RECIPROCITY ALGEBRAS AND BRANCHING FOR CLASSICAL SYMMETRIC PAIRS

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Abstract. We study branching laws for a classical group G and a symmetric subgroup H. Our approach is through the *branching algebra*, the algebra of covariants for H in the regular functions on the natural torus bundle over the flag manifold for G. We give concrete descriptions of (natural subalgebras of) the branching algebra using classical invariant theory. In this context, it turns out that the ten classes of classical symmetric pairs (G, H) are associated in pairs, (G, H) and (H', G'), and that the (partial) branching algebra for (G, H) also describes a branching law from H' to G'. (However, the second branching law may involve certain infinite-dimensional highest weight modules for H'.) To highlight the fact that these algebras describe two branching laws simultaneously, we call them *reciprocity algebras*. Our description of the reciprocity algebras reveals that they all are related to the tensor product algebra for GL_n. This relation is especially strong in the *stable range*. We give quite explicit descriptions of reciprocity algebras in the stable range in terms of the tensor product algebra for GL_n. This is the structure lying behind formulas for branching multiplicities in terms of Littlewood-Richardson coefficients.

1. Branching Algebras

A basic problem in the representation theory of compact Lie groups (mutatis mutandis reductive linear algebraic groups over $\mathbb{C}$) is to describe *branching laws* - how irreducible representations of a given group decomposes on restriction to a subgroup. Among the most important cases of branching laws are those for which the group G and the subgroup H form a *symmetric pair,* which means that H is the fixed point subgroup of an involutive (order two) automorphism of G. Since the diagonal subgroup $\Delta(G)$ of the product $G \times G$ of a group with itself is the fixed point set of the involution which exchanges the two copies of G, the problem of describing branching laws for symmetric pairs includes the problem of decomposing tensor products of representations.

Considerable work has been done to describe branching laws, including application of Weyl's character formula [Kn1] [Kn2], methods using Young diagrams [Ful] [JK] [Mac], approaches involving standard monomials [LL] [LS], and path models [Lit]. Here we consider another approach to these problems, one which exploits the fact that the representations have a natural product structure, embodied by the algebra of regular functions on the natural torus bundle over the flag manifold of the group.

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Following [Zhe], we propose to study the covariant algebra (the algebra of highest weight vectors) for a subgroup acting on this ring.

More precisely, for a reductive complex linear algebraic G, let U_G be a maximal unipotent subgroup of G. The group U_G is determined up to conjugacy in G [Bor]. Let A_G denote a maximal torus which normalizes U_G, so that $B_G = A_G \cdot U_G$ is a Borel subgroup of G. Also let $\widehat{A}_G^+$ be the set of dominant characters of A_G – the semigroup of highest weights of representations of G. It is well-known [Bor] [Ho4] and may be thought of as a geometric version of the theory of the highest weight, that the space of regular functions on the coset space G/U_G, denoted by $\mathcal{R}(G/U_G)$, decomposes (under the action of G by left translations) as a direct sum of one copy of each irreducible representation V_ψ (with highest weight ψ) of G (see [Tow]):

$$\mathcal{R}(G/U_G) \simeq \bigoplus_{\psi \in \widehat{A}_G^+} V_\psi. \tag{1.1}$$

We note that $\mathcal{R}(G/U_G)$ has the structure of an $\widehat{A}_G^+$-graded algebra, for which the V_ψ are the graded components. To be specific, we note that since A_G normalizes U_G, it acts on G/U_G by right translations, and this action commutes with the action of G by left translations. The following result is well-known, probably folklore. We provide the proof anyway, since we are not able to find a suitable reference.

Proposition 1.1. *The algebra of regular functions $\mathcal{R}(G/U_G)$ is an $\widehat{A}_G^+$-graded algebra, under the right action of A_G. More precisely, the decomposition (1.1) is the graded algebra decomposition under A_G, where V_ψ is the A_G-eigenspace corresponding to $\phi \in \widehat{A}_G^+$ with $\phi = w^*(\psi^{-1})$. Here w is the longest element of the Weyl group with respect to the root system determined by the Borel subgroup B_G.*

Proof. Since A_G is commutative and reductive, we can decompose $\mathcal{R}(G/U_G)$ into eigenspaces for the right action of A_G. These eigenspaces must be invariant under the action of G by left translations. Since V_ψ is irreducible for the action of G, it must belong to a single A_G eigenspace. Let f be the highest weight vector for B_G in V_ψ. Then by definition $L_a(f) = \psi(a)(f)$, where $a \in A_G$ and L_g indicates left translation by g: $L_g(f)(h) = f(g^{-1}h)$ for any h in G, representing a point in G/U_G. If w is the longest element of the Weyl group, then the B_G orbit $B_G w U_G$ is dense in G/U_G, so that f is determined by its restriction to $B_G w$. We compute that

$$\psi(a)f(w) = L_a(f)(w) = f(a^{-1}w) = f(ww^{-1}a^{-1}w) = R_{w^{-1}a^{-1}w}f(w),$$

where $R_g f(h) = f(hg)$ refers to the right translation of f by g. If V_ψ belongs to the ϕ eigenspace for the right action of A_G, then this equation implies that

$$\psi(a) = \phi(w^{-1}a^{-1}w), \qquad \text{or} \qquad \psi = w^*(\phi^{-1}), \tag{1.2}$$

where w^* indicates the action of w on $\widehat{A}_G$ resulting from conjugation of A_G by w. Thus, the V_ψ are exactly the eigenspaces for the right action of A_G, with the A_G-eigencharacter related to the highest weight by the equation (1.2). Since the right action by A_G (as well as the left action by G) is an action by algebra automorphisms of $\mathcal{R}(G/U_G)$, it is easy to check that if f_1 is a ϕ-eigenfunction for A_G, and f_2 is a θ-eigenfunction, then the product $f_1 f_2$ is a $\phi\theta$-eigenfunction for A_G. It follows that

$$V_\psi V_\chi = V_{\psi\chi},$$

so that, indeed, the decomposition (1.1) defines a structure of an $\widehat{A}_G^+$-graded algebra on $\mathcal{R}(G/U_G)$. $\square$

Now let $H \subset G$ be a reductive subgroup, and let U_H be a maximal unipotent subgroup of H. We consider the algebra $\mathcal{R}(G/U_G)^{U_H}$, of functions on G/U_G which are invariant under left translations by U_H. Let A_H be a maximal torus of H normalizing U_H, so that $B_H := A_H \cdot U_H$ is a Borel subgroup of H. Then $\mathcal{R}(G/U_G)^{U_H}$ will be invariant under the (left) action of A_H, and we may decompose $\mathcal{R}(G/U_G)^{U_H}$ into eigenspaces for A_H. Since the functions in $\mathcal{R}(G/U_G)^{U_H}$ are by definition (left) invariant under U_H, the (left) A_H-eigenfunctions will in fact be (left) B_H eigenfunctions. In other words, they are highest weight vectors for H. Hence, the characters of A_H acting on (the left of) $\mathcal{R}(G/U_G)^{U_H}$ will all be dominant with respect to B_H, and we may write $\mathcal{R}(G/U_G)^{U_H}$ as a sum of (left) A_H eigenspaces $(\mathcal{R}(G/U_G)^{U_H})^\chi$ for dominant characters χ of H:

$$\mathcal{R}(G/U_G)^{U_H} = \bigoplus_{\chi \in \widehat{A}_H^+} (\mathcal{R}(G/U_G)^{U_H})^\chi. \tag{1.3}$$

Since the spaces V_ψ of decomposition (1.1) are (left) G-invariant, they are a fortiori (left) H-invariant, so we have a decomposition of $\mathcal{R}(G/U_G)^{U_H}$ into (left) A_G eigenspaces $(\mathcal{R}(G/U_G)^{U_H})_\psi$:

$$\mathcal{R}(G/U_G)^{U_H} = \bigoplus_{\psi \in \widehat{A}_G^+} \mathcal{R}(G/U_G)^{U_H} \cap V_\psi := \bigoplus_{\psi \in \widehat{A}_G^+} \mathcal{R}(G/U_G)_\psi^{U_H}.$$

Combining this decomposition with the decomposition (1.3), we may write

$$\mathcal{R}(G/U_G)^{U_H} = \bigoplus_{\psi \in \widehat{A}_G^+,\ \chi \in \widehat{A}_H^+} (\mathcal{R}(G/U_G)_\psi^{U_H})^\chi. \tag{1.4}$$

To emphasize the key features of this algebra, we note the resulting consequences of decomposition (1.4) in the following proposition.

Proposition 1.2. (a) *The decomposition (1.4) is an* $(\widehat{A}_G^+ \times \widehat{A}_H^+)$*-graded algebra decomposition of* $\mathcal{R}(G/U_G)^{U_H}$.

(b) *The subspaces* $(\mathcal{R}(G/U_G)_\psi^{U_H})^\chi$ *tell us the* χ *highest weight vectors for* B_H *in the irreducible representation* V_ψ *of* G. *Therefore, the decomposition*

$$\mathcal{R}(G/U_G)_\psi^{U_H} = \bigoplus_{\chi \in \widehat{A}_H^+} (\mathcal{R}(G/U_G)_\psi^{U_H})^\chi$$

tells us how V_ψ *decomposes as a* H*-module.*

Thus, knowledge of $\mathcal{R}(G/U_G)^{U_H}$ as a $(\widehat{A}_G^+ \times \widehat{A}_H^+)$-graded algebra tell us how representations of G decompose when restricted to H, in other words, it describes the branching rule from G to H. We will call $\mathcal{R}(G/U_G)^{U_H}$ the (G, H) *branching algebra.* When $G \simeq H \times H$, and H is embedded diagonally in G, the branching algebra describes the decomposition of tensor products of representations of H, and we then call it the *tensor product algebra* for H. More generally, we would like to understand the (G, H) branching algebras for symmetric pairs (G, H).

1.1. **Overview.** We have just seen the construction of a branching algebra. We shall illustrate how the algebra structure emphasizes the coherence of the branching problem across representations of G, rather than simply providing a representation-by-representation answer. Further, the algebra structure goes beyond the numerical nature of multiplicities, and is the basis of some of the well-known multiplicity results.

There are three main parts to this paper:

(a) **Background:** preliminary knowledge on multiplicity-free spaces, the theory of dual pairs and the classification of classical symmetric pairs, leading to the concept of reciprocity algebra. One can group the classical symmetric pairs into 10 families, and it turns out that these families are associated in pairs, (G, H) and (H', G') (see Tables I and II), and that the branching algebra for (G, H) also describes a branching law from H' to G'. Hence, we call the algebra that describes the two related branching laws a *reciprocity algebra.*

(b) **Reciprocity:** explaining the underpinnings of the reciprocity of multiplicities for the 5 pairs of reciprocity algebras, with detailed presentations for certain representative pairs.

(c) **Stability:** describing the branching algebra for a symmetric pair (G, K) in terms of a GL_n tensor product algebra. This emphasizes the underlying importance of the GL_n branching algebras, since their structure relate to all others. This phenomenon is also the basis for the expression of multiplicity formulas for representations of classical groups in terms of Littlewood-Richardson coefficients - a recurring theme in the literature. The present paper brings more structure to these multiplicity results.

The first three sections provide the necessary background:

§2: We provide the notations for the parametrization of representations in §2.1. Next, we discuss some preliminary concepts on multiplicity-free spaces and prove a main theorem (see Theorem 2.3) on embedding the associated graded algebra of a graded G-multiplicity-free space into the algebra $\mathcal{R}(G/U_G)$. The final subsection discusses the rudiments of classical invariant theory, formulated in terms of the theory of dual pairs.

§3: Section 3 begins with the simplest example of a pair of branching algebras for GL_n, leading us to the concept of reciprocity algebras. This is where we bring forth the classification of classical symmetric pairs and the theory of dual pairs to introduce the 5 pairs (also called *see-saw pairs*) of reciprocity algebras.

For the second part of the paper, we organize discussions of the see-saw pairs as follows:

§4: discusses the branching from GL_n to O_n (note that branching from GL_{2n} to Sp_{2n} can be treated similarly);
§5: discusses the tensor product algebra for O_n (note that the tensor product algebra of Sp_{2n} can be treated similarly);
§6: provides a more general treatment of the reciprocity for (GL_n, GL_m).

For the sake of brevity, we will not discuss the branching from $Sp_{2(m+n)}$ to $Sp_{2m} \times Sp_{2n}$, O_{2n} to GL_n and Sp_{2n} to GL_n. They can be treated similarly.

The final part of this paper is perhaps quite involved. We begin with some important comments on stability results for branching in §7. We demonstrate the theoretical underpinnings for stability results, highlighting certain specific see-saw pairs:

§8: we interpret the associated graded of the branching algebra from GL_n to O_n as a $(0,1)$-subalgebra (see Definition 2.1) of the tensor product algebra of GL_m in the stable range $n > 2m$.
§9: we interpret the associated graded of the O_n tensor product algebra as a $(0,1)$-subalgebra of a triple product of tensor product algebras of GL_n, GL_m and GL_ℓ in the stable range $n > 2(m+\ell)$.
§10: we interpret the branching algebra of O_{n+m} to $O_n \times O_m$ as a $(0,1)$-subalgebra of a triple tensor product algebra of GL_ℓ in the stable range $\min(n,m) > 2\ell$.

In all these cases, we show that the branching algebras associated to symmetric pairs can all be identified with suitable branching algebras associated to the general linear groups. Thus, if we can have control of the solution in the general linear group case, we will have some control of the other classical groups. And indeed, we do (see [HTW2]). The other non-trivial examples will be important extensions of this work, and we hope to see them in further papers, for example, [HL].

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2. Preliminaries and Notations

2.1. **Parametrization of Representations.** Let G be a classical reductive algebraic group over $\mathbb{C}$: $G = GL_n(\mathbb{C}) = GL_n$, the general linear group; or $G = O_n(\mathbb{C}) = O_n$, the orthogonal group; or $G = Sp_{2n}(\mathbb{C}) = Sp_{2n}$, the symplectic group. We shall explain our notations on irreducible representations of G using integer partitions. In each of these cases, we select a Borel subalgebra of the classical Lie algebra and coordinatize it, as is done in [GW]. Consequently, all highest weights are parameterized in the standard way (see [GW]).

A non-negative integer *partition* λ, with k parts, is an integer sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$. We may sometimes refer to λ as a *Young* or *Ferrers diagram*. We use the same notation for partitions as is done in [Mac]. For example, we write $\ell(\lambda)$ to denote the *length* (or *depth*) of a partition, i.e., $\ell(\lambda) = k$ for the above partition. Also let $|\lambda| = \sum_i \lambda_i$ be the size of a partition and λ' denote the *transpose* (or *conjugate*) of λ (i.e., $(\lambda')_i = |\{\lambda_j : \lambda_j \geq i\}|$).

$\mathbf{GL_n}$ Representations: Given non-negative integers p, q and n such that $n \geq p + q$ and non-negative integer partitions λ^+ and λ^- with p and q parts respectively, let $F_{(n)}^{(\lambda^+,\lambda^-)}$ denote the irreducible rational representation of GL_n with highest weight given by the n-tuple:

$$(\lambda^+, \lambda^-) \quad = \quad \underbrace{\left(\lambda_1^+, \lambda_2^+, \cdots, \lambda_p^+, 0, \cdots, 0, -\lambda_q^-, \cdots, -\lambda_1^-\right)}_{n}$$

If $\lambda^- = (0)$ then we will write $F_{(n)}^{\lambda^+}$ for $F_{(n)}^{(\lambda^+,\lambda^-)}$. Note that if $\lambda^+ = (0)$ then $\left(F_{(n)}^{\lambda^-}\right)^*$ is equivalent to $F_{(n)}^{(\lambda^+,\lambda^-)}$. More generally, $\left(F_{(n)}^{(\lambda^+,\lambda^-)}\right)^*$ is equivalent to $F_{(n)}^{(\lambda^-,\lambda^+)}$.

$\mathbf{O_n}$ Representations: The complex orthogonal group has two connected components. Because the group is disconnected we cannot index irreducible representations by highest weights. There is however an analog of Schur-Weyl duality for the case of O_n in which each irreducible rational representation is indexed uniquely by a non-negative integer partition ν such that $(\nu')_1 + (\nu')_2 \leq n$. That is, the sum of the first two columns of the Young diagram of ν is at most n. We will call such a diagram O_n-admissible (see [GW] Chapter 10 for details). Let $E_{(n)}^{\nu}$ denote the irreducible representation of O_n indexed ν in this way.

The irreducible rational representations of SO_n may be indexed by their highest weight, since the group is a connected reductive linear algebraic group. In [GW] Section 5.2.2, the irreducible representations of O_n are determined in terms of their restrictions to SO_n (which is a normal subgroup having index 2). We note that if $l(\nu) \neq \frac{n}{2}$, then the restriction of $E_{(n)}^{\nu}$ to SO_n is irreducible. If $l(\nu) = \frac{n}{2}$ (n even), then $E_{(n)}^{\nu}$ decomposes into exactly two irreducible representations of SO_n. See [GW] Section 10.2.4 and 10.2.5 for the correspondence between this parametrization and the above parametrization by partitions.

The determinant defines an (irreducible) one-dimensional representation of O_n. This representation is indexed by the partition $\zeta = (1, 1, \cdots, 1)$. An irreducible representation of O_n will remain irreducible when tensored by $E_{(n)}^{\zeta}$, but the resulting representation *may* be inequivalent to the initial representation. We say that a pair of O_n-admissible partitions α and β are *associate* if $E_{(n)}^{\alpha} \otimes E_{(n)}^{\zeta} \cong E_{(n)}^{\beta}$. It turns out that α and β are associate exactly when $(\alpha')_1 + (\beta')_1 = n$ and $(\alpha')_i = (\beta')_i$ for all $i > 1$. This relation is clearly symmetric, and is related to the structure of the underlying

SO_n-representations. Indeed, when restricted to SO_n, $E^{\alpha}_{(n)} \cong E^{\beta}_{(n)}$ if and only if α and β are either associate or equal.

$\mathbf{Sp_{2n}}$ Representations: For a non-negative integer partition ν with p parts where $p \leq n$, let $V^{\nu}_{(2n)}$ denote the irreducible rational representation of Sp_{2n} where the highest weight indexed by the partition ν is given by the n tuple:

$$\underbrace{(\nu_1, \nu_2, \cdots, \nu_p, 0, \cdots, 0)}_{n}.$$

2.2. **Multiplicity-Free Actions.** Let G be a complex reductive algebraic group acting on a complex vector space V. We say V is a *multiplicity-free action* if the algebra $\mathcal{P}(V)$ of polynomial functions on V is multiplicity free as a G module. The criterion of Servedio-Vinberg [Ser] [Vin] says that V is multiplicity free if and only if a Borel subgroup B of G has a Zariski open orbit in V. In other words, B (and hence G) acts prehomogeneously on V (see [SK]). A direct consequence is that B eigenfunctions in $\mathcal{P}(V)$ have a very simple structure. Let $Q_\psi \in \mathcal{P}(V)$ be a B eigenfunction with eigencharacter ψ, normalized so that $Q_\psi(v_0) = 1$ for some fixed v_0 in a Zariski open B orbit in V. Then Q_ψ is completely determined by ψ: For $v = b^{-1}v_0$ in the Zariski open B orbit,

$$Q_\psi(v) = Q_\psi(b^{-1}v_0) = \psi(b)Q_\psi(v_0) = \psi(b), \qquad b \in B.$$

Q_ψ is then determined on all of V by continuity. Since $B = AU$, and $U = (B, B)$ is the commutator subgroup of B, we can identify a character of B with a character of A. Thus the B eigenfunctions are precisely the G highest weight vectors (with respect to B) in $\mathcal{P}(V)$. Further

$$Q_{\psi_1} Q_{\psi_2} = Q_{\psi_1 \psi_2}$$

and so the set of $\widehat{A}^+(V) = \{\psi \in \widehat{A}^+ \mid Q_\psi \neq 0\}$ forms a sub-semigroup of the cone $\widehat{A}^+$ of dominant weights of A.

An element $\psi (\neq 1)$ of a semigroup is *primitive* if it is not expressible as a non-trivial product of two elements of the semigroup. The algebra $P(V)^U$ has unique factorization (see [HU]). The eigenfunctions associated to the primitive elements of $\widehat{A}^+(V)$ are prime polynomials, and $P(V)^U$ is the polynomial ring on these eigenfunctions. If $\psi = \psi_1\psi_2$, then $Q_\psi = Q_{\psi_1} Q_{\psi_2}$. Thus, if ψ is not primitive, then the polynomial Q_ψ cannot be prime. An element

$$\psi = \Pi_{j=1}^{k} \psi_j^{c_j}$$

has c_j's uniquely determined, and hence the prime factorization

$$Q_\psi = \Pi_{j=1}^{k} Q_{\psi_j}^{c_j}.$$

Consider a multiplicity-free action of G on an algebra $\mathcal{W}$. In the general situation, we would like to associate this algebra $\mathcal{W}$ with a subalgebra of $\mathcal{R}(G/U)$. With this goal in mind, we introduce the following notion:

Definition 2.1. *Let $\mathcal{P} = \bigoplus_{\lambda \in \widehat{A}^+} \mathcal{P}_\lambda$ denote an algebra graded by an abelian semigroup $\widehat{A}^+$. If $\mathcal{W} \subseteq \mathcal{P}$ is a subalgebra of $\mathcal{P}$, then we say that $\mathcal{W}$ is a* $(0,1)$**-subalgebra of** $\mathcal{P}$ *if*

$$\mathcal{W} = \bigoplus_{\lambda \in Z} \mathcal{P}_\lambda$$

where Z is a sub-semigroup of $\widehat{A}^+$. Note that $\mathcal{W}$ is graded by Z.

In what is to follow, we will usually have $\mathcal{P} = \mathcal{P}(V)$ (polynomial function on a representation V) and $\widehat{A}^+$ will denote the dominant chamber of the character group of a maximal torus, A, acting on V.

Consider $\mathcal{W}$ which is a G-invariant and G-multiplicity-free subalgebra of a polynomial algebra $\mathcal{P}(V)$. Suppose that $\mathcal{W}^U$ has unique factorization. Then $\widehat{A}^+(\mathcal{W})$ is a sub-semigroup in $\widehat{A}^+$ generated by $P\widehat{A}^+(\mathcal{W})$. (We have abused the notation for $P\widehat{A}^+(V)$ here.)

Write the G decomposition as follows:

$$\mathcal{W} = \bigoplus_{\psi \in \widehat{A}^+(\mathcal{W})} \mathcal{W}_\psi$$

noting that $\mathcal{W}_\psi$ is an irreducible G module with highest weight ψ. Introduce an A^+ filtration on $\mathcal{W}$ as follows:

$$\mathcal{W}^{(\psi)} = \bigoplus_{\phi \leq \psi} \mathcal{W}_\phi$$

where the ordering $\leq$ is the ordering on $\widehat{A}^+$ given by (see [Pop])

$$\psi_1 \leq \psi_2 \qquad \text{if } \psi_1^{-1}\psi_2 \text{ is expressible as a product of rational powers of positive roots.}$$

Note that positive roots are weights of the adjoint representation of G on its Lie algebra $\mathfrak{g}$. We refer to the abelian group structure on the integral weights multiplicatively. Note also that for our purposes, we only need positive *integer* powers of the positive roots.

If δ occurs with positive multiplicity in the tensor product decomposition

$$\mathcal{W}_\phi \otimes \mathcal{W}_\psi = \bigoplus_\delta \dim Hom_G(\mathcal{W}_\delta, \mathcal{W}_\phi \otimes \mathcal{W}_\psi)\ \mathcal{W}_\delta,$$

then $\delta \leq \phi\psi$. If

$$\mathcal{W}_\eta \subset \mathcal{W}^{(\phi)} \quad \text{and} \quad \mathcal{W}_\gamma \subset \mathcal{W}^{(\psi)}, \quad \text{i.e., } \eta \leq \phi \quad \text{and} \quad \gamma \leq \psi,$$

then it follows that

$$\mathcal{W}_\eta \cdot \mathcal{W}_\gamma \hookrightarrow \mathcal{W}_\eta \otimes \mathcal{W}_\gamma \subset \mathcal{W}^{(\eta\gamma)} \subset \mathcal{W}^{(\phi\psi)}.$$

Thus

$$\mathcal{W}^{(\phi)} \cdot \mathcal{W}^{(\psi)} \subset \mathcal{W}^{(\phi\psi)}.$$

We have now an A^+-filtered algebra

$$\mathcal{W} = \bigoplus_{\psi \in \widehat{A}^+(\mathcal{W})} \mathcal{W}^{(\psi)},$$

and this filtration is known as the *dominance filtration* [Pop].

With a filtered algebra, we can form its associated algebra which is $\widehat{A}^+$ graded:

$$\mathrm{Gr}_{\widehat{A}^+}\mathcal{W} = \bigoplus_{\psi \in \widehat{A}^+(\mathcal{W})} (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\psi}$$

where

$$(\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\psi} = \mathcal{W}^{(\psi)} / \left(\bigoplus_{\phi < \psi} \mathcal{W}^{(\phi)} \right).$$

Theorem 2.2. *Consider a multiplicity-free G-module $\mathcal{W}$ with a A^+-filtered algebra structure such that $\mathcal{W}$ is a unique factorization domain. Assume that the zero degree subspace of $\mathcal{W}$ is $\mathbb{C}$. Then there is a canonical $\widehat{A}^+$-graded algebra injection:*

$$Gr_{\widehat{A}^+}\pi \ : \ Gr_{\widehat{A}^+}\mathcal{W} \hookrightarrow \mathcal{R}(G/U).$$

Proof. In [Ho4] it is shown that under the above hypothesis, $\mathcal{W}^U$ is a polynomial ring on a canonical set of generators. Now, $\mathcal{W}^U$ is a $\widehat{A}^+$-graded algebra and therefore, there exists an injective $\widehat{A}^+$-graded algebra homomorphism:

$$\alpha \ : \ \mathcal{W}^U \hookrightarrow \mathcal{R}(G/U)^U.$$

So $\mathcal{W}^U = \mathrm{Gr}_{\widehat{A}^+}(\mathcal{W}^U) = (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^U$.

There exists a unique G-module homomorphism $\overline{\alpha} : \mathrm{Gr}_{\widehat{A}^+}\mathcal{W} \hookrightarrow \mathcal{R}(G/U)$ such that the following diagram commutes:

$$\begin{array}{cccc} \alpha : & \mathcal{W}^U & \hookrightarrow & \mathcal{R}(G/U)^U \\ & \cap & & \cap \\ \overline{\alpha} : & \mathrm{Gr}_{\widehat{A}^+}\mathcal{W} & \hookrightarrow & \mathcal{R}(G/U) \end{array}$$

We wish to show that $\overline{\alpha}$ is an algebra homomorphism, i.e.,

$$\begin{array}{ccccc} (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\lambda} & \times & (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\mu} & \underset{m_{\mathcal{W}}}{\longrightarrow} & (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\lambda+\mu} \\ \overline{\alpha}\downarrow & & \overline{\alpha}\downarrow & & \overline{\alpha}\downarrow \\ \mathcal{R}(G/U)^{\lambda} & \times & \mathcal{R}(G/U)^{\mu} & \underset{m_{\mathcal{R}(G/U)}}{\longrightarrow} & \mathcal{R}(G/U)^{\lambda+\mu} \end{array}$$

commutes.

We have two maps:

$$f_i : (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\lambda} \otimes (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\mu} \to \mathcal{R}(G/U)^{\lambda+\mu}, \qquad i = 1, 2,$$

defined by: $f_1(v \otimes w) = m_{\mathcal{R}(G/U)}(\overline{\alpha}(v) \otimes \overline{\alpha}(w))$ and $f_2(v \otimes w) = \overline{\alpha}(m_{\mathcal{W}}(v \otimes w))$.

Each of f_1 and f_2 is G-equivariant and,

$$\dim \left\{ \beta \;\middle|\; \beta : (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\lambda} \otimes (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\mu} \to \mathcal{R}(G/U)^{\lambda+\mu} \right\} = 1$$

because the Cartan product has multiplicity one in the tensor product of two irreducible G-modules V_λ and V_μ [Pop].

Therefore, there exists a constant C such that $f_1 = Cf_2$. We know that $\overline{\alpha}|_{\mathcal{W}^U} = \alpha$ is an algebra homomorphism. So for highest weight vectors $v^\lambda \in \mathcal{W}_\lambda^U$ and $w^\mu \in \mathcal{W}_\mu^U$:

$$f_1(v^\lambda \otimes w^\mu) = \overline{\alpha}(v^\lambda)\overline{\alpha}(w^\mu) = \alpha(v^\lambda)\alpha(w^\mu) = \alpha(v^\lambda w^\mu) = \overline{\alpha}(v^\lambda w^\mu) = f_2(v^\lambda \otimes w^\mu).$$

(Note that $v^\lambda w^\mu$ is a highest weight vector.) So $C = 1$. □

2.3. **Dual Pairs and Duality Correspondence.** In our context, the theory of dual pairs may be cast in a purely algebraic language. In this section, we will describe three dual pairs $(K, \mathfrak{g})$, where K is a classical linear algebraic group defined over $\mathbb{C}$ and $\mathfrak{g}$ is a complex classical Lie algebra. In each case, we have a linear action of K on a finite dimensional complex vector space V, which is a finite sum of copies of the standard module for K or copies of the dual of the standard module for K. This action induces an action on the complex valued polynomial functions on V, upon which $\mathfrak{g}$ acts by polynomial coefficient differential operators. The actions of $\mathfrak{g}$ and K commute with each other. Furthermore, the algebra of polynomial coefficient differential operators which commute with the K-action (resp. $\mathfrak{g}$-action) on $\mathcal{P}(V)$ is generated as an algebra by the image of the $\mathfrak{g}$-action (resp. K-action). In light of this situation, we may regard $\mathcal{P}(V)$ as a representation of $\mathfrak{g}$ and K simultaneously. Theorem 2.4 describes, in part, the (multiplicity-free) decomposition of $\mathcal{P}(V)$ into irreducible modules for the joint action.

Of particular importance are the K-invariants in $\mathcal{P}(V)$. In each case, we may describe this invariant ring through the action of $\mathfrak{g}$. Indeed, $\mathfrak{g}$ may be decomposed into three subspaces denoted $\mathfrak{g}^{(2,0)}$, $\mathfrak{g}^{(1,1)}$ and $\mathfrak{g}^{(0,2)}$. Theorem 2.3 asserts that $\mathcal{P}(V)^K$ is generated as an algebra by $\mathfrak{g}^{(2,0)}$. Moreover, within a certain stable range (described in Theorem 2.3), $\mathcal{P}(V)^K$ is isomorphic to $\mathcal{S}(\mathfrak{g}^{(2,0)})$, the full symmetric algebra on $\mathfrak{g}^{(2,0)}$. These facts are the celebrated *fundamental theorems of classical invariant theory* (see [GW], [Wey]).

Note that at the same time, we obtain an action of K (by conjugation) on the constant coefficient differential operators on $\mathcal{P}(V)$, denoted $\mathcal{D}(V)$. In turn, the K-invariant subalgebra, $\mathcal{D}(V)^K$ is generated by $\mathfrak{g}^{(0,2)}$. This brings us to our next ingredient. Define the *K-harmonic polynomials* to be:

$$\mathcal{H} := \left\{ f \in \mathcal{P}(V) \mid \Delta f = 0 \text{ for all } \Delta \in \mathcal{D}(V)^K \right\}.$$

For each dual pair, we have a surjection, $\mathcal{P}(V)^K \otimes \mathcal{H} \xrightarrow{m} \mathcal{P}(V)$ defined by multiplication. Note that we may regard $\mathcal{P}(V)$ as a module over the algebra $\mathcal{P}(V)^K$. By definition, this module is free iff m is injective. Within a certain *stable range*, m is indeed injective, and this range is indicated as part of Theorem 2.3. We have provided a proof of the injectivity part of this theorem (also known as the Separation of Variables Theorem) as an appendix of this paper.

For each $(\mathfrak{g}, K)$, the subspace $\mathfrak{g}^{(1,1)}$ is *isomorphic* to the Lie algebra of a subgroup $G^{(1,1)} \subseteq GL(V)$ which commutes with the action of K. In Theorem 2.5 we describe the action of this group. Note that, in general, the differential of the action of $G^{(1,1)}$ on $\mathcal{P}(V)$ is not the same as the action of $\mathfrak{g}^{(1,1)}$, but differs only by a central shift. Under the joint action of $K \times G^{(1,1)}$, $\mathcal{H}$ is a multiplicity-free invariant subspace of $\mathcal{P}(V)$. The precise decomposition of $\mathcal{H}$ is provided in Theorem 2.5.

Finally, in each of the three dual pair settings we have $\mathcal{P}(V) = I(\mathcal{J}^+) \oplus \mathcal{H}$, where $I(\mathcal{J}^+)$ is the ideal in $\mathcal{P}(V)$ generated by $\mathfrak{g}^{(2,0)}$ (which is the same as the ideal generated by the homogeneous invariants of positive degree). We note that the natural map:

$$\mathcal{H} \longrightarrow \mathcal{P}(V)/I(\mathcal{J}^+) \tag{2.3.1}$$

is a linear isomorphism of representations.

Details in this section including all theorems stated can be found in [GW], [Ho2] or [Ho4].

2.3.1. *Definitions of the Three Dual Pair Actions.* We now describe the three dual pairs in detail as well as state Theorems 2.3, 2.4 and 2.5 on a case-by-case basis. For the following, we let $M_{n,m}$ be the complex vector space of n by m matrices. We shall select a coordinate system $\{x_{ij} | i = 1, \cdots, n,\ j = 1, \cdots, m\}$.

CASE A: $(\mathbf{O_n}, \mathfrak{sp}_{\mathbf{2m}})$ where $\mathbf{V} := \mathbf{M_{n,m}}$.

Define the following differential operators:

$$\Delta_{ij} := \textstyle\sum_{s=1}^{n} \frac{\partial^2}{\partial x_{si} \partial x_{sj}}, \qquad r_{ij}^2 := \sum_{s=1}^{n} x_{si} x_{sj}, \text{ and } \qquad E_{ij} := \sum_{s=1}^{n} x_{si} \frac{\partial}{\partial x_{sj}}.$$

We define three spaces:

$$\begin{aligned} \mathfrak{sp}_{2m}^{(1,1)} &:= \text{Span } \left\{ E_{ij} + \tfrac{k}{2}\, \delta_{i,j} \mid i, j = 1, \ldots, m \right\}, \\ \mathfrak{sp}_{2m}^{(2,0)} &:= \text{Span } \left\{ r_{ij}^2 \mid 1 \le i \le j \le m \right\}, \quad \text{and} \\ \mathfrak{sp}_{2m}^{(0,2)} &:= \text{Span } \left\{ \Delta_{ij} \mid 1 \le i \le j \le m \right\}. \end{aligned}$$

The direct sum, $\mathfrak{g} := \mathfrak{sp}_{2m}^{(2,0)} \oplus \mathfrak{sp}_{2m}^{(1,1)} \oplus \mathfrak{sp}_{2m}^{(0,2)}$, is preserved under the usual operator bracket and is isomorphic, as a Lie algebra, to the rank m symplectic Lie algebra, $\mathfrak{sp}_{2m}$. This presentation defines an action of $\mathfrak{sp}_{2m}$ on $\mathcal{P}(M_{n,m})$. The O_n action is defined by multiplication on the left: for $g \in O_n$ and $f \in \mathcal{P}(M_{n,m})$ we set $g \cdot f(x) = f(g^t x)$ for all $x \in M_{n,m}$.

CASE B: $(\mathbf{Sp_{2n}}, \mathfrak{so}_{\mathbf{2m}})$ where $\mathbf{V} = \mathbf{M_{2n,m}}$.
Define the following differential operators:

$$D_{ij} := \sum_{s=1}^{n} \left(\frac{\partial^2}{\partial x_{si} \partial x_{s+n,j}} - \frac{\partial^2}{\partial x_{s+n,i} \partial x_{sj}} \right), \qquad S_{ij}^2 := \sum_{s=1}^{n} (x_{si} x_{s+n,j} - x_{s+n,i} x_{sj}), \text{ and}$$
$$E_{ij} := \sum_{s=1}^{2n} x_{si} \frac{\partial}{\partial x_{sj}}.$$

We define three spaces:

$$\mathfrak{so}_{2m}^{(1,1)} := \text{Span } \{E_{ij} + k\,\delta_{i,j} \mid i,j = 1, \ldots, m\},$$
$$\mathfrak{so}_{2m}^{(2,0)} := \text{Span } \left\{S_{ij}^2 \mid 1 \le i < j \le m\right\}, \qquad \text{and}$$
$$\mathfrak{so}_{2m}^{(0,2)} := \text{Span } \{D_{ij} \mid 1 \le i < j \le m\}.$$

The direct sum, $\mathfrak{g} := \mathfrak{so}_{2m}^{(2,0)} \oplus \mathfrak{so}_{2m}^{(1,1)} \oplus \mathfrak{so}_{2m}^{(0,2)}$, is isomorphic to $\mathfrak{so}_{2m}$, the rank m orthogonal Lie algebra of type D, and this presentation defines an action of $\mathfrak{so}_{2m}$ on $\mathcal{P}(M_{2n,m})$. For $g \in Sp_{2n}$ and $f \in \mathcal{P}(M_{2n,m})$, we set $g \cdot f(x) = f(g^t x)$ for all $x \in M_{2n,m}$.

CASE C: $(\mathbf{GL_n}, \mathfrak{gl}_{\mathbf{m}+\ell})$ where $\mathbf{V} = \mathbf{M_{n,m}} \oplus \mathbf{M_{\ell,n}}$.
Let $\{x_{ab}\}$ and $\{y_{cd}\}$ be the coordinates on $M_{n,m}$ and $M_{\ell,n}$ respectively. Define the following differential operators:

$$\overline{\Delta}_{ij} := \sum_{s=1}^{n} \frac{\partial^2}{\partial x_{si} \partial y_{js}}, \quad \overline{r}_{ij}^2 := \sum_{s=1}^{n} x_{si} y_{js}, \quad E_{ij}^X := \sum_{s=1}^{n} x_{si} \frac{\partial}{\partial x_{sj}}, \quad \text{and} \quad E_{ij}^Y := \sum_{s=1}^{n} y_{is} \frac{\partial}{\partial y_{js}}.$$

(In the above, i and j range over the appropriate interval defined by the sizes of the matrices.) We define three spaces:

$$\mathfrak{gl}_{m,\ell}^{(1,1)} := \text{Span } \left\{E_{ij}^X + \tfrac{n}{2}\,\delta_{i,j} \mid i,j = 1, \ldots, m\right\} \oplus \text{Span } \left\{E_{ij}^Y + \tfrac{n}{2}\,\delta_{i,j} \mid i,j = 1, \ldots, \ell\right\},$$
$$\mathfrak{gl}_{m,\ell}^{(2,0)} := \text{Span } \left\{\overline{r}_{ij}^2 \mid i = 1, \ldots, m, j = 1, \ldots, \ell\right\}, \qquad \text{and}$$
$$\mathfrak{gl}_{m,\ell}^{(0,2)} := \text{Span } \left\{\overline{\Delta}_{ij} \mid i = 1, \ldots, m, j = 1, \ldots, \ell\right\}.$$

The direct sum, $\mathfrak{g} := \mathfrak{gl}_{m,\ell}^{(2,0)} \oplus \mathfrak{gl}_{m,\ell}^{(1,1)} \oplus \mathfrak{gl}_{m,\ell}^{(0,2)}$, is isomorphic to the rank $m + \ell$ general linear Lie algebra $\mathfrak{gl}_{m+\ell}$, and this presentation defines an action of $\mathfrak{gl}_{m+\ell}$ on $\mathcal{P}(M_{n,m} \oplus M_{\ell,n})$. For $g \in GL_n$ and $f \in \mathcal{P}(M_{n,m} \oplus M_{\ell,n})$, we set $g \cdot f(x,y) = f(g^t x, yg)$ for all $x \in M_{n,m}$, $y \in M_{\ell,n}$.

2.3.2. *Theorems on the Invariants, Decompositions and Harmonics.* Let SM_m and AM_m be the space of symmetric and anti-symmetric m by m matrices respectively. If V is a vector space, we denote the symmetric algebra on V by $\mathcal{S}(V)$. Note that in each of the dual pairs, we have defined the action of K on $\mathcal{P}(V)$ so that $\mathcal{P}(V) \cong \mathcal{S}(V)$ as K modules. (This is in contrast with the usual identification $\mathcal{P}(V) \cong \mathcal{S}(V^*)$.) Also, for a set S, we shall denote by $\mathbb{C}[S]$ by the algebra generated by elements in the set S.

Theorem 2.3. *(First Fundamental Theorem of Invariant Theory and Separation of Variables)*

(a) CASE A: $(\mathbf{O_n}, \mathfrak{sp}_{\mathbf{2m}})$ *The invariants*

$$\mathcal{J}_{n,m} := \mathcal{P}(M_{n,m})^{O_n} = \mathbb{C}[r_{ij}^2] \qquad \left(\cong \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)}) \cong \mathcal{P}(SM_m) \text{ if } n \geq m\right).$$

Let $\mathcal{H}_{n,m} \subseteq \mathcal{P}(M_{n,m})$ denote the O_n-harmonics. Further, if $n \geq 2m$, we have separation of variables

$$\mathcal{P}(M_{n,m}) \simeq \mathcal{H}_{n,m} \otimes \mathcal{J}_{n,m}.$$

(b) CASE B: $(\mathbf{Sp_{2n}}, \mathfrak{so}_{\mathbf{2m}})$ *The invariants*

$$\overline{\mathcal{J}}_{2n,m} := \mathcal{P}(M_{2n,m})^{Sp_{2n}} = \mathbb{C}[S_{ij}^2] \qquad \left(\cong \mathcal{S}(\mathfrak{so}_{2m}^{(2,0)}) \cong \mathcal{P}(AM_m) \text{ if } n \geq m\right).$$

Let $\overline{\mathcal{H}}_{2n,m} \subseteq \mathcal{P}(M_{2n,m})$ denote the Sp_{2n}-harmonics. Further, if $n \geq m$, we have separation of variables

$$\mathcal{P}(M_{2n,m}) \simeq \overline{\mathcal{H}}_{2n,m} \otimes \overline{\mathcal{J}}_{2n,m}.$$

(c) CASE C: $(\mathbf{GL_n}, \mathfrak{gl}_{\mathbf{m+\ell}})$ *The invariants*

$$\widetilde{\mathcal{J}}_{n,m,\ell} := \mathcal{P}(M_{n,m} \oplus M_{\ell,n})^{GL_n} = \mathbb{C}[\overline{r}_{ij}^2] \qquad \left(\cong \mathcal{S}(\mathfrak{gl}_{m,\ell}^{(2,0)}) \cong \mathcal{P}(M_{m,\ell}) \text{ if } n \geq \min(m,\ell)\right).$$

Let $\widetilde{\mathcal{H}}_{n,m,\ell} \subseteq \mathcal{P}(M_{n,m} \oplus M_{\ell,n})$ denote the GL_n-harmonics. Further, if $n \geq m+\ell$, we have separation of variables

$$\mathcal{P}(M_{n,m} \oplus M_{\ell,n}) \simeq \widetilde{\mathcal{H}}_{n,m,\ell} \otimes \widetilde{\mathcal{J}}_{n,m,\ell}.$$

We refer our readers to the Appendix for a new proof of the Separation of Variables Theorem.

Definition: The Lie algebra $\mathfrak{g}$ acts on $\mathcal{P}(V)$ via differential operators. Under this action $\mathcal{P}(V)$ decomposes into irreducible (infinite-dimensional, highest weight) representation of $\mathfrak{g}$. In each of the three cases, denote this representations by $\widetilde{E}^\lambda$, $\widetilde{V}^\lambda$ and $\widetilde{F}^\lambda$ respectively. The parametrization being made precise by the pairing defined in the following theorem:

Theorem 2.4. *(Multiplicity-Free Decomposition under* $\mathbf{K} \times \mathfrak{g}$*)*
For each case, we state the decomposition of $\mathcal{P}(V)$ into irreducible representations:

(a) CASE A: $(\mathbf{O_n}, \mathfrak{sp}_{\mathbf{2m}})$

$$\mathcal{P}(M_{n,m}) = \bigoplus E_{(n)}^\lambda \otimes \widetilde{E}_{(2m)}^\lambda \tag{2.3.2}$$

where the sum is over all partitions λ with length at most $\min(n,m)$, *and such that $(\lambda')_1 + (\lambda')_2 \leq n$.*

As a representation of GL_m,

$$\begin{aligned} \widetilde{E}^{\lambda}_{(2m)} &= \mathcal{J}_{n,m} \cdot F^{\lambda}_{(m)} && \text{for any } n, m \geq 0, \\ &\cong \mathcal{S}(SM_m) \otimes F^{\lambda}_{(m)} && \text{provided } n \geq 2m. \end{aligned} \tag{2.3.3}$$

(b) CASE B: $(\mathbf{Sp_{2n}}, \mathfrak{so}_{\mathbf{2m}})$

$$\mathcal{P}(M_{2n,m}) = \bigoplus V^{\lambda}_{(2n)} \otimes \widetilde{V}^{\lambda}_{(2m)} \tag{2.3.4}$$

where the sum is over all partitions λ with length at most $\min(n, m)$.

As a representation of GL_m,

$$\begin{aligned} \widetilde{V}^{\lambda}_{(2m)} &= \overline{\mathcal{J}}_{2n,m} \cdot F^{\lambda}_{(m)} && \text{for any } n, m \geq 0, \\ &\cong \mathcal{S}(AM_m) \otimes F^{\lambda}_{(m)} && \text{provided } n \geq m. \end{aligned} \tag{2.3.5}$$

(c) CASE C: $(\mathbf{GL_n}, \mathfrak{gl}_{\mathbf{m}+\ell})$

$$\mathcal{P}(M_{n,m} \oplus M_{\ell,n}) = \bigoplus F^{(\lambda^+,\lambda^-)}_{(n)} \otimes \widetilde{F}^{(\lambda^+,\lambda^-)}_{(m,\ell)} \tag{2.3.6}$$

where the sum is over all ordered pairs of partitions (λ^+, λ^-) such that $\ell(\lambda^+) + \ell(\lambda^-) \leq n$, $\ell(\lambda^+) \leq \min(n, m)$, and $\ell(\lambda^-) \leq \min(n, \ell)$.

As a representation of $GL_m \times GL_\ell$,

$$\begin{aligned} \widetilde{F}^{(\lambda^+,\lambda^-)}_{(m,\ell)} &= \widetilde{\mathcal{J}}_{n,m,\ell} \cdot \left(F^{\lambda^+}_{(m)} \otimes F^{\lambda^-}_{(\ell)} \right) && \text{for any } n, m, \ell \geq 0, \\ &\cong \mathcal{S}(M_{m,\ell}) \otimes \left(F^{\lambda^+}_{(m)} \otimes F^{\lambda^-}_{(\ell)} \right) && \text{provided } n \geq m + \ell. \end{aligned} \tag{2.3.7}$$

Remarks: In Case C, the representation $\widetilde{F}^{(\lambda^+,\lambda^-)}_{(m,\ell)}$ are (in general) complexifications of infinite-dimensional highest weight representations of $(\mathfrak{u}_{(m,\ell)})_{\mathbb{C}} \simeq \mathfrak{gl}_{m+\ell}$. Sometimes want to emphasize the interplay of the two pieces $M_{n,m}$ and $M_{\ell,n}$, by writing $\mathfrak{gl}_{m,\ell}$ instead of $\mathfrak{gl}_{m+\ell}$. The degenerate case when $\ell = 0$ is particularly interesting. This is the $\mathbf{GL_n} \times \mathbf{GL_m}$ **duality**:

$$\mathcal{P}(M_{n,m}) = \bigoplus_{\lambda} F^{\lambda}_{(n)} \otimes F^{\lambda}_{(m)} \tag{2.3.8}$$

where the sum is over all integer partitions λ such that $\ell(\lambda) \leq \min(n, m)$.

Theorem 2.5. *(Multiplicity-Free Decomposition of Harmonics under* $\mathbf{K} \times \mathbf{G^{(1,1)}}$*)*
We proceed in cases:

(a) CASE A: $(\mathbf{O_n}, \mathfrak{sp}_{\mathbf{2m}})$ *Let $\mathcal{H}_{n,m} \subseteq \mathcal{P}(M_{n,m})$ denote the O_n-harmonics. The group $O_n \times GL_m$ acts on $\mathcal{P}(M_{n,m})$ by $(g, h) \cdot f(x) = f(g^t x h)$, where $g \in O_n$, $h \in GL_m$ and $x \in M_{n,m}$. Then $\mathcal{H}_{n,m}$ is invariant under this action. As an $O_n \times GL_m$ representation,*

$$\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}) \cong \mathcal{H}_{n,m} = \bigoplus E^{\lambda}_{(n)} \otimes F^{\lambda}_{(m)}, \tag{2.3.9}$$

where the sum is over all partitions λ *with length at most* $\min(n,m)$ *and such that* $(\lambda')_1 + (\lambda')_2 \leq n$.

(b) CASE B: $(\mathbf{Sp_{2n}}, \mathfrak{so}_{\mathbf{2m}})$ *Let* $\overline{\mathcal{H}}_{2n,m} \subseteq \mathcal{P}(M_{2n,m})$ *denote the* Sp_{2n}*-harmonics. The group* $Sp_{2n} \times GL_m$ *acts on* $\mathcal{P}(M_{2n,m})$ *by* $(g,h)\cdot f(x) = f(g^t x h)$, *where* $g \in Sp_{2n}$, $h \in GL_m$ *and* $x \in M_{2n,m}$. *Then* $\mathcal{H}_{2n,m}$ *is invariant under this action. As a* $Sp_{2n} \times GL_m$ *representation,*

$$\mathcal{P}(M_{2n,m})/I(\overline{\mathcal{J}}^+_{2n,m}) \cong \overline{\mathcal{H}}_{2n,m} = \bigoplus V^{\lambda}_{(n)} \otimes F^{\lambda}_{(m)} \tag{2.3.10}$$

where the sum is over all partitions λ *with length at most* $\min(n,m)$.

(c) CASE C: $(\mathbf{GL_n}, \mathfrak{gl}_{\mathbf{m}+\ell})$ *Let* $\widetilde{\mathcal{H}}_{n,m,\ell} \subseteq \mathcal{P}(M_{n,m} \oplus M_{\ell,n})$ *denote the* GL_n*-harmonics. The group* $GL_n \times GL_m \times GL_\ell$ *acts on* $\mathcal{P}(M_{n,m} \oplus M_{\ell,n})$ *by*

$$(g,h_1,h_2)\cdot f(x,y) = f(g^t x h_1, h_2^t y (g^t)^{-1}),$$

for $g \in GL_n$, $h_1 \in GL_m$, $h_2 \in GL_\ell$, $x \in M_{n,m}$ *and* $y \in M_{\ell,n}$. *Then* $\widetilde{\mathcal{H}}_{n,m,\ell}$ *is invariant under this action. As a* $GL_n \times GL_m \times GL_\ell$ *representation,*

$$\mathcal{P}(M_{n,m} \oplus M_{\ell,n})/I(\widetilde{\mathcal{J}}^+_{n,m,\ell}) \cong \widetilde{\mathcal{H}}_{n,m,\ell} = \bigoplus F^{(\lambda^+,\lambda^-)}_{(n)} \otimes \left(F^{\lambda^+}_{(m)} \otimes F^{\lambda^-}_{(\ell)}\right) \tag{2.3.11}$$

where the sum is over all ordered pairs of partitions (λ^+,λ^-) *such that* $\ell(\lambda^+) + \ell(\lambda^-) \leq n$, $\ell(\lambda^+) \leq \min(n,m)$, *and* $\ell(\lambda^-) \leq \min(n,\ell)$.

Remarks. The three cases are summarized in the following table:

K	O_n	Sp_{2n}	GL_n
$\mathfrak{g}$	$\mathfrak{sp}_{2m}$	$\mathfrak{so}_{2m}$	$\mathfrak{gl}_{m,\ell}$
V	$M_{n,m}$	$M_{2n,m}$	$M_{n,m} \oplus M_{\ell,n}$
$G^{(1,1)}$	GL_m	GL_m	$GL_m \times GL_\ell$

3. Reciprocity Algebras

In this paper, we study branching algebras using classical invariant theory. The formulation of classical invariant theory in terms of dual pairs [Ho2] allows one to realize branching algebras for classical symmetric pairs as concrete algebras of polynomials on vector spaces. Furthermore, when realized in this way, the branching algebras have a double interpretation in which they solve two related branching problems simultaneously. Classical invariant theory also provides a flexible means which allows an inductive approach to the computation of branching algebras, and makes evident natural connections between different branching algebras.

The easiest illustration of the above assertions is the realization of the tensor product algebra for GL_n presented as follows.

3.1. **Illustration: Tensor Product Algebra for $\mathbf{GL}_n$.** Consider the joint action of $GL_n \times GL_m$ on the $\mathcal{P}(M_{n,m})$ by the rule

$$(g,h)\cdot f(x) = f(g^t x h), \qquad \text{for } g \in GL_n, h \in GL_m, x \in M_{n,m}.$$

For the corresponding action on polynomials, one has the decomposition (see Theorem 2.4(c) and (2.3.8))

$$\mathcal{P}(M_{n,m}) \simeq \bigoplus_{\lambda} F^{\lambda}_{(n)} \otimes F^{\lambda}_{(m)}, \tag{3.1.1}$$

of the polynomials into irreducible $GL_n \times GL_m$ representations. Note that the sum is over non-negative partitions λ with depth at most $\min(n,m)$.

Let $U_m = U_{GL_m}$ denote the upper triangular unipotent subgroup of GL_m. From decomposition (3.1.1), we can easily see that

$$\mathcal{P}(M_{n,m})^{U_m} \simeq \left(\bigoplus_{\lambda} F^{\lambda}_{(n)} \otimes F^{\lambda}_{(m)} \right)^{U_m} \simeq \bigoplus_{\lambda} F^{\lambda}_{(n)} \otimes (F^{\lambda}_{(m)})^{U_m}. \tag{3.1.2}$$

Since the spaces $(F^{\lambda}_{(m)})^{U_m}$ are one-dimensional, the sum in equation (3.1.2) consists of one copy of each $F^{\lambda}_{(n)}$. Just as in the discussion of §1, the algebra is graded by $\widehat{A}_m^+$, where A_m is the diagonal torus of GL_m, and one sees from (3.1.2) that the graded components are the $F^{\lambda}_{(n)}$.

By the arguments in §2.2, $\mathcal{P}(M_{n,m})^{U_m}$ can thus be associated to a graded subalgebra in $\mathcal{R}(GL_n/U_n)$, in particular, this is a $(0,1)$-subalgebra as in Definition 2.1. To study tensor products of representations of GL_n, we can take the direct sum of $M_{n,m}$ and $M_{n,\ell}$. We then have an action of $GL_n \times GL_m \times GL_\ell$ on $\mathcal{P}(M_{n,m} \oplus M_{n,\ell})$. Since $\mathcal{P}(M_{n,m} \oplus M_{n,\ell}) \simeq \mathcal{P}(M_{n,m}) \otimes P(M_{n,\ell})$, we may deduce from (3.1.1) that

$$\begin{aligned} \mathcal{P}(M_{n,m} \oplus M_{n,\ell})^{U_m \times U_\ell} &\simeq \mathcal{P}(M_{n,m})^{U_m} \otimes \mathcal{P}(M_{n,\ell})^{U_\ell} \\ &\simeq \bigoplus_{\mu,\nu} (F^{\mu}_{(n)} \otimes F^{\nu}_{(n)}) \otimes \left((F^{\mu}_{(m)})^{U_m} \otimes (F^{\nu}_{(\ell)})^{U_\ell} \right). \end{aligned} \tag{3.1.3}$$

Thus, this algebra is the sum of one copy of each tensor products $F^{\mu}_{(n)} \otimes F^{\nu}_{(n)}$. Hence, if we take the U_n-invariants, we will get a subalgebra of the tensor product algebra for GL_n. This results in the algebra

$$(\mathcal{P}(M_{n,m} \oplus M_{n,\ell})^{U_m \times U_\ell})^{U_n} \simeq \mathcal{P}(M_{n,m} \oplus M_{n,\ell})^{U_m \times U_\ell \times U_n}.$$

This shows that we can realize the tensor product algebra for GL_n, or more precisely, various $(0,1)$-subalgebras of it, as algebras of polynomial functions on matrices, specifically as the algebras $\mathcal{P}(M_{n,m} \oplus M_{n,\ell})^{U_m \times U_\ell \times U_n}$.

However, the algebra $\mathcal{P}(M_{n,m} \oplus M_{n,\ell})^{U_m \times U_\ell \times U_n}$ has a second interpretation, as a different branching algebra. We note that $M_{n,m} \oplus M_{n,\ell} \simeq M_{n,m+\ell}$. On this space we have the action of $GL_n \times GL_{m+\ell}$, which is described by the obvious adaptation of equation (3.1.1). The action of $GL_n \times GL_m \times GL_\ell$ arises by restriction of the action of $GL_{m+\ell}$ to the subgroup $GL_m \times GL_\ell$ embedded block diagonally in $GL_{m+\ell}$. By (the obvious analog of) decomposition (3.1.2), we see that

$$\mathcal{P}(M_{n,m+\ell})^{U_n} \simeq \bigoplus_\lambda (F^\lambda_{(n)})^{U_n} \otimes F^\lambda_{(m+\ell)}.$$

This algebra embeds as a subalgebra of $\mathcal{R}(GL_{m+\ell}/U_{m+\ell})$, in particular, this is a $(0,1)$-subalgebra as in Definition 2.1. If we then take the $U_m \times U_\ell$ invariants, we find that

$$(\mathcal{P}(M_{n,m+\ell})^{U_n})^{U_m\times U_\ell} \simeq \bigoplus_\lambda (F^\lambda_{(n)})^{U_n} \otimes (F^\lambda_{(m+\ell)})^{U_m\times U_\ell}$$

is (a $(0,1)$-subalgebra of) the $(GL_{m+\ell}, GL_m \times GL_\ell)$ branching algebra. Thus, we have established the following result.

Theorem 3.1. (a) *The algebra $\mathcal{P}(M_{n,m+\ell})^{U_n\times U_m\times U_\ell}$ is isomorphic to a $(0,1)$-subalgebra of the $(GL_n \times GL_n, GL_n)$ branching algebra (a.k.a. the GL_n tensor product algebra), and to a $(0,1)$-subalgebra of the $(GL_{m+\ell}, GL_m \times GL_\ell)$ branching algebra.*

(b) *In particular, the dimension of the $\psi^\lambda \times \psi^\mu \times \psi^\nu$ homogeneous component for $A_n \times A_m \times A_\ell$ of $\mathcal{P}(M_{n,m+\ell})^{U_n\times U_m\times U_\ell}$ records simultaneously*

(i) *the multiplicity of $F^\lambda_{(n)}$ in the tensor product $F^\mu_{(n)} \otimes F^\nu_{(n)}$, and*

(ii) *the multiplicity of $F^\mu_{(m)} \otimes F^\nu_{(\ell)}$ in $F^\lambda_{(m+\ell)}$,*

for appropriate diagrams μ, ν and λ.

Thus, we can not only realize the GL_n tensor product algebra concretely as an algebra of polynomials, we find that it appears simultaneously in two guises, the second being as the branching algebra for the pair $(GL_{m+\ell}, GL_m \times GL_\ell)$. We emphasize two features of this situation.

First, the pair $(GL_{m+\ell}, GL_m \times GL_\ell)$, as well as the pair $(GL_n \times GL_n, GL_n)$, is a symmetric pair. Hence, both the interpretations of $\mathcal{P}(M_{n,m+\ell})^{U_n\times U_m\times U_\ell}$ are as branching algebras for symmetric pairs.

Second, the relationship between the two situations is captured by the notion of "see-saw pair" of dual pairs [Kud]. Precisely, a context for understanding the decomposition law (3.1.1) is provided by observing that GL_n and GL_m (or more correctly, slight modifications of their Lie algebras) are mutual centralizers inside the Lie algebra $\mathfrak{sp}(M_{n,m})$ (of the metaplectic group) of polynomial coefficient differential operators of total degree two on $M_{n,m}$ [Ho2] [Ho4]. We say that they define a *dual pair* inside $\mathfrak{sp}(M_{n,m})$. The decomposition (3.1.1) then appears as the correspondence of representations associated to this dual pair [Ho2]. Further, the pairs of groups $(GL_n, GL_{m+\ell}) = (G_1, G_1')$ and $(GL_n \times GL_n, GL_m \times GL_\ell) = (G_2, G_2')$ both define dual pairs inside the Lie algebra $\mathfrak{sp}(M_{n,m+\ell})$. We evidently have the relations

$$G_1 = GL_n \subset GL_n \times GL_n = G_2, \tag{3.1.4}$$

and (hence)

$$G'_1 = GL_{m+\ell} \supset GL_m \times GL_\ell = G'_2. \tag{3.1.5}$$

We refer to a pair of dual pairs related as in inclusions (3.1.4) and (3.1.5), a *see-saw pair* of dual pairs.

In these terms, we may think of the symmetric pairs (G_1, G_2) and (G'_2, G'_1) as a "*reciprocal pair*" of symmetric pairs. If we do so, we see that the algebra $\mathcal{P}(M_{n,m+\ell})^{U_n \times U_m \times U_\ell}$ is describable as $\mathcal{P}(M_{n,m+\ell})^{U_{G_1} \times U_{G'_2}}$ – it has a description in terms of the see-saw pair, and in this description the two pairs of the see-saw, or alternatively, the two reciprocal symmetric pairs, enter equivalently into the description of the algebra that describes the branching law for both symmetric pairs. For this reason, we also call this algebra, which describes the branching law for both symmetric pairs, the *reciprocity algebra* of the pair of pairs.

It turns out that any branching algebra associated to a classical symmetric pair, that is, a pair (G, H) in which G is a product of classical groups, has an interpretation as a reciprocity algebra – an algebra that describes a branching law for two reciprocal symmetric pairs simultaneously. Sometimes, however, one of the branching laws involves infinite-dimensional representations.

3.2. **Symmetric Pairs and Reciprocity Pairs.** In the context of dual pairs, we would like to understand the (G, H) branching of irreducible representations of G to H, for symmetric pairs (G, H). Table I lists the symmetric pairs which we will cover in this paper.

Table I: Classical Symmetric Pairs

Description	**G**	**H**
Diagonal	$GL_n \times GL_n$	GL_n
Diagonal	$O_n \times O_n$	O_n
Diagonal	$Sp_{2n} \times Sp_{2n}$	Sp_{2n}
Direct Sum	GL_{n+m}	$GL_n \times GL_m$
Direct Sum	O_{n+m}	$O_n \times O_m$
Direct Sum	$Sp_{2(n+m)}$	$Sp_{2n} \times Sp_{2m}$
Polarization	O_{2n}	GL_n
Polarization	Sp_{2n}	GL_n
Bilinear Form	GL_n	O_n
Bilinear Form	GL_{2n}	Sp_{2n}

If G is a classical group over $\mathbb{C}$, then G can be embedded as one member of a dual pair in the symplectic group as described in [Ho2]. The resulting pairs of groups are (GL_n, GL_m) or (O_n, Sp_{2m}), each inside Sp_{2nm}, and are called *irreducible* dual pairs. In general, a dual pair of reductive groups in Sp_{2r} is a product of such pairs.

Proposition 3.2 *Let G be a classical group, or a product of two copies of a classical group. Let G belong to a dual pair (G, G') in a symplectic group Sp_{2m}. Let $H \subset G$ be a symmetric subgroup, and let H' be the centralizer of H in Sp_{2m}. Then (H, H') is also a dual pair in Sp_{2m}, and G' is a symmetric subgroup inside H'.*

Proof: This can be shown by fairly easy case-by-case checking. The basic reason that (H, H') form a dual pair is that, for any classical symmetric pair (G, H), the restriction of the standard module of G, or its dual, to H is a sum of standard modules of H, or their duals [Ho2]. This is very easy to check on a case-by-case basis. The see-saw relationship of symmetric pairs organizes the 10 series of symmetric pairs as given in Table I into five pairs of pairs. These are shown in Table II. $\square$

Table II: Reciprocity Pairs

Symmetric Pair $(\mathbf{G}, \mathbf{H})$	$(\mathbf{H}, \mathfrak{h}')$	$(\mathbf{G}, \mathfrak{g}')$
$(GL_n \times GL_n, GL_n)$	$(GL_n, \mathfrak{gl}_{m+\ell})$	$(GL_n \times GL_n, \mathfrak{gl}_m \oplus \mathfrak{gl}_l)$
$(O_n \times O_n, O_n)$	$(O_n, \mathfrak{sp}_{2(m+\ell)})$	$(O_n \times O_n, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2l})$
$(Sp_{2n} \times Sp_{2n}, Sp_{2n})$	$(Sp_{2n}, \mathfrak{so}_{2(m+\ell)})$	$(Sp_{2n} \times Sp_{2n}, \mathfrak{so}_{2m} \oplus \mathfrak{so}_{2\ell})$
$(GL_{n+m}, GL_n \times GL_m)$	$(GL_n \times GL_m, \mathfrak{gl}_\ell \oplus \mathfrak{gl}_\ell)$	$(GL_{n+m}, \mathfrak{gl}_\ell)$
$(O_{n+m}, O_n \times O_m)$	$(O_n \times O_m, \mathfrak{sp}_{2\ell} \oplus \mathfrak{sp}_{2\ell})$	$(O_{n+m}, \mathfrak{sp}_{2\ell})$
$(Sp_{2(n+m)}, Sp_{2n} \times Sp_{2m})$	$(Sp_{2n} \times Sp_{2m}, \mathfrak{so}_{2\ell} \oplus \mathfrak{so}_{2\ell})$	$(Sp_{2(n+m)}, \mathfrak{so}_{2\ell})$
(O_{2n}, GL_n)	$(GL_n, \mathfrak{gl}_{m,m})$	$(O_{2n}, \mathfrak{sp}_{2m})$
(Sp_{2n}, GL_n)	$(GL_n, \mathfrak{gl}_{m,m})$	$(Sp_{2n}, \mathfrak{so}_{2m})$
(GL_n, O_n)	$(O_n, \mathfrak{sp}_{2m})$	$(GL_n, \mathfrak{gl}_m)$
(GL_{2n}, Sp_{2n})	$(Sp_{2n}, \mathfrak{so}_{2m})$	$(GL_{2n}, \mathfrak{gl}_m)$

Remark: Note that when the second component of any pair in Table II is of Lie type A, then the action actually integrates to the group. Table II also amounts to another point of view on the structure on which [Ho3] is based.

We begin with discussions of reciprocity algebras in the next three sections. The discussions provided are ordered more in terms of complexity and do not follow the sequence given in Table I.

4. Branching from GL_n to O_n

Consider the problem of restricting irreducible representations of GL_n to the orthogonal group O_n. We consider the symmetric see-saw pair (GL_n, O_n) and (Sp_{2m}, GL_m). As in the discussion of §3.1, we can realize (a $(0,1)$-subalgebra of) the coordinate ring of the flag manifold GL_n/U_n as the algebra of U_m-invariants on $\mathcal{P}(M_{n,m})$. If we then look at the U_{O_n}-invariants in this algebra, then we will have (a certain $(0,1)$-subalgebra of) the (GL_n, O_n) branching algebra. Thus, we are interested in the algebra

$$\mathcal{P}(M_{n,m})^{U_{O_n} \times U_m}.$$

We note that, in analogy with the situation of §3.1, this is the algebra of invariants for the unipotent subgroups of the smaller member of each symmetric pair. Let us investigate what this algebra appears to be if we first take invariants with respect to U_{O_n}.

We have a decomposition of $\mathcal{P}(M_{n,m})$ as a joint $O_n \times \mathfrak{sp}_{2m}$-module (see Theorem 2.4 (a)):

$$\mathcal{P}(M_{n,m}) \simeq \bigoplus_{\mu} E^{\mu}_{(n)} \otimes \widetilde{E}^{\mu}_{(2m)}. \tag{4.1}$$

Recall that the sum runs through the set of all non-negative integer partitions μ such that $l(\mu) \leq \min(n, m)$ and $(\mu')_1 + (\mu')_2 \leq n$. Here $E^{\mu}_{(n)}$ denotes the irreducible O_n representation parameterized by μ. Recall from §2.3, the decomposition $\mathcal{P}(M_{n,m}) \simeq \bigoplus_{\mu} F^{\mu}_{(n)} \otimes F^{\mu}_{(m)}$. The module $E^{\mu}_{(n)}$ is generated by the GL_n highest weight vector in $F^{\mu}_{(n)}$. Further, $\widetilde{E}^{\mu}_{(2m)}$ is an irreducible infinite-dimensional representation of $\mathfrak{sp}_{2m}$ with lowest $\mathfrak{gl}_m$-type $F^{\mu}_{(m)}$.

Theorem 4.1. *Assume $n > 2m$.*

(a) *The algebra $\mathcal{P}(M_{n,m})^{U_{O_n} \times U_m}$ is isomorphic to a $(0,1)$-subalgebra of the (GL_n, O_n) branching algebra, and to a $(0,1)$-subalgebra of the $(\mathfrak{sp}_{2m}, GL_m)$ branching algebra.*

(b) *In particular, the dimension of the $\phi^{\mu} \times \psi^{\lambda}$ homogeneous component for $A_{O_n} \times A_m$ of $\mathcal{P}(M_{n,m})^{U_{O_n} \times U_m}$ records simultaneously*
   (i) *the multiplicity of $E^{\mu}_{(n)}$ in the representation $F^{\lambda}_{(n)}$, and*
   (ii) *the multiplicity of $F^{\lambda}_{(m)}$ in $\widetilde{E}^{\mu}_{(2m)}$.*
   for appropriate partitions μ and λ.

Proof. Taking the U_{O_n}-invariants for the decomposition (4.1), we find that

$$\mathcal{P}(M_{n,m})^{U_{O_n}} \simeq \bigoplus_{\mu} (E^{\mu}_{(n)})^{U_{O_n}} \otimes \widetilde{E}^{\mu}_{(2m)}. \tag{4.2}$$

The space $(E^{\mu}_{(n)})^{U_{O_n}}$ is the space of highest weight vectors for $(E^{\mu}_{(n)})^{U_{O_n}}$. We would like to say that it is one-dimensional, so that $\mathcal{P}(M_{n,m})^{U_{O_n}}$ would consist of one copy of each of the irreducible representations $\widetilde{E}^{\mu}_{(2m)}$. But, owing to the disconnectedness of O_n, this is not quite true, and when it is true, the highest weight may not completely determine $E^{\mu}_{(n)}$.

However, if $n > 2m$, then $(E^{\mu}_{(n)})^{U_{O_n}}$ is one-dimensional, and does single out $E^{\mu}_{(n)}$ among the representations which appear in the sum (4.1). Hence, let us make this restriction for the present discussion. Taking the U_m invariants in the sum (4.2), we find that

$$(\mathcal{P}(M_{n,m})^{U_{O_n}})^{U_m} \simeq \bigoplus_{\mu} (E^{\mu}_{(n)})^{U_{O_n}} \otimes (\widetilde{E}^{\mu}_{(2m)})^{U_m}. \tag{4.3}$$

The space $(\widetilde{E}^{\mu}_{(2m)})^{U_m}$ describes how the representation $\widetilde{E}^{\mu}_{(2m)}$ of $\mathfrak{sp}_{2m}$ decomposes as a $\mathfrak{gl}_m$ module, or equivalently, as a GL_m-module. In other words, $(\widetilde{E}^{\mu}_{(2m)})^{U_m}$ describes the branching rule from $\mathfrak{sp}_{2m}$ to $\mathfrak{gl}_m$ for the module $\widetilde{E}^{\mu}_{(2m)}$.

We know (thanks to our restriction to $n > 2m$) that the space $(E^{\mu}_{(n)})^{U_{O_n}}$ is one-dimensional. Let ϕ^μ be the A_{O_n} weight of $(E^{\mu}_{(n)})^{U_{O_n}}$. Thus, ϕ^μ is the restriction to the diagonal maximal torus A_{O_n} of the character ψ^μ of the group A_n of diagonal $n \times n$ matrices. Our assumption further implies that ϕ^μ determines $E^{\mu}_{(n)}$. Therefore, for a given dominant A_m weight ψ^λ, corresponding to the partition λ, the ψ^λ-eigenspace in $(\widetilde{E}^{\mu}_{(2m)})^{U_m}$ tells us the multiplicity of $F^{\lambda}_{(m)}$ in the restriction of $\widetilde{E}^{\mu}_{(2m)}$ to $\mathfrak{gl}_m$. This is the same as the dimension of the joint $(\phi^\mu \times \psi^\lambda)$-eigenspace in

$$(\mathcal{P}(M_{n,m})^{U_{O_n}})^{U_m} \simeq \mathcal{P}(M_{n,m})^{U_{O_n} \times U_m} \simeq (\mathcal{P}(M_{n,m})^{U_m})^{U_{O_n}}.$$

But we have already seen that this eigenspace describes the multiplicity of $E^{\mu}_{(n)}$ in $F^{\lambda}_{(n)}$. Thus, again the $A_{O_n} \times A_m$ homogeneous components of $\mathcal{P}(M_{n,m})^{U_{O_n} \times U_m}$ have a simultaneous interpretation, one for a branching law associated to each of the two symmetric pairs composing the symmetric see-saw pair. □

In this case, one of the branching laws involves infinite-dimensional representations. However, they are highest weight representations, which are the most tractable of infinite-dimensional representations, from an algebraic point of view.

5. Tensor Product Algebra for O_n

Using the symmetric see-saw pair $\big((O_n, O_n \times O_n), (Sp_{2m} \times Sp_{2\ell}, Sp_{2(m+\ell)})\big)$, we can construct ((0,1)-subalgebras of) the tensor product algebra for O_n. To prepare for this, we should explicate the decomposition (4.1) further.

Let us recall the basic setup as in §2.3.1 Case A. Recall that $\mathcal{J}_{n,m} = \mathcal{P}(M_{n,m})^{O_n}$ is the algebra of O_n-invariant polynomials. Theorem 2.3(a) implies that $\mathcal{J}_{n,m}$ is a quotient of $\mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m})$, the symmetric algebra on $\mathfrak{sp}^{(2,0)}_{2m}$.

The natural mapping

$$\mathcal{H}_{n,m} \to \mathcal{P}(M_{n,m})/I(\mathcal{J}^{+}_{n,m}) \tag{5.1}$$

is a linear $O_n \times GL_m$-module isomorphism. Further, the $O_n \times GL_m$ structure of $\mathcal{H}_{n,m}$ is as follows (see Theorem 2.5(a)):

$$\mathcal{H}_{n,m} \simeq \bigoplus_{\mu} E^{\mu}_{(n)} \otimes F^{\mu}_{(m)}. \tag{5.2}$$

Here μ ranges over the same diagrams as in (4.1).

From Theorem 2.4(a),

$$\widetilde{E}^{\mu}_{(2m)} \simeq F^{\mu}_{(m)} \cdot \mathcal{J}_{n,m} \simeq \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m}) \cdot F^{\mu}_{(m)}, \tag{5.3}$$

and it follows that
$$\widetilde{E}^{\mu}_{(2m)}/(\mathfrak{sp}_{2m}^{(2,0)}\cdot\widetilde{E}^{\mu}_{(2m)})\simeq F^{\mu}_{(m)}.$$
In other words, we can detect the $\mathfrak{sp}_{2m}$ isomorphism class of the module $\widetilde{E}^{\mu}_{(2m)}$ by the GL_m isomorphism class of the quotient $\widetilde{E}^{\mu}_{(2m)}/(\mathfrak{sp}_{2m}^{(2,0)}\cdot\widetilde{E}^{\mu}_{(2m)})$. Also, if $W\subset\mathcal{P}(M_{n,m})$ is any $\mathfrak{sp}_{2m}$-invariant subspace, then
$$W/(\mathfrak{sp}_{2m}^{(2,0)}\cdot W)\simeq W\cap\mathcal{H}_{n,m},$$
and this subspace also reveals the $\mathfrak{sp}_{2m}$ isomorphism type of W.

We can use the above to find a model for (a $(0,1)$-subalgebra of) the tensor product algebra of O_n. One consequence of the above discussion is that
$$\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m})\right)^{U_m}\simeq\bigoplus_{\mu}E^{\mu}_{(n)}\otimes(F^{\mu}_{(m)})^{U_m}\tag{5.4}$$
consists of one copy of each irreducible representation $E^{\mu}_{(n)}$.

If we repeat the above discussion for $M_{n,\ell}$, and combine the results, we find that
$$\begin{aligned}&\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m})\right)^{U_m}\otimes\left(\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell})\right)^{U_\ell}\\&\simeq\bigoplus_{\mu,\nu}\left(E^{\mu}_{(n)}\otimes E^{\nu}_{(n)}\right)\otimes\left((F^{\mu}_{(m)})^{U_m}\otimes(F^{\nu}_{(\ell)})^{U_\ell}\right)\end{aligned}\tag{5.5}$$
is a direct sum of one copy of each possible tensor product of an $E^{\mu}_{(n)}$ with an $E^{\nu}_{(n)}$. If we now take the U_{O_n}-invariants in equation (5.5), we will have (a $(0,1)$-subalgebra of) the tensor product algebra of O_n:
$$\begin{aligned}&\left((\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}))^{U_m}\otimes(\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell}))^{U_\ell}\right)^{U_{O_n}}\\&\simeq\bigoplus_{\mu,\nu}\left(E^{\mu}_{(n)}\otimes E^{\nu}_{(n)}\right)^{U_{O_n}}\otimes\left((F^{\mu}_{(m)})^{U_m}\otimes(F^{\nu}_{(\ell)})^{U_\ell}\right).\end{aligned}\tag{5.6}$$

We can describe this algebra in another way. Begin with the observation that $\mathcal{P}(M_{n,m})\otimes\mathcal{P}(M_{n,\ell})\simeq\mathcal{P}(M_{n,m+\ell})$, and
$$\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m})\otimes\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell})\simeq\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m}\oplus\mathcal{J}^+_{n,\ell}).$$
Thus
$$(\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}))^{U_m}\otimes(\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell}))^{U_\ell}\simeq(\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m}\oplus\mathcal{J}^+_{n,\ell}))^{U_m\times U_\ell},$$
and
$$\begin{aligned}&\left((\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}))^{U_m}\otimes(\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell}))^{U_\ell}\right)^{U_{O_n}}\\&\simeq\left((\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m}\oplus\mathcal{J}^+_{n,\ell}))^{U_m\times U_\ell}\right)^{U_{O_n}}\\&\simeq\left((\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m}\oplus\mathcal{J}^+_{n,\ell}))^{U_{O_n}}\right)^{U_m\times U_\ell}.\end{aligned}$$

Theorem 5.1. (a) *The algebra*

$$\big((\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}))^{U_m} \otimes (\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell}))^{U_\ell}\big)^{U_{O_n}}$$
$$\simeq \big((\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell}))^{U_{O_n}}\big)^{U_m \times U_\ell}$$

is isomorphic to a $(0,1)$-subalgebra of the $(O_n \times O_n, O_n)$ branching algebra (a.k.a. the O_n tensor product algebra), and to a $(0,1)$-subalgebra of the $(\mathfrak{sp}_{2(m+\ell)}, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2\ell})$ branching algebra.

(b) *Specifically, the dimension of the $(\phi^\lambda \times \psi^\mu \times \psi^\nu)$-eigenspace for $A_{O_n} \times A_m \times A_\ell$ of $\big((\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell}))^{U_{O_n}}\big)^{U_m \times U_\ell}$ records simultaneously*
  (i) *the multiplicity of $E^\lambda_{(n)}$ in $E^\mu_{(n)} \otimes E^\nu_{(n)}$, as well as*
  (ii) *the multiplicity of $\widetilde{E}^\mu_{(2m)} \otimes \widetilde{E}^\nu_{(2\ell)}$ in the restriction of $\widetilde{E}^\lambda_{(2(m+\ell))}$.*

Proof. Let us now compute the ring expressed in this way. From Theorem 2.4(a), we know that

$$\mathcal{P}(M_{n,m})^{U_{O_n}} \simeq \left(\bigoplus_\mu E^\mu_{(n)} \otimes \widetilde{E}^\mu_{(2m)}\right)^{U_{O_n}} \simeq \bigoplus_\mu (E^\mu_{(n)})^{U_{O_n}} \otimes \widetilde{E}^\mu_{(2m)}.$$

Now repeat this with m replaced by $m+\ell$:

$$\mathcal{P}(M_{n,m+\ell})^{U_{O_n}} \simeq \left(\bigoplus_\mu E^\mu_{(n)} \otimes \widetilde{E}^\mu_{(2(m+\ell))}\right)^{U_{O_n}} \simeq \bigoplus_\mu (E^\mu_{(n)})^{U_{O_n}} \otimes \widetilde{E}^\mu_{(2(m+\ell))}.$$

Hence

$$(\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell}))^{U_{O_n}} \simeq \left(\left(\bigoplus_\lambda E^\lambda_{(n)} \otimes \widetilde{E}^\lambda_{(2(m+\ell))}\right)/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell})\right)^{U_{O_n}}$$
$$\simeq \bigoplus_\lambda (E^\lambda_{(n)})^{U_{O_n}} \otimes \left(\widetilde{E}^\lambda_{(2(m+\ell))}/(\mathfrak{sp}^{(2,0)}_{2m} \oplus \mathfrak{sp}^{(2,0)}_{2\ell}) \cdot \widetilde{E}^\lambda_{(2(m+\ell))}\right).$$

From this we finally get

$$\big((\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell}))^{U_{O_n}}\big)^{U_m \times U_\ell}$$
$$\simeq \bigoplus_\lambda (E^\lambda_{(n)})^{U_{O_n}} \otimes \left(\widetilde{E}^\lambda_{(2(m+\ell))}/(\mathfrak{sp}^{(2,0)}_{2m} \oplus \mathfrak{sp}^{(2,0)}_{2\ell}) \cdot \widetilde{E}^\lambda_{(2(m+\ell))}\right)^{U_m \times U_\ell}.$$

From the discussion following equation (5.1), we see that the factor

$$\left(\widetilde{E}^\lambda_{(2(m+\ell))}/(\mathfrak{sp}^{(2,0)}_{2m} \oplus \mathfrak{sp}^{(2,0)}_{2\ell}) \cdot \widetilde{E}^\lambda_{(2(m+\ell))}\right)^{U_m \times U_\ell}$$

tells us the $\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2\ell}$ decomposition of $\widetilde{E}^\lambda_{(2(m+\ell))}$. □

Hence, again the algebra has a double interpretation, one in terms of decomposing tensor products of O_n representations, and one in terms of branching from $\mathfrak{sp}_{2(m+\ell)}$

to $\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2\ell}$ (although the second branching law involves infinite-dimensional representations).

6. More Reciprocity Algebras for (GL_n, GL_m)

Whereas our first example of a reciprocity algebra in §3.1 involved only finite-dimensional representations, the others all involve infinite-dimensional representations in some respect. It turns out that the apparently exceptional nature of the reciprocity algebra for the pair (GL_n, GL_m) is somewhat deceptive. In fact, we can associate several reciprocity algebras to (GL_n, GL_m), and nearly all of them will involve infinite-dimensional representations.

We shall refer to §2.3.1 and consider the action of GL_n on $\mathcal{P}(M_{n,m} \oplus M_{\ell,n})$ by the rule

$$g \cdot f(x, y) = f(g^t x, y(g^t)^{-1}) \tag{6.1}$$

for $x \in M_{n,m}$, $y \in M_{\ell,n}$ and $g \in GL_n$. Recall from Theorem 2.3(c) that the algebra $\widetilde{\mathcal{J}}_{n,m,\ell}$ generated by $\mathfrak{gl}_{m,\ell}^{(2,0)}$. It is the space of all polynomials on $\mathcal{P}(M_{n,m} \oplus M_{\ell,n})$ invariant under GL_n. Let $\widetilde{\mathcal{H}}_{n,m,\ell}$ be the space of GL_n-harmonics and recall the $GL_n \times GL_m \times GL_\ell$ isomorphism (see Theorem 2.5(c)):

$$\mathcal{P}(M_{n,m} \oplus M_{\ell,n})/I(\widetilde{\mathcal{J}}^+_{n,m,\ell}) \simeq \widetilde{\mathcal{H}}_{n,m,\ell}.$$

Theorem 6.1. (a) *The algebra*

$$\Big((\mathcal{P}(M_{n,m} \oplus M_{\ell,n})/I(\widetilde{\mathcal{J}}^+_{n,m,\ell}))^{U_m \times U_\ell} \otimes (\mathcal{P}(M_{n,m'} \oplus M_{\ell',n})/I(\widetilde{\mathcal{J}}^+_{n,m',\ell'}))^{U_{m'} \times U_{\ell'}}\Big)^{U_n}$$

is isomorphic to a $(0,1)$-subalgebra of the $(GL_n \times GL_n, GL_n)$ branching algebra as well as to a $(\mathfrak{gl}_{m+m',\ell+\ell'}, \mathfrak{gl}_{m,\ell} \oplus \mathfrak{gl}_{m',\ell'})$ branching algebra.

(b) *In particular, the dimension of the $(A_n \times A_m \times A_{m'} \times A_\ell \times A_{\ell'})$-eigenspace of $\mathcal{P}(M_{n,m+m'} \oplus M_{\ell+\ell',n})$ describes simultaneously*
   (i) *the multiplicity of $F_{(n)}^{(\lambda^+,\lambda^-)}$ in $F_{(n)}^{(\mu^+,\mu^-)} \otimes F_{(n)}^{(\nu^+,\nu^-)}$, and*
   (ii) *the multiplicity of the representation $\widetilde{F}_{(m,\ell)}^{(\mu^+,\mu^-)} \otimes \widetilde{F}_{(m',\ell')}^{(\nu^+,\nu^-)}$ of $\mathfrak{gl}_{m,\ell} \oplus \mathfrak{gl}_{m',\ell'}$ in the restriction of the representation $\widetilde{F}_{(m+m',\ell+\ell')}^{(\lambda^+,\lambda^-)}$ of $\mathfrak{gl}_{(m+m'),(\ell+\ell')}$.*

Remarks: Recall from the remarks after Theorem 2.4 that we have written $\mathfrak{gl}_{m,\ell}$ instead of $\mathfrak{gl}_{m+\ell}$ to emphasize the interplay of the two components $M_{n,m}$ and $M_{\ell,n}$.

Proof. From the above description of $\mathcal{P}(M_{n,m} \oplus M_{\ell,n})$, we can see using (2.3.11) that

$$(\mathcal{P}(M_{n,m} \oplus M_{\ell,n})/I(\widetilde{\mathcal{J}}^+_{n,m,\ell}))^{U_m \times U_\ell} \simeq \bigoplus_{\mu^+,\mu^-} F_{(n)}^{(\mu^+,\mu^-)} \otimes (F_{(m)}^{\mu^+})^{U_m} \otimes (F_{(\ell)}^{\mu^-})^{U_\ell}$$

is a multiplicity-free sum of representations $F_{(n)}^{(\mu^+,\mu^-)}$ of GL_n. Again, this can be embedded as a $(0,1)$-subalgebra of the coordinate ring of GL_n/U_n.

Now repeat this with m' in place of m and ℓ' in place of ℓ. We again get a multiplicity-free sum of a family of representations of GL_n. If we take the tensor product of the two sums, and look at highest weight vectors for GL_n, we will get a $(0,1)$-subalgebra of the tensor algebra for GL_n:

$$\Big((\mathcal{P}(M_{n,m} \oplus M_{\ell,n})/I(\widetilde{\mathcal{J}}^{+}_{n,m,\ell}))^{U_m \times U_\ell} \otimes (\mathcal{P}(M_{n,m'} \oplus M_{\ell',n})/I(\widetilde{\mathcal{J}}^{+}_{n,m',\ell'}))^{U_{m'} \times U_{\ell'}}\Big)^{U_n}$$

$$\simeq \bigoplus_{\mu^+,\mu^-,\nu^+,\nu^-} (F_{(n)}^{(\mu^+,\mu^-)} \otimes F_{(n)}^{(\nu^+,\nu^-)})^{U_n} \otimes \Big((F_{(m)}^{\mu^+})^{U_m} \otimes (F_{(\ell)}^{\mu^-})^{U_\ell} \otimes (F_{(m')}^{\nu^+})^{U_{m'}} \otimes (F_{(\ell')}^{\nu^-})^{U_{\ell'}}.\Big)$$

where the sum is over partitions μ^+, μ^-, ν^+ and ν^- such that $l(\mu^+) \leq m$, $l(\mu^-) \leq \ell$, $l(\mu^+) + l(\mu^-) \leq n$, $l(\nu^+) \leq m'$, $l(\nu^-) \leq \ell'$, $l(\nu^+) + l(\nu^-) \leq n$.

On the other hand,

$$\begin{aligned}
&\Big((\mathcal{P}(M_{n,m} \oplus M_{\ell,n})/I(\widetilde{\mathcal{J}}^{+}_{n,m,\ell}))^{U_m \times U_\ell} \otimes (\mathcal{P}(M_{n,m'} \oplus M_{\ell',n})/I(\widetilde{\mathcal{J}}^{+}_{n,m',\ell'}))^{U_{m'} \times U_{\ell'}}\Big)^{U_n} \\
&\simeq \Big((\mathcal{P}(M_{n,m} \oplus M_{\ell,n})/I(\widetilde{\mathcal{J}}^{+}_{n,m,\ell})) \otimes (\mathcal{P}(M_{n,m'} \oplus M_{\ell',n})/I(\widetilde{\mathcal{J}}^{+}_{n,m',\ell'}))\Big)^{U_n \times U_m \times U_\ell \times U_{m'} \times U_{\ell'}} \\
&\simeq \Big(\mathcal{P}(M_{n,m} \oplus M_{\ell,n} \oplus M_{n,m'} \oplus M_{\ell',n})/I(\widetilde{\mathcal{J}}^{+}_{n,m,\ell} \oplus \widetilde{\mathcal{J}}^{+}_{n,m',\ell'})\Big)^{U_n \times U_m \times U_\ell \times U_{m'} \times U_{\ell'}} \\
&\simeq \Big(\mathcal{P}(M_{n,m+m'} \oplus M_{\ell+\ell',n})/I(\widetilde{\mathcal{J}}^{+}_{n,m,\ell} \oplus \widetilde{\mathcal{J}}^{+}_{n,m',\ell'})\Big)^{U_n \times U_m \times U_\ell \times U_{m'} \times U_{\ell'}} \\
&\simeq \left(\bigoplus_{\lambda^+,\lambda^-} (F_{(n)}^{(\lambda^+,\lambda^-)})^{U_n} \otimes \widetilde{F}^{(\lambda^+,\lambda^-)}_{(m+m',\ell+\ell')}/I(\widetilde{\mathcal{J}}^{+}_{n,m,\ell} \oplus \widetilde{\mathcal{J}}^{+}_{n,m',\ell'})\right)^{U_m \times U_\ell \times U_{m'} \times U_{\ell'}} \\
&\simeq \bigoplus_{\lambda^+,\lambda^-} (F_{(n)}^{(\lambda^+,\lambda^-)})^{U_n} \otimes \Big(\widetilde{F}^{(\lambda^+,\lambda^-)}_{(m+m',\ell+\ell')}/(\widetilde{\mathcal{J}}^{+}_{n,m,\ell} \oplus \widetilde{\mathcal{J}}^{+}_{n,m',\ell'}) \cdot \widetilde{F}^{(\lambda^+,\lambda^-)}_{(m+m',\ell+\ell')}\Big)^{U_m \times U_\ell \times U_{m'} \times U_{\ell'}}
\end{aligned}$$

which tells us about the $\mathfrak{gl}_{m,\ell} \oplus \mathfrak{gl}_{m',\ell'}$ decomposition in the representation $\widetilde{F}^{(\lambda^+,\lambda^-)}_{(m+m',\ell+\ell')}$ of $\mathfrak{gl}_{(m+m'),(\ell+\ell')}$. This completes the proof. □

The construction of §3.1 of course is just the case $\ell = 0 = \ell'$ of the current discussion. That case is notable for staying completely in the context of finite-dimensional representation theory. Another case of interest is when $\ell = 0 = m'$. Then, although the representations of $\mathfrak{gl}_{m,\ell'}$ are infinite dimensional, the representations of the subalgebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_{\ell'}$ are finite dimensional. This case is analogous to branching from GL_n to O_n (or from GL_{2n} to Sp_{2n}).

7. The Stable Range and Relations Between Reciprocity Algebras

Let us summarize our discussions this far. Given any classical symmetric pair, we can embed it in a (family of) see-saw symmetric pair(s). Doing this, we find that (a $(0,1)$-subalgebra of) the branching algebra for the pair can equally well be interpreted as the branching algebra for a dual family of representations of the dual symmetric pair. The representations of the dual symmetric pair will frequently be infinite dimensional, but they are always highest weight modules.

An immediate consequence of this isomorphism of algebras is the isomorphisms of intertwining spaces and hence equality of multiplicities, which we have collectively described as *reciprocity laws*. These reciprocity laws are of the same nature as Frobenius Reciprocity for induced representations of groups.

From §3.2, we see that the see-saw symmetric pairs actually come in two parameter families. If one of the pairs involves many more variables than the other, then certain features of the discussions above become simpler.

Take the results of Theorem 3.1 as an illustration: The Littlewood-Richardson coefficients for GL_n,

$$\begin{aligned} c^{\lambda}_{\mu\nu} &= \dim \operatorname{Hom}_{GL_n}(F^{\lambda}_{(n)}, F^{\mu}_{(n)} \otimes F^{\nu}_{(n)}) \\ &= \dim \operatorname{Hom}_{GL_m \times GL_\ell}(F^{\mu}_{(m)} \otimes F^{\nu}_{(\ell)}, F^{\lambda}_{(m+\ell)}) \end{aligned}$$

are independent of n, if $n \geq m + l$, and depends only on the shape of the partitions μ, ν and λ.

Consider another example: branching from GL_{2n} to Sp_{2n}. If we let these groups act on $\mathcal{P}(M_{2n,m})$, we get the see-saw pairs $(Sp_{2n}, \mathfrak{so}_{2m})$ and $(GL_{2n}, \mathfrak{gl}_m)$. The branching coefficients d^{μ}_{λ} from GL_{2n} to Sp_{2n} can be described as follows:

$$F^{\lambda}_{(2n)} \mid_{Sp_{2n}} = \sum_{\mu} d^{\mu}_{\lambda} \, V^{\mu}_{(2n)}$$

where

$$\begin{aligned} d^{\mu}_{\lambda} &= \dim \operatorname{Hom}_{Sp_2n}(V^{\mu}_{2n}, F^{\lambda}_{2n}) \\ &= \dim \operatorname{Hom}_{GL_m}\left(F^{\lambda}_{(m)}, F^{\mu}_{(m)} \otimes \mathcal{S}(\mathfrak{so}^{(2,0)}_{2m})\right) \\ &= \dim \operatorname{Hom}_{GL_m}\left(F^{\lambda}_{(m)}, F^{\mu}_{(m)} \otimes \mathcal{S}(\wedge^2 \mathbb{C}^m)\right) \end{aligned}$$

is independent of n, if $n \geq m$, and only depends on the diagrams λ and μ. This allows one to create a theory of "stable characters" for Sp_{2n}. Similar considerations apply to GL_n and O_n and this idea has been actively pursued by [KT], amongst others.

These are all instances of stability laws. The well-known one-step branching from GL_n to GL_{n-1} is another instance. This branching can be described entirely by diagrams, with no mention of the size n, if n is large. Iteration of this branching also shows that when n is large, the weight multiplicities of dominant weights of an irreducible GL_n representation are independent of n. See [BBL] for the other classical groups, which don't share this stability property.

In the sections that follow, we will illustrate the simplifications that occur in the stable range, highlighting certain specific see-saw pairs. In all these cases, we show that the branching algebras associated to symmetric pairs can all be described by use of suitable branching algebras associated to the general linear groups. Thus, if we can have control of the solution in the general linear group case, we will have some control of the other classical groups. The other non-trivial examples will be important extensions of this work, and we hope to see them in further papers, for example, [HTW1,2] and [HL].

8. Stability for Branching from GL_n to O_n

We begin with a detailed discussion of the case of (GL_n, O_n) and $(\mathfrak{sp}_{2m}, GL_m)$. Here we have already encountered the stable range, without the name. It is when $n > 2m$. Several things happen in the stable range:

(a) The representations $E^{\mu}_{(n)}$ of the orthogonal group remain irreducible when restricted to the special orthogonal group, and furthermore, no two of them are equivalent.

(b) Recall the algebra $\mathcal{J}_{n,m}$ of O_n-invariant polynomials on $M_{n,m}$ generated by the quadratic invariants, which is the abelian subalgebra $\mathfrak{sp}^{(2,0)}_{2m}$ of $\mathfrak{sp}_{2m}$. In the stable range (in fact it holds true whenever $n \geq m$), the natural surjective homomorphism

$$\mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m}) \to \mathcal{J}_{n,m}$$

is an isomorphism. See Theorem 2.3(a).

(c) In the stable range, the multiplication map

$$\mathcal{H}_{n,m} \otimes \mathcal{J}_{n,m} \simeq \mathcal{H}_{n,m} \otimes \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m}) \to \mathcal{P}(M_{n,m})$$

is also an isomorphism of $O_n \times GL_m$-modules. See Theorem 2.3(a).

Of course, the subspace $\mathcal{H}_{n,m}$ of harmonic polynomials is not an algebra – it is not closed under multiplication. This is quite clear, since $\mathcal{H}_{n,m}$ contains all the linear functions, which generate the whole polynomial ring. However, to form the reciprocity algebra associated to the symmetric see-saw pairs (GL_n, O_n) and $(\mathfrak{sp}_{2m}, GL_m)$, we need to take the U_{O_n}-invariants. Thus, our reciprocity algebra is a subalgebra of

$$\begin{aligned}(\mathcal{H}_{n,m} \otimes \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m}))^{U_{O_n}} &= \mathcal{H}^{U_{O_n}}_{n,m} \otimes \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m}) \\ &\simeq \left(\bigoplus_{\mu} (E^{\mu}_{(n)})^{U_{O_n}} \otimes F^{\mu}_{(m)} \right) \otimes \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m}). \end{aligned} \tag{8.1}$$

Theorem 8.1. *When $n > 2m$, the space $\mathcal{H}^{U_{O_n}}_{n,m}$ is a subalgebra of $\mathcal{P}(M_{n,m})$. Hence, the algebra $\mathcal{P}(M_{n,m})^{U_{O_n}}$ is isomorphic to a tensor product*

$$\mathcal{P}(M_{n,m})^{U_{O_n}} \simeq \mathcal{H}^{U_{O_n}}_{n,m} \otimes \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m})$$

of the algebras $\mathcal{H}^{U_{O_n}}_{n,m}$ and $\mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m})$. Furthermore, the algebra $\mathcal{H}^{U_{O_n}}_{n,m}$ is isomorphic (as a representation) to the subalgebra $\mathcal{R}^{+}(GL_m/U_m)$ of $\mathcal{R}(GL_m/U_m)$ defined by the polynomial representations.

Proof. Note that $\mathcal{H}_{n,m}^{U_{O_n}}$ can be identified with a subalgebra $\mathcal{R}^+(GL_m/U_m)$ of $\mathcal{R}(GL_m/U_m)$ defined by the polynomial representations, from our discussion in §2.2. Consider the space of polynomials belonging to the sum in the last expression of equation (8.1). Let $\{x_{jk} \mid j = 1, \ldots, n, k = 1, \ldots, m\}$ be the standard matrix entries on $M_{n,m}$. In order to make the unipotent group U_{O_n} of O_n maximally compatible with (in fact, contained in) the unipotent subgroup U_n of GL_n, we should choose the inner product on $\mathbb{C}^n$ defining O_n to be $\sum_{j=1}^n x_j x_{n+1-j}$. If we do so, then the joint $O_n \times GL_m$ harmonic highest weight vectors are monomials in the determinants

$$\delta_j = \det \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1j} \\ x_{21} & x_{22} & \ldots & x_{1j} \\ \vdots & \vdots & \vdots & \vdots \\ x_{j1} & x_{j2} & \ldots & x_{jj} \end{bmatrix} \quad \text{for } j = 1, \ldots, m. \tag{8.2}$$

From this, we can see that the space $\sum_\mu (E_{(n)}^\mu)^{U_{O_n}} \otimes F_{(m)}^\mu$ is spanned by the monomials in the determinants

$$\det \begin{bmatrix} x_{1,b_1} & x_{1,b_2} & \ldots & x_{1,b_j} \\ x_{2,b_1} & x_{2,b_2} & \ldots & x_{2,b_j} \\ \vdots & \vdots & \vdots & \vdots \\ x_{j,b_1} & x_{j,b_2} & \ldots & x_{j,b_j} \end{bmatrix} \tag{8.3}$$

as $\{b_1, b_2, b_3, \ldots, b_j\}$ ranges over all j-tuples of integers from 1 to m. Indeed, the span of such monomials is clearly invariant under $\mathfrak{gl}_m$, and consists of highest weight vectors for O_n. Finally, we see that these monomials will all be harmonic, because the partial Laplacians spanning $\mathfrak{sp}_{2m}^{(0,2)}$ have the form

$$\Delta_{ab} = \sum_{j=1}^n \frac{\partial^2}{\partial x_{j,a} \partial x_{n+1-j,b}}. \tag{8.4}$$

Since every term of Δ_{ab} involves differentiating with respect to a variable x_{jk} with $j > n/2$, and the determinants (8.3) do not depend on these variables, we see that they will be annihilated by the Δ_{ab}, which means that they are harmonic. This shows that $\mathcal{H}_{n,m}^{U_{O_n}}$ is a subalgebra of $\mathcal{P}(M_{n,m})$.

We have thus completed the proof of the theorem. □

We can use the description in Theorem 8.1 of $\mathcal{P}(M_{n,m})^{U_{O_n}}$ to relate the branching algebra $\mathcal{P}(M_{n,m})^{U_{O_n} \times U_m}$ to the tensor product algebra for GL_m. As a GL_m-module, the space $\mathfrak{sp}_{2m}^{(2,0)}$ is isomorphic to $\mathcal{S}^2(\mathbb{C}^m)$, the space of symmetric $m \times m$ matrices. It is well known that the symmetric algebra $\mathcal{S}(\mathcal{S}^2(\mathbb{C}^m))$ is multiplicity-free as a representation of GL_m, and decomposes into a sum of one copy of each polynomial representation corresponding to a diagram with rows of even length (or a partition of even parts):

$$\mathcal{S}(\mathcal{S}^2(\mathbb{C}^m)) \simeq \bigoplus_\nu F_{(m)}^{2\nu}. \tag{8.5}$$

As a GL_m-module, $\mathcal{S}(\mathcal{S}^2(\mathbb{C}^m))$ could be embedded in $\mathcal{R}(GL_m/U_m)$, but the algebra structures on these two algebras are quite different. The algebra $\mathcal{S}(\mathcal{S}^2(\mathbb{C}^m))$ is filtered by $\widehat{A}_m^+$, and each homogeneous component is an irreducible representation for GL_m.

Using the dominance filtration (see §2.2), we have a canonical $\widehat{A}_m^+$-algebra filtration on $\mathcal{S}(\mathcal{S}^2(\mathbb{C}^m))$ so that the associated graded algebra is isomorphic to the subalgebra of $\mathcal{R}(GL_m/U_m)$ spanned by the representations corresponding to diagrams with rows of even length.

Let us denote the associated graded algebra of $\mathcal{S}(\mathcal{S}^2(\mathbb{C}^m))$ by $\mathrm{Gr}_{\widehat{A}^+}\mathcal{S}(\mathcal{S}^2(\mathbb{C}^m))$. Let us denote the subalgebra of $\mathcal{R}(GL_m/U_m)$ spanned by the representations attached to diagrams with even length rows by $\mathcal{R}^{+2}(GL_m/U_m)$.

We can filter the tensor product $\mathcal{H}_{n,m}^{U_{O_n}} \otimes \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)})$ by means of the filtration on $\mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)})$. The associated graded algebra will then be $\mathcal{H}_{n,m}^{U_{O_n}} \otimes \mathrm{Gr}_{\widehat{A}^+}\mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)})$. This discussion has indicated that the following result holds.

Theorem 8.2. *When $n > 2m$, the associated graded algebra of $\mathcal{P}(M_{n,m})^{U_{O_n}}$ with respect to the dominance filtration on the factor $\mathcal{J}_{n,m}$ is isomorphic to the tensor product of the graded subalgebras $\mathcal{R}^+(GL_m/U_m)$ and $\mathcal{R}^{+2}(GL_m/U_m)$ of $\mathcal{R}(GL_m/U_m)$:*

$$Gr_{\widehat{A}_m^+}(\mathcal{P}(M_{n,m})^{U_{O_n}}) \simeq \mathcal{R}^+(GL_m/U_m) \otimes \mathcal{R}^{+2}(GL_m/U_m).$$

Of course, $\mathrm{Gr}_{\widehat{A}_m^+}(\mathcal{P}(M_{n,m})^{U_{O_n}})$ is isomorphic as a GL_m-module to $\mathcal{P}(M_{n,m})^{U_{O_n}}$. Also $\mathrm{Gr}_{\widehat{A}_m^+}(\mathcal{P}(M_{n,m})^{U_{O_n}})$ inherits the $\widehat{A}_{O_n}^+$ grading from $\mathcal{P}(M_{n,m})^{U_{O_n}}$ – it becomes identified with the $\widehat{A}_m^+$ grading on the first factor $\mathcal{R}^+(GL_m/U_m)$ in the tensor product of Theorem 8.2. On the other hand, the second factor is also $\widehat{A}_m^+$-graded in the obvious way, since it is the factor which defines the associated graded. When we take the U_m invariants, we get another grading by $\widehat{A}_m^+$, associated to the A_m action on the U_m invariants. This triply $\widehat{A}_m^+$-graded algebra is evidently a $(0,1)$-subalgebra of the tensor product algebra of GL_m.

On the other hand, we could take the U_m invariants inside $\mathcal{P}(M_{n,m})^{U_{O_n}}$, and then pass to the associated graded. It is not hard to convince oneself that these two processes commute with each other. Hence, we finally have:

Corollary 8.3 *When $n > 2m$, the associated graded algebra of U_m invariants in $\mathcal{P}(M_{n,m})^{U_{O_n}}$,*

$$Gr_{\widehat{A}_m^+}\left(\left(\mathcal{P}(M_{n,m})^{U_{O_n}}\right)^{U_m}\right) \simeq \left(Gr_{\widehat{A}_m^+}(\mathcal{P}(M_{n,m})^{U_{O_n}})\right)^{U_m}$$

$$\simeq \left(\mathcal{R}^+(GL_m/U_m) \otimes \mathcal{R}^{+2}(GL_m/U_m)\right)^{U_m}$$

is a triply-graded $(0,1)$-subalgebra of the tensor product algebra of GL_m. The restrictions on the gradings which define $Gr_{\widehat{A}_m^+}\left(\left(\mathcal{P}(M_{n,m})^{U_{O_n}}\right)^{U_m}\right)$ are:

(a) *the weight on the first factor of* $(\mathcal{R}(GL_m/U_m)\otimes\mathcal{R}(GL_m/U_m))^{U_m}$ *should correspond to a partition (i.e., it should be a polynomial weight), and*
(b) *the weight on the second factor should correspond to a partition with even parts.*

Remark: The content of Corollary 8.3 in terms of multiplicities is the Littlewood Restriction Formula [EW], [HTW1; see formula (2.4.1)], [Ki1; see (5.7) with (4.19)], [KT; see Theorem 1.5.3 and 2.3.1], [Li1] and [Li2]. With this result it is possible to compute a basis of the reciprocity algebra for (GL_n, O_n) using [HTW2]; see second preprint of [HL].

9. Tensor Products for O_n

According to Theorem 5.1, we can compute tensor products for the orthogonal group via the algebra

$$\left(\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m})\right)^{U_m}\otimes\left(\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell})\right)^{U_\ell}\right)^{U_{O_n}}.$$

Here the stable range is $n > 2(m+\ell)$. Then we have

$$\mathcal{P}(M_{n,m+\ell})\simeq\mathcal{H}_{n,m+\ell}\otimes\mathcal{S}(\mathfrak{sp}^{(2,0)}_{2(m+\ell)}).$$

Furthermore,

$$\begin{aligned}\mathcal{S}(\mathfrak{sp}^{(2,0)}_{2(m+\ell)}) &= \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m}\oplus\mathfrak{sp}^{(2,0)}_{2\ell}\oplus(\mathbb{C}^m\otimes\mathbb{C}^\ell))\\ &\simeq\mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m})\otimes\mathcal{S}(\mathfrak{sp}^{(2,0)}_{2\ell})\otimes\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell)\end{aligned}$$

Since $\mathcal{J}_{n,m}\simeq\mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m})$ and $\mathcal{J}_{n,\ell}\simeq\mathcal{S}(\mathfrak{sp}^{(2,0)}_{2\ell})$, we see that

$$\begin{aligned}\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m})\otimes\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell}) &\simeq\mathcal{P}(M_{n,m}\oplus M_{n,\ell})/I(\mathcal{J}^+_{n,m}\oplus\mathcal{J}^+_{n,\ell})\\ &\simeq\mathcal{H}_{n,m+\ell}\otimes\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell).\end{aligned}\tag{9.1}$$

Thus, using equation (9.1), we see that

$$\begin{aligned}&\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m})\otimes\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell})\right)^{U_{O_n}}\\ \simeq&\left(\mathcal{H}_{n,m+\ell}\otimes\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell)\right)^{U_{O_n}}\simeq\mathcal{H}^{U_{O_n}}_{n,m+\ell}\otimes\ \mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell)\\ \simeq&\left(\bigoplus_\lambda E^\lambda_{(n)}\otimes F^\lambda_{(m+\ell)}\right)^{U_{O_n}}\otimes\ \mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell)\\ \simeq&\left(\bigoplus_\lambda (E^\lambda_{(n)})^{U_{O_n}}\otimes F^\lambda_{(m+\ell)}\right)\otimes\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell)\\ \simeq&\left(\bigoplus_\lambda (F^\lambda_{(n)})^{U_n}\otimes F^\lambda_{(m+\ell)}\right)\otimes\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell).\end{aligned}$$

Note that $F^{\lambda}_{(n)}$ is the GL_n representation generated by the highest weight of the O_n representation $E^{\lambda}_{(n)}$ and both $(F^{\lambda}_{(n)})^{U_n}$ and $(E^{\lambda}_{(n)})^{O_n}$ are one dimensional.

Hence, finally we get

$$\Big(\big(\mathcal{P}(M_{n,m})/I(\mathcal{J}^{+}_{n,m})\otimes\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^{+}_{n,\ell})\big)^{U_{O_n}}\Big)^{U_m\times U_\ell}$$
$$\simeq\left(\left(\bigoplus_{\lambda}(F^{\lambda}_{(n)})^{U_n}\otimes F^{\lambda}_{(m+\ell)}\right)\otimes\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell)\right)^{U_m\times U_\ell}. \tag{9.2}$$

We can interpret this algebra in term of tensor product algebras for general linear groups. According to Theorem 2.4(c) and (2.3.8), as a $GL_m\times GL_\ell$ module, we have

$$\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell)\simeq\bigoplus_{\delta}F^{\delta}_{(m)}\otimes F^{\delta}_{(\ell)}. \tag{9.3}$$

We also know that

$$\mathcal{R}(GL_m/U_m\times GL_\ell/U_\ell)\simeq\bigoplus_{\mu,\nu}F^{\mu}_{(m)}\otimes F^{\nu}_{(\ell)}. \tag{9.4}$$

Since this algebra is bigraded by μ and ν, we can consider the "diagonal" $(0,1)$-subalgebra

$$\Delta\mathcal{R}(GL_m/U_m\times GL_\ell/U_\ell)\simeq\bigoplus_{\delta}F^{\delta}_{(m)}\otimes F^{\delta}_{(\ell)} \tag{9.5}$$

resulting from requiring the two partitions to be the same. Evidently, the algebra $\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell)$ is isomorphic to $\Delta\mathcal{R}(GL_m/U_m\times GL_\ell/U_\ell)$ as $GL_m\times GL_\ell$-module. They are not isomorphic as algebras, since $\Delta\mathcal{R}(GL_m/U_m\times GL_\ell/U_\ell)$ is graded by $\widehat{A}^{+}_{m}\times\widehat{A}^{+}_{\ell}$, while $\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell)$ is not. However, we may filter $\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell)$ by the representations of $GL_m\times GL_\ell$ (or of either factor) using the dominance filtration (see §2.2), and then the associated graded algebra will be isomorphic to $\Delta\mathcal{R}(GL_m/U_m\times GL_\ell/U_\ell)$:

$$\mathrm{Gr}_{\widehat{A}^{+}_{m}\times\widehat{A}^{+}_{\ell}}(\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^\ell))\simeq\Delta\mathcal{R}(GL_m/U_m\times GL_\ell/U_\ell).$$

Now turn to the first factor $\bigoplus_{\lambda}\left((F^{\lambda}_{(n)})^{U_n}\otimes F^{\lambda}_{(m+\ell)}\right)$ on the right hand side of equation (9.2). According to Theorem 3.1, we can write this as

$$\bigoplus_{\lambda}(F^{\lambda}_{(n)})^{U_n}\otimes F^{\lambda}_{(m+\ell)}\simeq\bigoplus_{\alpha,\beta}\left(F^{\alpha}_{(n)}\otimes F^{\beta}_{(n)}\right)^{U_n}\otimes F^{\alpha}_{(m)}\otimes F^{\beta}_{(\ell)}. \tag{9.6}$$

Combining equations (9.2), (9.3) and (9.6), we see that

$$\Big(\big(\mathcal{P}(M_{n,m})/I(\mathcal{J}^{+}_{n,m})\otimes\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^{+}_{n,\ell})\big)^{U_{O_n}}\Big)^{U_m\times U_\ell}$$
$$\simeq\left(\left(\bigoplus_{\alpha,\beta}\left(F^{\alpha}_{(n)}\otimes F^{\beta}_{(n)}\right)^{U_n}\otimes F^{\alpha}_{(m)}\otimes F^{\beta}_{(\ell)}\right)\otimes\left(\bigoplus_{\delta}F^{\delta}_{(m)}\otimes F^{\delta}_{(\ell)}\right)\right)^{U_m\times U_\ell}$$

$$\simeq \left(\bigoplus_{\alpha,\beta,\delta} \left(F^{\alpha}_{(n)} \otimes F^{\beta}_{(n)}\right)^{U_n} \otimes \left(F^{\alpha}_{(m)} \otimes F^{\delta}_{(m)}\right) \otimes \left(F^{\beta}_{(\ell)} \otimes F^{\delta}_{(\ell)}\right)\right)^{U_m \times U_\ell} \tag{9.7}$$

$$\simeq \bigoplus_{\alpha,\beta,\delta} \left(F^{\alpha}_{(n)} \otimes F^{\beta}_{(n)}\right)^{U_n} \otimes \left(F^{\alpha}_{(m)} \otimes F^{\delta}_{(m)}\right)^{U_m} \otimes \left(F^{\beta}_{(\ell)} \otimes F^{\delta}_{(\ell)}\right)^{U_\ell}$$

At this point, this is an isomorphism of graded vector spaces, not an algebra isomorphism.

We may interpret the last expression in (9.7) analogously to (9.4) and (9.5). We have the (polynomial) tensor product algebras

$$(\mathcal{R}^+(GL_k/U_k) \otimes \mathcal{R}^+(GL_k/U_k))^{U_k} \simeq \bigoplus_{\lambda,\mu} \left(F^{\lambda}_{(k)} \otimes F^{\mu}_{(k)}\right)^{U_k}$$

for $k = n, m$ and ℓ. If we form the tensor product of these, we get

$$(\mathcal{R}^+(GL_n/U_n) \otimes \mathcal{R}^+(GL_n/U_n))^{U_n} \otimes (\mathcal{R}^+(GL_m/U_m) \otimes \mathcal{R}^+(GL_m/U_m))^{U_m}$$
$$\otimes\ (\mathcal{R}^+(GL_\ell/U_\ell) \otimes \mathcal{R}^+(GL_\ell/U_\ell))^{U_\ell}$$
$$\simeq \bigoplus_{\alpha,\beta,\delta,\lambda,\mu,\nu} \left(F^{\alpha}_{(n)} \otimes F^{\beta}_{(n)}\right)^{U_n} \otimes \left(F^{\delta}_{(m)} \otimes F^{\lambda}_{(m)}\right)^{U_m} \otimes \left(F^{\mu}_{(\ell)} \otimes F^{\nu}_{(\ell)}\right)^{U_\ell}$$

Let us denote this algebra by $\mathbb{T}_{n,m,\ell}$. The algebra $\mathbb{T}_{n,m,\ell}$ is $(\widehat{A}^+_n)^3 \times (\widehat{A}^+_m)^3 \times (\widehat{A}^+_\ell)^3$-graded. If we require that $\lambda = \alpha$, or that $\mu = \beta$, or that $\nu = \delta$, then we obtain $(0,1)$-subalgebras of $\mathbb{T}_{n,m,\ell}$. If $\delta = \alpha$, we will denote it by $\Delta_{1,3}\mathbb{T}_{n,m,\ell}$, and so forth. The subalgebra obtained by requiring that all three diagonal conditions occur at once will be denoted by using all three Δ's. Thus we will write

$$\Delta_{1,3}\Delta_{2,5}\Delta_{4,6}\mathbb{T}_{n,m,\ell}$$
$$= \sum_{\alpha,\beta,\delta} \left(F^{\alpha}_{(n)} \otimes F^{\beta}_{(n)}\right)^{U_n} \otimes \left(F^{\alpha}_{(m)} \otimes F^{\delta}_{(m)}\right)^{U_m} \otimes \left(F^{\beta}_{(\ell)} \otimes F^{\delta}_{(\ell)}\right)^{U_\ell} \tag{9.8}$$

We see from equations (9.7) and (9.8), that $\Delta_{1,3}\Delta_{2,5}\Delta_{4,6}\mathbb{T}_{n,m,\ell}$ and $\left(\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}) \otimes \mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell})\right)^{U_{O_n}}\right)^{U_m \times U_\ell}$ are isomorphic as multigraded vector spaces. They may not be isomorphic as algebras, because

$$\left(\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}) \otimes \mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell})\right)^{U_{O_n}}\right)^{U_m \times U_\ell}$$

is not graded, while we see from equations (9.7) and (9.8), that $\Delta_{1,3}\Delta_{2,5}\Delta_{4,6}\mathbb{T}_{n,m,\ell}$ is. However, if we pass to the associated graded of $\mathcal{S}(\mathbb{C}^m \otimes \mathbb{C}^\ell)$, then the two algebras do become isomorphic. We record this fact.

Theorem 9.1. *Assume the stable range $n > 2(m+\ell)$. We have the following isomorphisms of $(\widehat{A}_n^+)^3 \times (\widehat{A}_m^+)^3 \times (\widehat{A}_\ell^+)^3$-graded algebras:*

$$Gr_{(\widehat{A}_n^+)^3 \times (\widehat{A}_m^+)^3 \times (\widehat{A}_\ell^+)^3}\left(\left(\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}) \otimes \mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell})\right)^{U_{O_n}}\right)^{U_m \times U_\ell}\right)$$
$$\simeq \Delta_{1,3}\Delta_{2,5}\Delta_{4,6}\mathbb{T}_{n,m,\ell}.$$

Remark: The content of Theorem 9.1 in terms of multiplicities can be found in [HTW1; see formula (2.1.2)], [Ki2; see Theorem 4.1] and [Ne].

10. Restriction from O_{n+m} to $O_n \times O_m$

Consider now branching from O_{n+m} to the product $O_n \times O_m$. We look at the action of O_{n+m} on $M_{n+m,\ell}$ by multiplication on the left. Here the stable range is $\min(n,m) > 2\ell$.

For $k = n$, m or $m+n$, and ℓ in the stable range, we have the equation (see Theorem 2.3(a))

$$\mathcal{P}(M_{k,\ell}) \simeq \mathcal{H}_{k,l} \otimes \mathcal{S}(\mathfrak{sp}_{2\ell}^{(2,0)}). \tag{10.1}$$

We shall use $\mathfrak{sp}_{2\ell,1}$ and $\mathfrak{sp}_{2\ell,2}$ to denote the symplectic algebras isomorphic to $\mathfrak{sp}_{2\ell}$ acting on $\mathcal{P}(M_{n,\ell})$ and $\mathcal{P}(M_{m,\ell})$ respectively. The action on $\mathcal{P}(M_{n+m,\ell})$ is the diagonal action of the two, and is denoted by $\Delta\mathfrak{sp}_{2\ell}$.

Theorem 10.1. *Assume the stable range* $\min(m,n) > 2\ell$. *We have the following isomorphisms of $(\widehat{A}_\ell^+)^3 \times (\widehat{A}_\ell^+)^3 \times (\widehat{A}_\ell^+)^3$-graded algebras:*

$$Gr_{(\widehat{A}_\ell^+)^3 \times (\widehat{A}_\ell^+)^3 \times (\widehat{A}_\ell^+)^3}\left(\mathcal{P}(M_{n+m,\ell})/I(\mathcal{J}^+_{n+m,\ell})\right)^{U_{O_n} \times U_{O_m} \times U_\ell}$$
$$\simeq \left(\mathcal{R}^+(GL_\ell/U_\ell) \otimes \mathcal{R}^+(GL_\ell/U_\ell) \otimes \mathcal{R}^{+2}(GL_\ell/U_\ell)\right)^{U_\ell}.$$

Remark: The content of Theorem 10.1 in terms of multiplicities can be found in [HTW1; see formula (2.2.2)], [Ki2; see (2.16)] and [KT; see Theorem 2.5 and corollary 2.6]. With this result it is possible to compute a basis of the reciprocity algebra for $(O_{n+m}, O_n \times O_m)$ using [HTW2]; see second preprint of [HL].

Proof. In the stable range, we have

$$\mathcal{P}(M_{n+m,\ell}) \simeq \left(\bigoplus_\lambda E^\lambda_{(n+m)} \otimes F^\lambda_{(\ell)}\right) \otimes \mathcal{S}(\Delta\mathfrak{sp}_{2\ell}^{(2,0)})$$
$$\simeq \mathcal{P}(M_{n,\ell} \oplus M_{m,\ell})$$
$$\simeq \left(\left(\bigoplus_\mu E^\mu_{(n)} \otimes F^\mu_{(\ell)}\right) \otimes \mathcal{S}(\mathfrak{sp}_{2\ell,1}^{(2,0)})\right) \otimes \left(\left(\bigoplus_\nu E^\nu_{(m)} \otimes F^\nu_{(\ell)}\right) \otimes \mathcal{S}(\mathfrak{sp}_{2\ell,2}^{(2,0)})\right)$$
$$\simeq \left(\bigoplus_{\mu,\nu} E^\mu_{(n)} \otimes E^\nu_{(m)} \otimes F^\mu_{(\ell)} \otimes F^\nu_{(\ell)}\right) \otimes \mathcal{S}(\mathfrak{sp}_{2\ell,1}^{(2,0)} \oplus \mathfrak{sp}_{2\ell,2}^{(2,0)}). \tag{10.2}$$

From this, we form the reciprocity algebra

$$\left(\mathcal{P}(M_{n+m,\ell})/I(\mathcal{J}^+_{n+m,\ell})\right)^{U_{O_n}\times U_{O_m}\times U_\ell}$$

$$\simeq \bigoplus_{\mu,\nu}(E^\mu_{(n)})^{U_{O_n}} \otimes (E^\nu_{(m)})^{U_{O_m}} \otimes \left((\widetilde{E}^\mu_{(2\ell)} \otimes \widetilde{E}^\nu_{(2\ell)})/\left(\Delta\mathfrak{sp}^{(2,0)}_{2\ell} \cdot (\widetilde{E}^\mu_{(2\ell)} \otimes \widetilde{E}^\nu_{(2\ell)})\right)\right)^{U_\ell}.$$

In the stable range $\min(n,m) > 2\ell$, we have

$$\widetilde{E}^\mu_{(2\ell)}\mid_{GL_\ell} \simeq F^\mu_{(\ell)} \otimes \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2\ell,1}) \simeq F^\mu_{(\ell)} \otimes \mathcal{S}(\mathcal{S}^2\mathbb{C}^\ell),$$

and

$$\widetilde{E}^\nu_{(2\ell)}\mid_{GL_\ell} \simeq F^\nu_{(\ell)} \otimes \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2\ell,2}) \simeq F^\nu_{(\ell)} \otimes \mathcal{S}(\mathcal{S}^2\mathbb{C}^\ell).$$

Thus, the GL_ℓ isomorphism classes in

$$(\widetilde{E}^\mu_{(2\ell)} \otimes \widetilde{E}^\nu_{(2\ell)})/\left(\Delta\mathfrak{sp}^{(2,0)}_{2\ell} \cdot (\widetilde{E}^\mu_{(2\ell)} \otimes \widetilde{E}^\nu_{(2\ell)})\right)$$

are given by the GL_ℓ isomorphism classes of

$$F^\mu_{(\ell)} \otimes F^\nu_{(\ell)} \otimes \mathcal{S}(\mathcal{S}^2\mathbb{C}^\ell).$$

This is simply the branching for tensor products of holomorphic discrete series as in [Rep]. The (associated graded of the) branching algebra of O_{m+n} to $O_m \times O_n$ can thus be identified with a triple tensor product algebra of GL_ℓ:

$$\left(\mathcal{R}^+(GL_\ell/U_\ell) \otimes \mathcal{R}^+(GL_\ell/U_\ell) \otimes \mathcal{R}^{+2}(GL_\ell/U_\ell)\right)^{U_\ell}. \qquad \square$$

Appendix: A Proof of the Separation of Variables Theorem

We provide a simple proof here for $G = O_n$. It could be adapted easily for Sp_{2n} acting on copies of $\mathbb{C}^{2n}$ or GL_n acting on copies of $\mathbb{C}^n$ and $\mathbb{C}^{n*}$.

Let O_n act by the usual left multiplication on $M_{n,m}$, the $n\times m$ matrices. We assume that we are in the stable range, which is $n > 2m$. Let r_{ij} be the invariant pairing between the i-th column and the j-th column. Recall that $\mathcal{J}_{n,m} = \mathcal{P}(M_{n,m})^{O_n}$ is generated freely by the homogeneous quadratic polynomials $\{r_{ij}\}$ in this range (see [GW] Theorem 5.2.7). Also, the space of harmonics, i.e., polynomials annihilated by all the differential operators dual to the r_{ij}'s, is denoted by $\mathcal{H}_{n,m}$ as in Theorem 2.3(a).

Theorem (Separation of Variables) *If $n \geq 2m$, then*

$$\mathcal{P}(M_{n,m}) \simeq \mathcal{H}_{n,m} \otimes \mathcal{P}(M_{n,m})^{O_n}$$

Remarks. Proofs of this result for orthogonal groups (see Theorem 2.5 of [TT1]) and for symplectic groups (see Theorem 1.10 of [TT2]) are given by Ton-That using results of [Kos].

Proof of Separation of Variables Theorem: Let $\mathcal{I}(\mathcal{J}_{n,m}^+)$ be the ideal in $\mathcal{P}(M_{n,m})$ generated by r_{ij}'s with zero constant terms, and consider $\mathcal{P}(M_{n,m})$ as an $\mathcal{I}(\mathcal{J}_{n,m}^+)$ module by multiplication.

We shall need the notion of a regular sequence. First, denote the ideal in $\mathcal{P}(M_{n,m})$ generated by $\{f_1, \dots, f_s\}$ by the symbol $< f_1, \dots, f_s > \mathcal{P}(M_{n,m})$. A sequence $\{f_1, \dots, f_k\} \subset \mathcal{I}(\mathcal{J}_{n,m}^+)$ forms a *regular sequence* for $\mathcal{P}(M_{n,m})$ if

(a) f_i is not a zero-divisor on $\mathcal{P}(M_{n,m})/ < f_1, \dots, f_{i-1} > \mathcal{P}(M_{n,m})$ for all $i = 1, \dots, k$, and

(b) $\mathcal{P}(M_{n,m})/ < f_1, \dots, f_k > \mathcal{P}(M_{n,m})$ is non-zero.

Geometrically, saying that a given function f is not a zero-divisor on $\mathcal{P}(M_{n,m})/ < f_1, \dots, f_{i-1} >$ is the same as saying that f does not vanish identically on any irreducible component of the zero set of $\{f_1, \dots, f_{i-1}\}$. This in turn is the same as saying that each irreducible component of $\{f_1, \dots, f_{i-1}, f\}$ has dimension one less that the component of the zero set of $\{f_1, \dots, f_{i-1}\}$.

Separation of variables would follow from knowing that the r_{ij}'s (in some order) form a regular sequence for $\mathcal{P}(M_{n,m})$. In fact, you can take any order you want. For an ideal I in a commutative ring S, we have the chain:

$$S \supseteq I \supseteq I^2 \supseteq I^3 \supseteq \dots$$

and we can thus form the associated graded algebra

$$\mathrm{Gr}_I S = S/I \oplus I/I^2 \oplus I^2/I^3 \oplus \dots$$

with multiplication (setting $S = I^0$)

$$I^i/I^{i+1} \otimes I^j/I^{j+1} \longrightarrow I^{i+j}/I^{i+j+1}$$

induced by multiplication on S. In our context, $S = \mathcal{P}(M_{n,m})$ and $\mathcal{P}(M_{n,m})/\mathcal{I}(\mathcal{J}_{n,m}^+)$ is the coordinate ring of the null-cone and as a linear space (in particular, as a $O_n \times GL_m$ module), it is isomorphic to $\mathcal{H}_{n,m}$. If $\{r_{ij}\} \subset \mathcal{I}(\mathcal{J}_{n,m}^+)$ is a regular sequence of $\mathcal{P}(M_{n,m})$, then we have a nice presentation of $\mathrm{Gr}_{\mathcal{I}(\mathcal{J}_{n,m}^+)}\mathcal{P}(M_{n,m})$:

Ree's Theorem (see Theorem 2.1 of [Ree]) *If I is generated by a regular sequence $f_1, \dots, f_n$, then the map $\phi : (S/I)[x_1, \dots, x_n] \leftarrow Gr_I S$, sending x_i to the class f_i in I/I^2 is an isomorphism.*

Ree's Theorem thus implies that as vector spaces, we have $S = S/I \otimes \mathbb{C}[f_1, \dots, f_n]$, and in our context, $\mathcal{P}(M_{n,m}) \cong \mathcal{H}_{n,m} \otimes \mathcal{J}_{n,m}$.

Thus we want to show that indeed $\{r_{ij}\}$ form a regular sequence in $\mathcal{I}(\mathcal{J}_{n,m}^+)$. To show this, consider the map from $M_{n,m}$ to the $m \times m$ symmetric matrices $\mathcal{S}^2(\mathbb{C}^m)$ by putting the r_{ij} in a matrix:

$$Q : M_{n,m} \longrightarrow \mathcal{S}^2(\mathbb{C}^m) \qquad \text{where} \qquad Q(T) = T^t T, \qquad T \in M_{n,m}.$$

First observe that this map is $O_n \times GL_m$ equivariant:

$$Q(gTh) = h^t Q(T) h, \qquad g \in O_n, h \in GL_m.$$

Further, the O_n and GL_m actions commute.

Let us study the fibers of the map Q. Define the following rank m matrices in $M_{n,m}$: For $k = 0, 1, \ldots, m$, let

$$T_k = [c_1 c_2 \ldots c_k c_{k+1} \ldots c_m] \in M_{n,m}$$

where $\{c_1, c_2, \ldots, c_k\}$ is an *orthonormal* set of *non-isotropic* vectors in $\mathbb{C}^n$ and $\{c_{k+1}, \ldots, c_m\}$ is an *orthogonal* set of *isotropic* vectors in $\mathbb{C}^n$. It is easy to see that

$$Q(T_k) = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix. Next, for $i = 0, 1, \ldots, k$, define the sets $\Theta_k \subset M_{n,m}$ as follows:

$$\Theta_k = \{X \in M_{n,m} \mid Q(X) = Q(T_k) \text{ and } \operatorname{rank} X = \operatorname{rank} T_k = m\}.$$

Let us remind our readers on the following version of Witt's Theorem (see Theorem 3.7.1 of [Ho4]):

Witt's Theorem *Given two $n \times m$ matrices T_1 and T_2, there is an orthogonal $n \times n$ matrix g such that $gT_1 = T_2$ if and only if $Q(T_1) = Q(T_2)$ and $\ker T_1 = \ker T_2$.*

Since X and T_k are of full rank, i.e., $\operatorname{rank} X = \operatorname{rank} T_k = m$, thus $\ker X = \ker T_k = \{0\}$. By Witt's Theorem, Θ_k is an O_n orbit in the fiber $Q^{-1}(Q(T_k)) \subset M_{n,m}$. The full rank condition gives the openness and denseness of this orbit. The *null cone* (or null fiber) $NCQ = \Theta_0$ corresponds to $k = 0$.

We claim that the fibers of the map Q are all varieties of the same dimension, i.e., Q is an *equi-dimensional* map:

Proposition *Consider the $O_n \times GL_m$-equivariant map Q. Then for each $Y \in \mathcal{S}^2(\mathbb{C}^m)$,*

(a) *the fiber $Q^{-1}(Y)$ is an irreducible variety, invariant under O_n, and contains an open dense orbit of the form $\Theta_k \cdot h$, for some $h \in GL_m$ and some $k = 0, 1, \ldots, m$.*
(b) *the dimension of each fiber $Q^{-1}(Y)$ is $f = nm - \frac{m(m+1)}{2}$, which is independent of the fiber.*

Proof of Proposition: For $Y \in \mathcal{S}^2(\mathbb{C}^m)$, we can find $h \in GL_m$ such that $h^t Y h = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$, for some $k = 0, 1, \ldots, m$. If $X \in M_{n,m}$ is such that $Q(X) = Y$, then $Q(Xh) = h^t Q(X) h = h^t Y h = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$. Further, $\operatorname{rank} Xh = \operatorname{rank} X$. In other words, the fiber $Q^{-1}(Y)$ associated to $Y \in \mathcal{S}^2(\mathbb{C}^m)$ is the closure of an open dense orbit given by $\Theta_k \cdot h$, and hence an irreducible variety.

The fact that the orbits Θ_k (and their translates) have the same dimension follows from the computation of the dimension of the pointwise stabilizer in O_n of $T_k \in M_{n,m}$. The pointwise stabilizer can be easily computed for each $k = 0, 1, \ldots, m$. The dimension of each fiber is $f = nm - \frac{m(m+1)}{2}$. (You can see this dimension from the null cone pretty easily because the ring of O_n invariants is freely generated by $\frac{m(m+1)}{2}$ polynomials r_{ij}'s.) □

Since the mapping Q has equi-dimensional fibres, if $V \subset \mathcal{S}^2(\mathbb{C}^m)$ is any irreducible variety of dimension e, then $Q^{-1}(V)$ will be an irreducible variety of dimension $e+f$. The $Q(r_{ij})$'s are coordinates on $\mathcal{S}^2(\mathbb{C}^m)$, so the variety defined by d of them is a subspace of codimension d. It follows that the pullback of this subspace by Q is also irreducible and of codimension d. Therefore, the dimension of the zero set of r_{ij}'s decrease by 1 at each stage, making $\{r_{ij}\}$ a regular sequence (see Lemma 4 on page 105 of [Mat]). □

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