $U_0\backslash U_0(\mathbb{A})$ (resp. $T_0\backslash T_0(\mathbb{A})^1$) and $a\in A_0^+$. By reduction theory, $\mathfrak{S}$ can be identified with an open subset of $G\backslash G(\mathbb{A})^1$ for $t_0\gg 0$.

In particular, $\|\Lambda^T E(\cdot,\varphi,s)\|_2^2$ is bounded below by

$$\int_{\mathbf{K}}\int_{A_0^+}\int_{T_0\backslash T_0(\mathbb{A})^1}\int_{U_0\backslash U_0(\mathbb{A})}\delta_0(a)^{-1}\left|\Lambda^T E(uatk)\right|^2\ du\ dt\ da\ dk.$$

By Fourier analysis and Lemma 4

$$\int_{U_0\backslash U_0(\mathbb{A})}\left|\Lambda^T E(uatk)\right|^2\ du\ge\left|(\Lambda^T E)^\psi(atk)\right|^2=\left|E^\psi(atk)\right|^2.$$

In fact, we could have taken the sum over all non-degenerate ψ. Fix $a_0\in A_0^+$ and choose a small neighborhood V of 1 in $T_0(\mathbb{A})$. Using (6) we get

$$\text{(13)}\quad \|\Lambda^T E(\cdot,\varphi,s)\|_2^2 \geq |\lambda|^2 \prod_{j=1}^{m} \left|L^S(1+js,\tilde{\pi},r_j)\right|^{-2} \int_{K_S}\int_V \left|\mathcal{W}_S^{\psi}(\varphi,a_0vk,s)\right|^2 \, dv\, dk$$

for a suitable S.

5. Conclusion of proof of Theorem 1

We now combine the upper bound (Proposition 2) and the lower bound (13) for $\|\Lambda^T E(\cdot,\varphi,s)\|_2^2$. Upon multiplying the ensuing inequality by $\left|L^S(1+js,\tilde{\pi},r_j)\right|^{-1}\prod_{k=1}^m \left|L^S(1+ks,\tilde{\pi},r_k)\right|^2$, and taking into account the upper bound for the L-function on $\mathrm{Re}(s)=1$ (Proposition 1), we obtain for any $j=1,\dots,m$ the inequality

$$\text{(14)}\quad \left|L^S(1+js,\tilde{\pi},r_j)\right|^{-1} \int_{K_S}\int_V \left|\mathcal{W}_S^{\psi}(\varphi,a_0vk,s)\right|^2 \, dv\, dk \leq c(1+|s|)^n \mu_N^2(\varphi)$$

for $s \in i\mathbb{R}$ with $|\mathrm{Im}\, s| \geq 1$ for appropriate c, n and N. The inequality (14) is based on the theory of Eisenstein series, and in principle it applies only to K_∞-finite sections. However, both sides of the equality extend continuously to all smooth sections. Hence the inequality is valid for smooth sections as well.

We are going to apply this inequality for a special section described below. From now on F will be a local field of characteristic 0 and π will be a ψ_M-generic irreducible representation of $M(F)$. Let $\phi \mapsto \phi^{\psi_M}$ be a (continuous) Whittaker functional on π. For any $\varphi \in I(\pi)$ we consider the Jacquet integral

$$\mathcal{W}^{\psi}(\varphi,g,s) = \int_{U'(F)} \varphi_s(w^{-1}u'g)^{\psi_M}\psi(u')\, du'.$$

5.1. **A special section.** Fix $a_0 \in A_0$ and a smooth compactly supported function ϕ on $U'(F)$ such that

$$\alpha = \int_{U'(F)} \phi(u')\psi(a_0'u'a_0'^{-1})\, du' \neq 0.$$

where $a_0' = wa_0w^{-1}$. Also, fix $v_0 \in \pi$ with $v_0^{\psi_M} \neq 0$. We take φ to be supported inside the big Bruhat cell where it is defined by

$$\varphi(umw^{-1}u') = |\varpi|^{-s}(w^{-1}u')\phi(u')\delta_P^{\frac{1}{2}}(m)\pi(ma_0^{-1})v_0.$$

Note that φ depends on s. Here are two crucial properties of the sections φ.

Lemma 6. (1) *We have*

$$\mathcal{W}^{\psi}(\varphi, a_0', s) = \alpha \cdot v_0^{\psi_M} \left|\varpi\right|^s (a_0)\delta_P^{-\frac{1}{2}}(a_0). \tag{15}$$

In particular,

$$\left|\mathcal{W}^{\psi}(\varphi, a_0', s)\right| = \beta$$

is independent of $s \in i\mathbb{R}$.

(2) *Suppose that F is archimedean. For any $X \in U(\mathfrak{k}_{\mathbb{C}})$ there exist constants c, n such that $\|X\varphi\|_\infty \le c(1+|s|)^n$ for $s \in i\mathbb{R}$.*

Proof. For the first part, we write the left-hand side of (15) as

$$\int_{U'(F)} \left|\varpi\right|^s (w^{-1}u'a_0')\varphi(w^{-1}u'a_0')^{\psi_M}\psi(u')\, du'.$$

After a change of variable this becomes

$$\delta_{P'}(a_0')\int_{U'(F)} \left|\varpi\right|^s (w^{-1}a_0'u')\varphi(w^{-1}a_0'u')^{\psi_M}\psi(a_0'u'a_0'^{-1})\, du',$$

or

$$\delta_{P'}(a_0')\left|\varpi\right|^s (a_0)\delta_P^{\frac{1}{2}}(a_0)\int_{U'(F)} \left|\varpi\right|^s (w^{-1}u')\left[\pi(a_0)\varphi(w^{-1}u')\right]^{\psi_M}\psi(a_0'u'a_0'^{-1})\, du'$$
$$= \left|\varpi\right|^s (a_0)\delta_P^{-\frac{1}{2}}(a_0)\int_{U'(F)} v_0^{\psi_M}\phi(u')\psi(a_0'u'a_0'^{-1})\, du'.$$

This is equal to the right-hand side of (15).

The second part of the Lemma is elementary. The only occurrence of s in φ is as an exponent. Thus, when we take derivatives the dependence of s will be a polynomial times an exponential which is bounded if $\mathrm{Re}(s)$ is bounded. □

5.2. **Continuity in the group variable.** For the next Lemma we consider G as a subgroup of some SL_N, and take the usual norm of matrices.

Lemma 7. *Suppose that F is archimedean. Then there exist constants c, n such that for all $s \in i\mathbb{R}$ and all $g \in G(F)$ satisfying $\|g-1\| < c(1+|s|)^{-n}$ we have $|\mathcal{W}(\varphi, a_0'g, s)| \ge \beta/2$.*

Proof. Write $w^{-1}u'g = u_1 m w^{-1} u'_1$ according to Bruhat decomposition. Then u'_1 is close to u' and m is close to 1. As before, we have

$$\mathcal{W}^\psi(\varphi, a'_0 g, s) = \int_{U'(F)} |\varpi|^s (w^{-1}u'a'_0 g)\varphi(w^{-1}u'a'_0 g)^{\psi_M}\psi(u')\, du'$$
$$= \delta_{P'}(a'_0)\int_{U'(F)} |\varpi|^s (w^{-1}a'_0 u' g)\varphi(w^{-1}a'_0 u' g)^{\psi_M}\psi(a'_0 u' a'^{-1}_0)\, du'$$
$$= \delta_{P'}(a'_0)\,|\varpi|^s(a_0)\delta_P^{\frac{1}{2}}(a_0)\int_{U'(F)} |\varpi|^s (w^{-1}u'g)\left[\pi(a_0)\varphi(w^{-1}u'g)\right]^{\psi_M}\psi(a'_0 u' a'^{-1}_0)\, du'$$
$$= |\varpi|^s(a_0)\delta_P^{-\frac{1}{2}}(a_0)\int_{U'(F)} |\varpi|^s(m)\left[\pi(a_0 m a_0^{-1})v_0\right]^{\psi_M}\phi(u'_1)\psi(a'_0 u' a'^{-1}_0)\, du'.$$

Hence,

$$\left|\mathcal{W}^\psi(\varphi, a'_0 g, s) - \mathcal{W}^\psi(\varphi, a'_0, s)\right| \le \delta_P^{-\frac{1}{2}}(a_0) \times \int_{U'(F)} \left| |\varpi|^s(m)\left[\pi(a_0 m a_0^{-1})v_0\right]^{\psi_M}\phi(u'_1) - v_0^{\psi_M}\phi(u')\right|\, du'. \tag{16}$$

We may assume at the outset that g lies in a fixed small compact set. Hence, u_1, m, u'_1 are algebraic functions of g and u' and in particular, they are confined to a fixed compact set as well. Thus the right-hand side of (16) is bounded by a constant times the maximum over u' of

$$|\phi(u'_1) - \phi(u')| + \left|\left[\pi(a_0 m a_0^{-1})v_0\right]^{\psi_M} - v_0^{\psi_M}\right| + \left||\varpi|^s(m) - 1\right|.$$

Thanks to the smoothness of ϕ and the continuity of the Whittaker functional the first two summands can be made arbitrarily small (independently of s) provided that g is chosen sufficiently close to 1. The last summand can be made arbitrarily small provided that $(1 + |s|)\,|\log(|\varpi|(m))|$ is small. This will be the case if $\|m-1\| < c(1+|s|)^{-n}$ for certain $c, n > 0$. In turn, this will hold if a similar qualitative inequality holds for g. □

The same proof shows that in the p-adic case $\mathcal{W}(\varphi, g, s)$ is right-K_0-invariant for some open subgroup K_0 which is independent of s. In the archimedean case, we note that

$$\mathrm{vol}\{v \in T_0(F) : \|v - 1\| < c\epsilon^n\} \ge \epsilon$$

for some c, n. Similarly for K.

We can now conclude Theorem 1 from the inequality (14) taking into account Lemma 6 and Lemma 7. (Note that for $\mathrm{Re}(s) = 0$, $L(s, \pi, r_j)^{-1}$ differs from $L_S(s, \pi, r_j)^{-1}$ by a bounded factor.)

5.3. **Proof of Corollary 2.** By the automatic continuity theorem of Casselman and Wallach (cf. [Wal92, Ch. 11]) for any given $s \in \mathbb{C}$ the Eisenstein series extend to a continuous map (of Frechet spaces) from $I(\pi, s)$ into the space of smooth functions of $G\backslash G(\mathbb{A})$ of moderate growth. We first prove that

$$|E(g, I(f, \pi, s)\varphi, s)| \leq c \cdot (1 + \|g\|)^n \cdot (1 + |s|)^n. \tag{17}$$

for any $f \in C_c(G(\mathbb{A})^1)$ (not necessarily smooth!). Indeed, we can assume that g is in the Siegel set of $G(\mathbb{A})^1$. By Lemma 5 of [Lap] we can write

$$\begin{aligned} E(g, I(f, s)\varphi, s) &= \int_{G(\mathbb{A})^1} f(x)E(gx, \varphi, s)\; dx \\ &= \int_{G(\mathbb{A})^1} f(x)\Lambda^T E(gx, \varphi, s)\; dx = \int_{G(\mathbb{A})^1} f(g^{-1}x)\Lambda^T E(x, \varphi, s)\; dx \\ &= \int_{G\backslash G(\mathbb{A})^1} K_f(g, x)\Lambda^T E(x, \varphi, s)\; dx \end{aligned}$$

where $K_f(x, y) = \sum_{\gamma \in G} f(x^{-1}\gamma y)$ provided that $T > c_1\|g\| + c_2$ where c_1, c_2 are constants and c_2 depends on the support of f. By [MW95, I.2.4], $|K_f(x, y))| \leq c\,|f|_\infty\,\|x\|^N$ uniformly in x for some $c, N > 0$ (with c depending on the support of f). Hence, by Cauchy-Schwartz

$$|E(g, I(f, \pi, s)\varphi, s)| \leq c'\|g\|^N\|f\|_\infty\|\Lambda^T E(x, \varphi, s)\|_2.$$

We conclude (17) from Theorem 1 and Proposition 2. More generally, using remark 2 we have

$$|E(g, I(f, \pi, s)I(X, \pi, s)\varphi, s)| \leq c \cdot (1 + \|g\|)^n \cdot (1 + |s|)^n. \tag{18}$$

for any $X \in U(\mathfrak{g}_{\infty,\mathbb{C}})$.

By [Art78, §4] there exist $f_1 \in C_c^\infty(G(F_\infty))$, $f_2 \in C_c(G(F_\infty))$ (in fact, we can choose $f_2 \in C_c^m(G(F_\infty))$ for any given m), and $Z \in U(\mathfrak{g}_{\infty,\mathbb{C}})$ such that $f_1 + f_2 \star Z$ is equal to the Dirac distribution at the identity. Hence, if $F_i = \mathrm{vol}(K_0)^{-1} \cdot f_i \otimes 1_{K_0}$, $i = 1, 2$ where 1_{K_0} is the characteristic function of a small open compact subgroup K_0 of $G(\mathbb{A}_{fin})$ then we have

$$E(g, \varphi, s) = E(g, I(F_1, s)\varphi, s) + E(g, I(F_2, s)I(Z, s)\varphi, s).$$

It remains to invoke (18).

References

[Art78] James G. Arthur. A trace formula for reductive groups. I. Terms associated to classes in $G(\mathbf{Q})$. *Duke Math. J.*, 45(4):911–952, 1978.

[Art80] James Arthur. A trace formula for reductive groups. II. Applications of a truncation operator. *Compositio Math.*, 40(1):87–121, 1980.

[Art89] James Arthur. Intertwining operators and residues. I. Weighted characters. *J. Funct. Anal.*, 84(1):19–84, 1989.

[Bru] Farrell Brumley. Effective Multiplicity One for GL_m. *arXiv:math.NT/0306052*.

[GLS] Stephen Gelbart, Erez Lapid, and Peter Sarnak. A new method for lower bounds of L-functions. *C. R. Acad. Sci. Paris Sér. I Math.*, to appear.

[GS88] Stephen Gelbart and Freydoon Shahidi. *Analytic properties of automorphic L-functions*, volume 6 of *Perspectives in Mathematics*. Academic Press Inc., Boston, MA, 1988.

[GS01] Stephen Gelbart and Freydoon Shahidi. Boundedness of automorphic L-functions in vertical strips. *J. Amer. Math. Soc.*, 14(1):79–107 (electronic), 2001.

[HL94] Jeffrey Hoffstein and Paul Lockhart. Coefficients of Maass forms and the Siegel zero. *Ann. of Math. (2)*, 140(1):161–181, 1994. With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman.

[HR95] Jeffrey Hoffstein and Dinakar Ramakrishnan. Siegel zeros and cusp forms. *Internat. Math. Res. Notices*, (6):279–308, 1995.

[JS81] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic representations. I. *Amer. J. Math.*, 103(3):499–558, 1981.

[JS77] Hervé Jacquet and Joseph A. Shalika. A non-vanishing theorem for zeta functions of GL_n. *Invent. Math.*, 38(1):1–16, 1976/77.

[Kim02] Henry Kim. On local L-functions and normalized intertwining opeartors. *Preprint*, 2002.

[Kim03] Henry H. Kim. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. *J. Amer. Math. Soc.*, 16(1):139–183 (electronic), 2003. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

[KS02a] Henry H. Kim and Freydoon Shahidi. Cuspidality of symmetric powers with applications. *Duke Math. J.*, 112(1):177–197, 2002.

[KS02b] Henry H. Kim and Freydoon Shahidi. Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and the symmetric cube for GL_2. *Ann. of Math. (2)*, 155(3):837–893, 2002. With an appendix by Colin J. Bushnell and Guy Henniart.

[Lan70] R. P. Langlands. Problems in the theory of automorphic forms. In *Lectures in modern analysis and applications, III*, pages 18–61. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[Lan71] Robert P. Langlands. *Euler products*. Yale University Press, New Haven, Conn., 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1967, Yale Mathematical Monographs, 1.

[Lan94] Serge Lang. *Algebraic number theory*. Springer-Verlag, New York, second edition, 1994.

[Lap] Erez Lapid. On the spectral expansion of Jacquet's relative trace formula. *preprint*.

[Mor85] Carlos J. Moreno. Analytic proof of the strong multiplicity one theorem. *Amer. J. Math.*, 107(1):163–206, 1985.

[Mül89] Werner Müller. The trace class conjecture in the theory of automorphic forms. *Ann. of Math. (2)*, 130(3):473–529, 1989.

[Mül00] W. Müller. On the singularities of residual intertwining operators. *Geom. Funct. Anal.*, 10(5):1118–1170, 2000.

[MW95] C. Mœglin and J.-L. Waldspurger. *Spectral decomposition and Eisenstein series.* Cambridge University Press, Cambridge, 1995. Une paraphrase de l'Écriture [A paraphrase of Scripture].

[Sar98] Peter Sarnak. L-functions. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, number Extra Vol. I, pages 453–465 (electronic), 1998.

[Sar01] P. Sarnak. A letter to S. Gelbart and F. Shahidi. 2001.

[Sar03] P. Sarnak. Non-vanishing of L-functions on $\mathrm{Re}(s) = 1$. *A Supplemental volume to the Amer. J. Math.*, Contributions to Automorphic forms, Geometry and Number Theory: Shalikafest 2002, 2003.

[Sha78] Freydoon Shahidi. Functional equation satisfied by certain L-functions. *Compositio Math.*, 37(2):171–207, 1978.

[Sha81] Freydoon Shahidi. On certain L-functions. *Amer. J. Math.*, 103(2):297–355, 1981.

[Sha88] Freydoon Shahidi. On the Ramanujan conjecture and finiteness of poles for certain L-functions. *Ann. of Math. (2)*, 127(3):547–584, 1988.

[Sha90] Freydoon Shahidi. A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups. *Ann. of Math. (2)*, 132(2):273–330, 1990.

[Tit32] E. C. Titchmarsh. *The theory of functions.* Oxford, Clarendon Press. , 1932.

[Wal92] Nolan R. Wallach. *Real reductive groups. II*, volume 132 of *Pure and Applied Mathematics.* Academic Press Inc., Boston, MA, 1992.

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