Rooted edges of a minimal directed spanning tree on random points

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Abstract

For n i.i.d. uniform points in $[0,1]^d$, $d \geq 2$, let L_n be the total distance from the origin to all the minimal points under the coordinate-wise partial order (this is also the total length of rooted edges of a minimal directed spanning tree on the given n random points). For $d \geq 3$, we establish the asymptotics of the mean and the variance of L_n, and show that L_n satisfies a central limit theorem, unlike in the case $d = 2$.

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1 Introduction and statement of results.

For $d \geq 2$, let $\prec$ be the coordinate-wise partial order on $\mathbf{R}^d$; $\mathbf{x} \prec \mathbf{y}$ if and only if all coordinates of $\mathbf{y} - \mathbf{x}$ are nonnegative and $\mathbf{x} \neq \mathbf{y}$. For $S \subset \mathbf{R}^d$ and $\mathbf{x} \in S$, we say $\mathbf{x}$ is a minimal element of S if no $\mathbf{y} \in S$ satisfies $\mathbf{y} \prec \mathbf{x}$, and $\mathbf{x}$ is a maximal element of S if no $\mathbf{y} \in S$ satisfies $\mathbf{x} \prec \mathbf{y}$. Let $\mathcal{M}(S)$ denote the set of minimal elements of S. In this paper our major interest is $\mathcal{M}(S)$ where S is a random set $\mathcal{X}_n$ consisting of n i.i.d.

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uniform points in $[0,1]^d$, $d \geq 3$. More precisely, in this paper we study the asymptotics of the random variables L_n given by

$$L_n := \sum_{\mathbf{x} \in \mathcal{M}(\mathcal{X}_n)} |\mathbf{x}| \tag{1.1}$$

where $|\cdot|$ denotes the Euclidean norm.

The quantity L_n arises in the context of a certain spanning tree problem, which we now describe. Suppose S is a finite subset of $[0,1]^d$, and let $\mathbf{0}$ denote the origin of $\mathbf{R}^d$. Then $\mathbf{0}$ is the only minimal element of $S \cup \{\mathbf{0}\}$. A directed spanning tree on $S \cup \{\mathbf{0}\}$ is a directed graph G with vertex set $S \cup \{\mathbf{0}\}$, such that (i) all directed edges are of the form $(\mathbf{x}, \mathbf{y})$ with $\mathbf{y} \prec \mathbf{x}$, and (ii) for every $\mathbf{x} \in S$, there is a unique directed path in G from $\mathbf{x}$ to $\mathbf{0}$. The length of G, denoted $L(G)$, is the sum of the Euclidean lengths of its edges. A minimal directed spanning tree (MDST) on $S \cup \{\mathbf{0}\}$ is a directed spanning tree G with the property that $L(G) \leq L(G')$ for every other directed spanning tree G' on $S \cup \{\mathbf{0}\}$. It can be shown that the minimal directed spanning tree on $\mathcal{X}_n \cup \{\mathbf{0}\}$ is almost surely unique.

The study of minimal directed spanning trees on random points was initiated by Bhatt and Roy [6], motivated by applications to communications and drainage networks. The construction of the MDST resembles that of other graphs where edges are drawn between nearby points in Euclidean space, such as the 'ordinary' minimal spanning tree, the nearest neighbor graph and the geometric graph. The probability theory of graphs of this type on random points is well-developed; see for example [12, 13, 15, 16, 17, 18]. However, the MDST has some distinctive features, notably the fact that there is no uniform bound on vertex degrees, and the presence of significant boundary effects. In view of these features, it is a reasonable first step to consider the rooted edges of the MDST, i.e., those edges that are incident to the origin.

For $\mathbf{x} \in \mathcal{M}(S)$, the edge $(\mathbf{x}, \mathbf{0})$ is in any directed spanning tree on $S \cup \{\mathbf{0}\}$. Conversely, if $\mathbf{x} \in S$ with $(\mathbf{x}, \mathbf{0})$ an edge of a MDST G on $S \cup \{\mathbf{0}\}$ then $\mathbf{x}$ must be in $\mathcal{M}(S)$ (since otherwise, one could find $\mathbf{y} \in \mathcal{M}(S)$ with $\mathbf{y} \prec \mathbf{x}$ and improve on the length of G by replacing the edge $(\mathbf{x}, \mathbf{0})$ by an edge $(\mathbf{x}, \mathbf{y})$). Consequently, the set of rooted edges of a MDST on $S \cup \{\mathbf{0}\}$ is precisely the set of edges $(\mathbf{x}, \mathbf{0}), \mathbf{x} \in \mathcal{M}(S)$.

So, the number of rooted edges is precisely the number of minimal elements of S, which we denote $|\mathcal{M}(S)|$. This quantity is of interest in multivariate extreme value theory, and the probability theory of $|\mathcal{M}(\mathcal{X}_n)|$ has received a degree of recent attention (see [1], [2] and references therein). In particular, Bai et al. [3] recently established that $|\mathcal{M}(\mathcal{X}_n)|$ satisfies a central limit theorem for $d \geq 2$. (Actually they consider the number of maxima in $\mathcal{X}_n$, which obviously has the same distribution as the number of minima.)

In the present work we are concerned instead with the quantity L_n defined at (1.1), which is the total length of the rooted edges of the MDST on $\mathcal{X}_n$. In the case $d = 2$, Bhatt and Roy [6] showed that the distribution of L_n converges weakly to a certain limiting distribution with corresponding convergence of all moments; subsequently, Penrose and Wade [14] identified the limiting distribution as a type of Dickman distribution. It is clear that this limiting distribution is non-normal since it is supported by the half-line $[0, \infty)$ (no re-scaling or centering of L_n is required in Bhatt and Roy's result).

Thus, for $d = 2$ there is a distinction between the limiting distribution of L_n (which is not normal) and that of a renormalized version of $|\mathcal{M}_n|$, which on the contrary is normal. This distinction is essentially due to the effect of long edges. It is natural to ask whether this distinction persists into higher dimensions, and in this paper we answer this question in the negative by showing that for $d \geq 3$, the limiting distribution of L_n (suitably scaled and centered) is indeed normal, using a method related to that of [3]. Moreover, we give precise asymptotic expressions for the mean and variance of L_n.

As a final introductory remark, we note that there is a resemblance between the study of minimal elements of a random sample, as in the present paper, and the study of convex hulls of random samples. In the latter subject, quite a lot is known [7, 8, 10] for $d = 2$, but much less is known in higher dimensions, so far as we are aware.

In this paper we use the notation $A_n \asymp B_n$ to denote $A_n = B_n(1 + O((\ln n)^{-1}))$. Here are the precise asymptotic expressions for the mean and variance of L_n.

Theorem 1. For $d \geq 3$, as $n \to \infty$

$$EL_n \asymp \frac{d}{(d-2)!}(\ln n)^{d-2}. \tag{1.2}$$

Theorem 2. For $d \geq 3$, as $n \to \infty$

$$Var(L_n) \asymp \left(\frac{1}{2}\frac{d}{(d-2)!} + 2\sum_{k=1}^{d-1}\binom{d}{k}kh_k - \gamma_d\right)(\ln n)^{d-2} \tag{1.3}$$

where h_k, $1 \leq k \leq d-1$, and $\gamma_d (< d/(2 \cdot (d-2)!))$ are strictly positive finite constants given by for $k = 1$,

$$h_1 = \int_0^1 dw_1 \int_0^1 dw_2 w_1 \left((w_1 + w_2 - w_1 w_2)^{-2} - (w_1 + w_2)^{-2}\right) \frac{1}{2} \frac{1}{(d-2)!} \frac{(-\ln w_2)^{d-2}}{(d-2)!} \tag{1.4}$$

for $2 \le k \le d - 1$,

$$h_k = \int_0^1 du_1 \int_0^{u_1} dw_1 \int_0^1 dw_2 \left((w_1 + w_2 - w_1 w_2)^{-2} - (w_1 + w_2)^{-2}\right) \frac{1}{2} \frac{1}{(d-2)!} \frac{(-\ln w_1 + \ln u_1)^{k-2}}{(k-2)!} \frac{(-\ln w_2)^{d-k-1}}{(d-k-1)!} \tag{1.5}$$

and

$$\begin{aligned} \gamma_d &= \frac{d}{((d-2)!)^2} \left(\int_0^1 dv_1 \int_0^1 ds \frac{1}{(1 + v_1 s)^2} (\ln \frac{1}{s})^{d-2} v_1 \right) \\ &< \frac{d}{((d-2)!)^2} \left(\int_0^1 dv_1 \int_0^1 ds (\ln \frac{1}{s})^{d-2} v_1 \right) \\ &= \frac{1}{2} \frac{d}{(d-2)!}. \end{aligned} \tag{1.6}$$

Our final result, Theorem 3, is a central limit theorem for L_n. To state it, we introduce the following notation: we write $Y_n \in CLT(r_n)$ if

$$\sup_x \left| P\left(\frac{Y_n - EY_n}{(Var(Y_n))^{1/2}} \le x \right) - \Phi(x) \right| = O(r_n) \text{ and } r_n \to 0 \tag{1.7}$$

where $\Phi(x)$ is the cumulative distribution function for the standard normal distribution.

Theorem 3. For $d \ge 3$, as $n \to \infty$

$$\frac{L_n - EL_n}{(Var(L_n))^{1/2}} \to N(0, 1)$$

in distribution. In fact, we have $L_n \in CLT((\ln n)^{-(d-2)/4} (\ln \ln n)^{(d+1)/2})$.

In Sections 2 and 3 we prove Theorems 1 and 2, respectively. We write the mean and the variance of L_n as the explicit integrals and using two elementary but useful inequalities (2.3) and (2.7) we approximate the explicit integrals as the tractable integrals. By evaluating these tractable integrals we obtain the asymptotic expressions (1.2) and (1.3).

In Section 4 we prove Theorem 3. With the help of a certain transformation, we approximate L_n by a space-truncated random variable conditioned on a very likely event. Then we approximate this conditioned space-truncated random variable by a random variable L''_n generated by a Poisson point process. Decomposing L''_n as a sum of locally dependent random variables, we can apply Stein's method to L''_n and obtain the central limit theorem for L''_n. Since our approximation errors turn out to be small, we can dig out the central limit theorem for L_n, Theorem 3, from the central limit theorem for L''_n.

2 Expectation.

Let $\mathcal{X}_n$ be the collection $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of i.i.d. uniform points on $[0,1]^d$, $d \geq 3$. Given $\mathcal{X}_n$, denote the event that $\mathbf{x}_i$ is minimal in $\mathcal{X}_n$ by G_i. Then, we can rewrite L_n as

$$L_n = \sum_{i=1}^{n} |\mathbf{x}_i| 1_{G_i}. \tag{2.1}$$

In this section we prove Theorem 1. Using (2.1) we write EL_n as an explicit integral (2.2). By two elementary but useful inequalities (2.3) and (2.7) we approximate the explicit integral as a tractable integral, the right hand side of (2.4) below. By evaluating this tractable integral we have Theorem 1.

By (2.1) and by the exchangeability of $\mathbf{x}_i$'s, we have $EL_n = nE|\mathbf{x}_1|1_{G_1}$ and hence

$$EL_n = n \int_0^1 \cdots \int_0^1 (x_1^2 + \cdots + x_d^2)^{1/2} (1 - \prod_{i=1}^{d} x_i)^{n-1} dx_1 \cdots dx_d. \tag{2.2}$$

To evaluate the length of a vector, we use the following elementary inequalities

$$(\sum_{i=1}^{d} x_i) \left(1 - \frac{\sum_{i \neq j} x_i x_j}{(\sum_{i=1}^{d} x_i)^2}\right) \leq (\sum_{i=1}^{d} x_i^2)^{1/2} \leq \sum_{i=1}^{d} x_i. \tag{2.3}$$

By the second inequality of (2.3), we have

$$\begin{aligned} EL_n &\leq n \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_d)(1 - \prod_{i=1}^{d} x_i)^{n-1} dx_1 \cdots dx_d \\ &= dn \int_0^1 \cdots \int_0^1 x_1 (1 - \prod_{i=1}^{d} x_i)^{n-1} dx_1 \cdots dx_d \\ &\leq dn \int_0^1 \cdots \int_0^1 x_1 e^{-(n-1)\prod_{i=1}^{d} x_i} dx_1 \cdots dx_d. \end{aligned} \tag{2.4}$$

We further replace $e^{-(n-1)\prod_{i=1}^d x_i}$ by $e^{-n\prod_{i=1}^d x_i}$ in the right hand side of (2.4). One can easily see that the error caused by this replacement is bounded by n^{-1} times the order of the right hand side of (2.4). So, with the following change of variable; $x_1 = x$, $x_i = e^{-y_i}$, $i = 2, \ldots, d$, we see that the right hand side of (2.4) is asymptotically equivalent to

$$\begin{aligned}
& dn \int_0^1 \cdots \int_0^1 x_1 e^{-n\prod_{i=1}^d x_i} dx_1 \cdots dx_d \\
= \;& dn \int_0^1 dx \int_0^\infty \cdots \int_0^\infty x e^{-nxe^{-\sum_{j=2}^d y_j}} e^{-\sum_{j=2}^d y_j} dy_2 \cdots dy_d \text{ (by setting } \textstyle\sum_{j=2}^d y_j = z) \\
= \;& dn \int_0^1 dx \int_0^\infty x e^{-nxe^{-z}} e^{-z} \frac{z^{d-2}}{(d-2)!} dz \text{ (by setting } z - \ln nx = u) \\
= \;& d \int_0^1 dx \int_{-\ln nx}^\infty e^{-e^{-u}} e^{-u} \frac{(u + \ln nx)^{d-2}}{(d-2)!} du \text{ (by setting } e^{-u} = v) \\
= \;& d \int_0^1 dx \int_0^{nx} e^{-v} \frac{(-\ln v + \ln nx)^{d-2}}{(d-2)!} dv. \qquad (2.5)
\end{aligned}$$

Now, we expand the term $(-\ln v + \ln nx)^{d-2} = (-\ln v + \ln n + \ln x)^{d-2}$ and integrate term by term. Then, one can easily see that the integration with the $(\ln n)^{d-2}$ term is the leading term. So, as $n \to \infty$

$$\begin{aligned}
dn \int_0^1 \cdots \int_0^1 x_1 e^{-n\prod_{i=1}^d x_i} dx_1 \cdots dx_d \;&\asymp\; d \int_0^1 dx \int_0^{nx} e^{-v} \frac{(\ln n)^{d-2}}{(d-2)!} dv \\
&\asymp\; d \int_0^1 dx \int_0^{\infty} e^{-v} \frac{(\ln n)^{d-2}}{(d-2)!} dv \\
&=\; \frac{d}{(d-2)!} (\ln n)^{d-2}. \qquad (2.6)
\end{aligned}$$

Before we continue our presentation, we would like to point out that many integral calculations in Sections 2 and 3 are very similar to those in (2.4), (2.5), and (2.6). So, we refer to the integral calculations similar to those in (2.4), (2.5), and (2.6) as *the usual calculation* and sometimes we denote the usual calculation as $\cdots$ in the equations.

Note that to get an asymptotic upper bound (2.6) of EL_n, we use two elementary but useful inequalities, the second inequality of (2.3) and the second inequality of (2.7);

$$(1 - nx^2)e^{-nx} \leq (1 - x)^n \leq e^{-nx}. \qquad (2.7)$$

So, the difference between EL_n and the asymptotic upper bound (2.6) consists of two parts; the error which is caused by the use of the second inequality of (2.7) and the error which is caused by the use of the second inequality of (2.3).

By the usual calculation we see that the error which is caused by the use of the second inequality of (2.7) is bounded by (using the first inequality of (2.7))

$$dn^2\int_0^1\cdots\int_0^1 x_1\prod_{i=1}^d x_i^2 e^{-n\prod_{i=1}^d x_i}dx_1\cdots dx_d = O(n^{-1}(\ln n)^{d-2}). \qquad (2.8)$$

By the usual calculation again we also see that The error which is caused by the use of the second inequality of (2.3) is bounded by (using the first inequality of (2.3) with notion $\mathbf{x}=(x_1,\ldots,x_d)$)

$$\begin{aligned}
& n\int_0^1\cdots\int_0^1 \frac{\sum_{i\neq j}x_ix_j}{\sum_{i=1}^d x_i}e^{-n\prod_{i=1}^d x_i}d\mathbf{x}\\
= \;& d(d-1)n\int_0^1\cdots\int_0^1 \frac{x_1x_2}{\sum_{i=1}^d x_i}e^{-n\prod_{i=1}^d x_i}d\mathbf{x}\\
\leq \;& d^2n\int_0^1\cdots\int_0^1 \frac{x_1x_2}{x_1+x_2}e^{-n\prod_{i=1}^d x_i}d\mathbf{x} \text{ (GM-HM inequality)}\\
\leq \;& d^2n\int_0^1\cdots\int_0^1 \sqrt{x_1x_2}e^{-n\prod_{i=1}^d x_i}d\mathbf{x}\\
\cdots &\\
= \;& O((\ln n)^{d-3}). \qquad (2.9)
\end{aligned}$$

Therefore, Theorem 1 follows from (2.6), (2.8) and (2.9).

3 Variance.

In this section we prove Theorem 2. The basic idea of the proof of Theorem 2 is the same as that of Theorem 1. Using (3.1) we write $Var(L_n)$ as the explicit integrals. By inequalities (2.3) and (2.7) we approximate the explicit integrals as the tractable integrals. Then, by evaluating these tractable integrals we have Theorem 2. Comparing with the proof of Theorem 1, in the proof of Theorem 2 there are many integrals and the error estimates are more complicated.

We start with an obvious observation; by (2.1),

$$\begin{aligned}
Var(L_n) &= \sum_{i=1}^n Var(|\mathbf{x}_i|1_{G_i}) + \sum_{i\neq j} Cov(|\mathbf{x}_i|1_{G_i},|\mathbf{x}_j|1_{G_j})\\
&= nVar(|\mathbf{x}_1|1_{G_1}) + n(n-1)Cov(|\mathbf{x}_1|1_{G_1},|\mathbf{x}_2|1_{G_2})\\
&\asymp nVar(|\mathbf{x}_1|1_{G_1}) + n^2Cov(|\mathbf{x}_1|1_{G_1},|\mathbf{x}_2|1_{G_2}). \qquad (3.1)
\end{aligned}$$

Since $nVar(|\mathbf{x}_1|1_{G_1}) = n(E|\mathbf{x}_1|^21_{G_1} - (E|\mathbf{x}_1|1_{G_1})^2)$, we estimate $nE|\mathbf{x}_1|^21_{G_1}$ first.

By the usual calculation, we have

$$
\begin{aligned}
nE|\mathbf{x}_1|^2 1_{G_1} &= n\int_0^1 \cdots \int_0^1 (x_1^2+\cdots+x_d^2)(1-\prod_{i=1}^d x_i)^{n-1} dx_1\cdots dx_d \\
&= dn \int_0^1 \cdots \int_0^1 x_1^2 (1-\prod_{i=1}^d x_i)^{n-1} dx_1 \cdots dx_d \\
&\cdots \\
&\asymp \frac{1}{2}\frac{d}{(d-2)!}(\ln n)^{d-2}. \qquad (3.2)
\end{aligned}
$$

So, by (3.2) and (1.2)

$$
nVar(|\mathbf{x}_1|1_{G_1}) = nE|\mathbf{x}_1|^2 1_{G_1} - n(E|\mathbf{x}_1|1_{G_1})^2 \asymp \frac{1}{2}\frac{d}{(d-2)!}(\ln n)^{d-2}. \qquad (3.3)
$$

Now, let's look at the crossing term $n^2 Cov(|\mathbf{x}_1|1_{G_1}, |\mathbf{x}_2|1_{G_2})$. Let us say $\mathbf{x}$ dominates $\mathbf{y}$ if $\mathbf{y} \prec \mathbf{x}$, and let

$$
D = \left\{(\mathbf{x},\mathbf{y}) \in [0,1]^d \times [0,1]^d;\ \mathbf{x} \text{ does not dominate } \mathbf{y} \text{ and } \mathbf{y} \text{ does not dominate } \mathbf{x}\right\}.
$$

Then by symmetry, with the notation $\mathbf{x} = (x_1,\ldots,x_d)$ and $\mathbf{y} = (y_1,\ldots,y_d)$,

$$
\begin{aligned}
& n^2 Cov(|\mathbf{x}_1|1_{G_1}, |\mathbf{x}_2|1_{G_2}) \\
= \ & n^2 \Bigg(\int_D |\mathbf{x}||\mathbf{y}| \left(1-\prod_{i=1}^d x_i - \prod_{i=1}^d y_i + \prod_{i=1}^d (x_i \wedge y_i)\right)^{n-2} d\mathbf{x}d\mathbf{y} \\
& - \int_{[0,1]^d\times[0,1]^d} |\mathbf{x}||\mathbf{y}|(1-\prod_{i=1}^d x_i)^{n-1}(1-\prod_{i=1}^d y_i)^{n-1} d\mathbf{x}d\mathbf{y}\Bigg) \\
= \ & n^2 \left(\int_D |\mathbf{x}||\mathbf{y}| f(\mathbf{x},\mathbf{y}) d\mathbf{x}d\mathbf{y} - 2\int_{\mathbf{x}\prec\mathbf{y}} |\mathbf{x}||\mathbf{y}|(1-\prod_{i=1}^d x_i)^{n-1}(1-\prod_{i=1}^d y_i)^{n-1} d\mathbf{x}d\mathbf{y}\right) \\
:= \ & I_1 - I_2 \qquad (3.4)
\end{aligned}
$$

where

$$
f(\mathbf{x},\mathbf{y}) = \left(1-\prod_{i=1}^d x_i - \prod_{i=1}^d y_i + \prod_{i=1}^d (x_i\wedge y_i)\right)^{n-2} - (1-\prod_{i=1}^d x_i)^{n-1}(1-\prod_{i=1}^d y_i)^{n-1}.
$$

Since $1-\prod_{i=1}^d x_i - \prod_{i=1}^d y_i + \prod_{i=1}^d (x_i \wedge y_i) \geq (1-\prod_{i=1}^d x_i)(1-\prod_{i=1}^d y_i)$, we have $f(\mathbf{x},\mathbf{y}) \geq 0$ and hence $I_1 \geq 0$. To get the asymptotics of I_1, we decompose D according to the number k of the components of $\mathbf{x}$ which are larger than the corresponding components of $\mathbf{y}$. Then we have

$$\begin{aligned}
I_1 &= n^2 \sum_{k=1}^{d-1} \binom{d}{k} \int_{[0,1]^d \times [0,1]^d} (\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{d} x_i^2 u_i^2)^{1/2} (\sum_{i=1}^{k} x_i^2 u_i^2 + \sum_{i=k+1}^{d} x_i^2)^{1/2} \\
&\quad \Big(\big(1 - \prod_{i=1}^{d} x_i \prod_{j=k+1}^{d} u_j - \prod_{i=1}^{d} x_i \prod_{j=1}^{k} u_j + \prod_{i=1}^{d} x_i \prod_{j=1}^{d} u_j\big)^{n-2} \\
&\quad - \big(1 - \prod_{i=1}^{d} x_i \prod_{j=k+1}^{d} u_j\big)^{n-1} \big(1 - \prod_{i=1}^{d} x_i \prod_{j=1}^{k} u_j\big)^{n-1} \Big) \prod_{i=1}^{d} x_i d\mathbf{x} d\mathbf{u}.
\end{aligned}$$

We replace the product terms by the exponential terms as we did in (2.4)-(2.5). With these replacements there will be two errors of order $n^{-1/2}(\ln n)^{d-1}$: Still using the first inequality of (2.7), we see that the error caused by the replacement of $(1 - \prod_{i=1}^{d} x_i \prod_{j=k+1}^{d} u_j - \prod_{i=1}^{d} x_i \prod_{j=1}^{k} u_j + \prod_{i=1}^{d} x_i \prod_{j=1}^{d} u_j)^{n-2}$ is bounded (by the usual calculation) by

$$\begin{aligned}
& Cn^3 \int_{[0,1]^d \times [0,1]^d} \Big(\prod_{i=1}^{d} x_i\Big)^3 \big(\prod_{j=k+1}^{d} u_j + \prod_{j=1}^{k} u_j - \prod_{j=1}^{d} u_j \big)^2 \\
& \exp\Big(-(n-2) \prod_{i=1}^{d} x_i \big(\prod_{j=k+1}^{d} u_j + \prod_{j=1}^{k} u_j - \prod_{j=1}^{d} u_j \big) \Big) d\mathbf{x} d\mathbf{u} \ (x^3 \le x^{5/2} \text{ for } 0 \le x \le 1) \\
\le\ & Cn^3 \int_{[0,1]^d \times [0,1]^d} \Big(\prod_{i=1}^{d} x_i\Big)^{5/2} \big(\prod_{j=k+1}^{d} u_j + \prod_{j=1}^{k} u_j - \prod_{j=1}^{d} u_j \big)^2 \\
& \exp\Big(-(n-2) \prod_{i=1}^{d} x_i \big(\prod_{j=k+1}^{d} u_j + \prod_{j=1}^{k} u_j - \prod_{j=1}^{d} u_j \big) \Big) d\mathbf{x} d\mathbf{u}. \\
\cdots & \\
\le\ & Cn^{-1/2} (\ln n)^{d-1} \int_{[0,1]^d} \big(\prod_{j=k+1}^{d} u_j + \prod_{j=1}^{k} u_j - \prod_{j=1}^{d} u_j \big)^{-3/2} d\mathbf{u} \\
=\ & O(n^{-1/2} (\ln n)^{d-1}). \qquad (3.5)
\end{aligned}$$

Here, we have used the fact that

$$\begin{aligned}
& \int_{[0,1]^d} \big(\prod_{j=k+1}^{d} u_j + \prod_{j=1}^{k} u_j - \prod_{j=1}^{d} u_j \big)^{-3/2} d\mathbf{u} \ (\text{by } x + y \ge 2\sqrt{xy} \text{ for } x, y > 0) \\
\le\ & \int_{[0,1]^d} \big(2(\prod_{j=1}^{d} u_j)^{1/2} - \prod_{j=1}^{d} u_j \big)^{-3/2} d\mathbf{u} \ (\text{by } x \le \sqrt{x} \text{ for } 0 \le x \le 1) \\
\le\ & \int_{[0,1]^d} \big(\prod_{j=1}^{d} u_j \big)^{-3/4} d\mathbf{u} < \infty. \qquad (3.6)
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{[0,1]^d} \frac{\left(\prod_{j=k+1}^d u_j\right)^2}{\left(\prod_{j=k+1}^d u_j + \prod_{j=1}^k u_j\right)^{7/2}} d\mathbf{u} \text{ (by } (x+y)^4 \geq 4xy^3 \text{ for } x, y > 0) \\
\leq\ & C \int_{[0,1]^d} \Big(\prod_{j=k+1}^d u_j \Big)^{-5/8} \Big(\prod_{j=1}^k u_j \Big)^{-7/8} d\mathbf{u} < \infty, \qquad (3.7)
\end{aligned}$$

and since (by the argument of (3.7))

$$\int_{[0,1]^d} \frac{\left(\prod_{j=1}^k u_j\right)^2}{\left(\prod_{j=k+1}^d u_j + \prod_{j=1}^k u_j\right)^{7/2}} d\mathbf{u} < \infty, \qquad (3.8)$$

by the argument of (3.5) we see that the error caused by the replacement of $(1 - \prod_{i=1}^d x_i \prod_{j=k+1}^d u_j)^{n-1}(1 - \prod_{i=1}^d x_i \prod_{j=1}^k u_j)^{n-1}$ is also of order $n^{-1/2}(\ln n)^{d-1}$.

We further replace the factors $(n-2)$ and $(n-1)$ by n in the corresponding exponents without any harm to the leading term. Then, we replace $(\sum_{i=1}^k x_i^2 + \sum_{i=k+1}^d x_i^2 u_i^2)^{1/2}$ and $(\sum_{i=1}^k x_i^2 u_i^2 + \sum_{i=k+1}^d x_i^2)^{1/2}$ by $(\sum_{i=1}^k x_i + \sum_{i=k+1}^d x_i u_i)$ and $(\sum_{i=1}^k x_i u_i + \sum_{i=k+1}^d x_i)$, respectively. With this replacement there will be an error a_n and we will show that this error a_n is lower order than the main term. So,

$$\begin{aligned}
I_1 \ \asymp\ & n^2 \sum_{k=1}^{d-1} \binom{d}{k} \int_{[0,1]^d \times [0,1]^d} (\sum_{i=1}^k x_i + \sum_{i=k+1}^d x_i u_i)(\sum_{i=1}^k x_i u_i + \sum_{i=k+1}^d x_i) \\
& e^{-n \prod_{i=1}^d x_i \left(\prod_{j=k+1}^d u_j + \prod_{j=1}^k u_j \right)} \left(e^{n \prod_{i=1}^d x_i \prod_{j=1}^d u_j} - 1 \right) \prod_{i=1}^d x_i d\mathbf{x} d\mathbf{u} + a_n \\
=\ & n^2 \sum_{k=1}^{d-1} \binom{d}{k} \int_{[0,1]^d \times [0,1]^d} \Big[k x_1^2 u_1 + k(k-1) x_1 u_1 x_2 + k(d-k) x_1 x_d \\
& + (d-k) x_d^2 u_d + (d-k)(d-k-1) x_d u_d x_{d-1} + (d-k) k x_1 u_1 x_d u_d \Big] \\
& e^{-n \prod_{i=1}^d x_i \left(\prod_{j=k+1}^d u_j + \prod_{j=1}^k u_j \right)} \left(e^{n \prod_{i=1}^d x_i \prod_{j=1}^d u_j} - 1 \right) \prod_{i=1}^d x_i d\mathbf{x} d\mathbf{u} + a_n \\
:=\ & H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + a_n. \qquad (3.9)
\end{aligned}$$

We first look at the terms with $k \geq 2$ in H_1. Starting with the obvious change of variable $x_i = e^{-y_i}$ and $u_i = e^{-v_i}$, $2 \leq i \leq d$, and $\sum_{i=2}^d y_i = z$, $\sum_{j=2}^k v_j = w_1$, $\sum_{j=k+1}^d v_j = w_2$, by the usual argument we have

$$n^2 \int_{[0,1]^d \times [0,1]^d} x_1^2 u_1 e^{-n \prod_{i=1}^d x_i \left(\prod_{j=k+1}^d u_j + \prod_{j=1}^k u_j \right)} \left(e^{n \prod_{i=1}^d x_i \prod_{j=1}^d u_j} - 1 \right) \prod_{i=1}^d x_i d\mathbf{x} d\mathbf{u}$$

$$
\begin{aligned}
&\dots \\
&= \int_0^1 dx_1 \int_0^1 du_1 \int_0^{nx_1} dz \int_0^{u_1} dw_1 \int_0^1 dw_2 x_1 e^{-zw_2-zw_1}(e^{zw_1w_2}-1)z \\
&\quad \frac{(-\ln z+\ln nx_1)^{d-2}}{(d-2)!}\frac{(-\ln w_1+\ln u_1)^{k-2}}{(k-2)!}\frac{(-\ln w_2)^{d-k-1}}{(d-k-1)!} \\
&\asymp \int_0^1 dx_1 \int_0^1 du_1 \int_0^{\infty} dz \int_0^{u_1} dw_1 \int_0^1 dw_2 x_1 e^{-zw_2-zw_1}\Big(e^{zw_1w_2}-1\Big)z \\
&\quad \frac{(\ln n)^{d-2}}{(d-2)!}\frac{(-\ln w_1+\ln u_1)^{k-2}}{(k-2)!}\frac{(-\ln w_2)^{d-k-1}}{(d-k-1)!} \\
&= \int_0^1 du_1 \int_0^{u_1} dw_1 \int_0^1 dw_2 ((w_1+w_2-w_1w_2)^{-2}-(w_1+w_2)^{-2}) \\
&\qquad \frac{1}{2}\frac{(\ln n)^{d-2}}{(d-2)!}\frac{(-\ln w_1+\ln u_1)^{k-2}}{(k-2)!}\frac{(-\ln w_2)^{d-k-1}}{(d-k-1)!} \\
&:= h_k(\ln n)^{d-2}.
\end{aligned} \tag{3.10}
$$

In the above, the last asymptotic follows from the following; since for $0 \le w_1, w_2 \le 1$ and for very small but strictly positive ε

$$
1-(1-w_1)^{1-\varepsilon} \le w_1(1-w_1)^{-\varepsilon}, \tag{3.11}
$$

$$
w_1 \le w_1+w_2-w_1w_2, w_2 \le w_1+w_2-w_1w_2, (w_1+w_2)^{-1} \le (w_1+w_2-w_1w_2)^{-1}, \tag{3.12}
$$

we have

$$
\begin{aligned}
&\Big|\int_0^1 dx_1 \int_0^1 du_1 \int_{nx_1}^{\infty} dz \int_0^{u_1} dw_1 \int_0^1 dw_2 x_1 e^{-zw_2-zw_1}(e^{zw_1w_2}-1)z \\
&\quad \frac{(-\ln z+\ln nx_1)^{d-2}}{(d-2)!}\frac{(-\ln w_1+\ln u_1)^{k-2}}{(k-2)!}\frac{(-\ln w_2)^{d-k-1}}{(d-k-1)!}\Big| \\
&\le C(\ln n)^{d-2}\int_0^1 dx_1 \int_0^1 du_1 \int_{nx_1}^{\infty} dz \int_0^{u_1} dw_1 \int_0^1 dw_2 \\
&\quad e^{-zw_2-zw_1}(e^{zw_1w_2}-1)x_1 z\left(\frac{z}{x_1}\right)^{\varepsilon} w_1^{-\varepsilon}u_1^{-\varepsilon}w_2^{-\varepsilon} \\
&\le C(\ln n)^{d-2}\int_0^1 du_1 \int_0^{u_1} dw_1 \int_0^{\infty}\left(\min\left(\frac{z}{n},1\right)\right)^{1-\varepsilon} dz \int_0^1 dw_2 \\
&\quad e^{-zw_2-zw_1}(e^{zw_1w_2}-1)z^{1+\varepsilon}w_1^{-\varepsilon}u_1^{-\varepsilon}w_2^{-\varepsilon} \\
&\le C(\ln n)^{d-2}n^{-1+\varepsilon}\int_0^1 du_1 \int_0^{u_1} dw_1 \int_0^{\infty} dz \int_0^1 dw_2 \\
&\quad e^{-zw_2-zw_1}(e^{zw_1w_2}-1)z^2 w_1^{-\varepsilon}u_1^{-\varepsilon}w_2^{-\varepsilon} \\
&= C(\ln n)^{d-2}n^{-1+\varepsilon}\int_0^1 du_1 \int_0^{u_1} dw_1 \int_0^1 dw_2 \\
&\quad \Big(\frac{1}{(w_1+w_2-w_1w_2)^3}-\frac{1}{(w_1+w_2)^3}\Big)w_1^{-\varepsilon}u_1^{-\varepsilon}w_2^{-\varepsilon}
\end{aligned}
$$

$$
\begin{aligned}
&= C(\ln n)^{d-2}n^{-1+\varepsilon}\int_0^1 dw_1\int_0^1 dw_2 \\
&\qquad (1-(1-w_1)^{1-\varepsilon})\Big(\frac{1}{(w_1+w_2-w_1w_2)^3}-\frac{1}{(w_1+w_2)^3}\Big)w_1^{-\varepsilon}w_2^{-\varepsilon} \text{ (by (3.11))} \\
&\le C(\ln n)^{d-2}n^{-1+\varepsilon}\int_0^1 dw_1\int_0^1 dw_2\frac{w_1^{2-2\varepsilon}w_2^{1-\varepsilon}(1-w_1)^{-\varepsilon}}{(w_1+w_2-w_1w_2)^3(w_1+w_2)} \text{ (by (3.12))} \\
&\le C(\ln n)^{d-2}n^{-1+\varepsilon}\int_0^1 dw_1\int_0^1 dw_2\frac{1}{(1-w_1)^{\varepsilon}(w_1+w_2-w_1w_2)^{1+3\varepsilon}} \\
&= C(\ln n)^{d-2}n^{-1+\varepsilon}\int_0^1\frac{1}{(1-w_1)^{1+\varepsilon}}\Big(w_1^{-3\varepsilon}-1\Big)dw_1 \\
&= O((\ln n)^{d-2}n^{-1+\varepsilon}). \qquad (3.13)
\end{aligned}
$$

Now, we consider the term with $k=1$ in H_1. Recalling the above calculation, we see that there is no need to make the change of variables $\sum_{j=2}^k v_j = w_1$. With this in mind we just follow the above calculation and we see that the term with $k=1$ in H_1 is evaluated as

$$
\begin{aligned}
& n^2\int_{[0,1]^d\times[0,1]^d} x_1^2u_1e^{-n\prod_{i=1}^d x_i\left(\prod_{j=2}^d u_j+u_1\right)}\left(e^{n\prod_{i=1}^d x_i\prod_{j=1}^d u_j}-1\right)\prod_{i=1}^d x_i d\mathbf{x}d\mathbf{u} \\
\asymp\ & \int_0^1 du_1\int_0^1 dw_2u_1((u_1+w_2-u_1w_2)^{-2}-(u_1+w_2)^{-2})\frac{1}{2}\frac{(\ln n)^{d-2}}{(d-2)!}\frac{(-\ln w_2)^{d-2}}{(d-2)!} \\
=:\ & h_1(\ln n)^{d-2}. \qquad (3.14)
\end{aligned}
$$

Therefore, by (3.14) and (3.10) we have

$$
H_1 \asymp (\sum_{k=1}^{d-1}\binom{d}{k}kh_k)(\ln n)^{d-2}, \qquad (3.15)
$$

where h_k are given by (1.4)-(1.5). By the similar calculations, one can see that

$$
H_2 = O((\ln n)^{d-3}), \qquad (3.16)
$$

and

$$
H_3 = O((\ln n)^{d-3}). \qquad (3.17)
$$

By symmetry, we have

$$
H_4 = H_1,\ H_5 = H_2,\ H_6 = H_3. \qquad (3.18)
$$

Therefore, by (3.9), (3.15)-(3.18)

$$
I_1 \asymp 2\sum_{k=1}^{d-1}\binom{d}{k}kh_k(\ln n)^{d-2} + a_n. \qquad (3.19)
$$

Next, we sketch how to estimate the error term a_n. By the argument used in the proof of (2.9), we see that

$$\begin{aligned} a_n &\le Cn^2 \sum_{k=1}^{d-1} \binom{d}{k} \int_{[0,1]^d\times[0,1]^d} \\ &\quad \Big((x_1x_2)^{1/2} + (x_1x_du_d)^{1/2} + (x_{d-1}u_{d-1}x_du_d)^{1/2}\Big) \big(\sum_{i=1}^{k} x_iu_i + \sum_{i=k+1}^{d} x_i\big) \\ &\quad e^{-n\prod_{i=1}^d x_i\left(\prod_{j=k+1}^d u_j + \prod_{j=1}^k u_j\right)} \left(e^{n\prod_{i=1}^d x_i \prod_{j=1}^d u_j} - 1\right) \prod_{i=1}^{d} x_i d\mathbf{x}d\mathbf{u}. \end{aligned} \tag{3.20}$$

Now we just follow the argument for (3.10)-(3.19). Since each term in the expansion of $((x_1x_2)^{1/2} + (x_1x_du_d)^{1/2} + (x_{d-1}u_{d-1}x_du_d)^{1/2})(\sum_{i=1}^k x_iu_i + \sum_{i=k+1}^d x_i)$ has at least two different x_j's, by the argument for (3.10)-(3.19) we see that each integration in (3.20) is of order $(\ln n)^{d-3}$ and that

$$a_n = O((\ln n)^{d-3}). \tag{3.21}$$

We will show a similar calculation for I_2 in detail below.

Now let us look at $I_2 = 2n^2 \int_{\mathbf{x}\prec\mathbf{y}} |\mathbf{x}||\mathbf{y}|(1-\prod_{i=1}^d x_i)^{n-1}(1-\prod_{i=1}^d y_i)^{n-1} d\mathbf{x}d\mathbf{y}$. We make the change of variables $x_i = u_iv_i$ and $y_i = u_i$, $i = 1,\dots,d$. We also replace $(1-\prod_{i=1}^d x_i)^{n-1}$ and $(1-\prod_{i=1}^d y_i)^{n-1}$ by $e^{-n\prod_{i=1}^d x_i}$ and $e^{-n\prod_{i=1}^d y_i}$, respectively. As we see in (2.5), this approximation is harmless. Also, by using (2.7) we replace $(\sum_{i=1}^d u_i^2v_i^2)^{1/2}(\sum_{i=1}^d u_i^2)^{1/2}$ by $(\sum_{i=1}^d u_iv_i)(\sum_{j=1}^d u_j)$. So, we have

$$\begin{aligned} I_2 &\asymp 2n^2 \int_{[0,1]^d\times[0,1]^d} \sum_{i=1}^{d}\sum_{j=1}^{d} u_iv_iu_j e^{-n\prod_{i=1}^d u_i(1+\prod_{i=1}^d v_i)} \prod_{i=1}^d u_i d\mathbf{u}d\mathbf{v} \\ &= 2n^2 d \int_{[0,1]^d\times[0,1]^d} u_1^2v_1 e^{-n\prod_{i=1}^d u_i(1+\prod_{i=1}^d v_i)} \prod_{i=1}^d u_i d\mathbf{u}d\mathbf{v} \\ &\quad +2n^2d(d-1) \int_{[0,1]^d\times[0,1]^d} u_1v_1u_2 e^{-n\prod_{i=1}^d u_i(1+\prod_{i=1}^d v_i)} \prod_{i=1}^d u_i d\mathbf{u}d\mathbf{v} \\ &:= J_1 + J_2. \end{aligned} \tag{3.22}$$

We first deal with J_2, the easier term:

$$\begin{aligned} J_2 &= 2n^2d(d-1) \int_{[0,1]^d\times[0,1]^d} u_1v_1u_2 e^{-n\prod_{i=1}^d u_i(1+\prod_{i=1}^d v_i)} \prod_{i=1}^d u_i d\mathbf{u}d\mathbf{v} \\ &\le 2n^2d(d-1) \int_{[0,1]^d\times[0,1]^d} u_1u_2 e^{-n\prod_{i=1}^d u_i} \prod_{i=1}^d u_i d\mathbf{u}d\mathbf{v} \\ &= 2n^2d(d-1) \int_{[0,1]^d} u_1u_2 e^{-n\prod_{i=1}^d u_i} \prod_{i=1}^d u_i d\mathbf{u}. \end{aligned}$$

Now, we make the change of variables $u_i = e^{-y_i}$, $i = 3, \ldots, d$. Then, by the usual calculation we have

$$J_2 = O((\ln n)^{d-3}). \tag{3.23}$$

For J_1 we make the change of variables $u_i = e^{-x_i}$, $v_i = e^{-y_i}$, $i = 2, \ldots, d$. Then we have

$$\begin{aligned} J_1 &= 2n^2 d \int_{[0,1]^d \times [0,1]^d} u_1^2 v_1 e^{-n \prod_{i=1}^d u_i (1+\prod_{i=1}^d v_i)} \prod_{i=1}^d u_i d\mathbf{u} d\mathbf{v} \\ &= 2n^2 d \int_0^1 du_1 \int_0^1 dv_1 \int\int u_1^3 v_1 e^{-nu_1 e^{-\sum_{i=2}^d x_i}(1+v_1 e^{-\sum_{i=2}^d y_i})} e^{-2\sum_{i=2}^d x_i - \sum_{i=2}^d y_i} d\mathbf{x} d\mathbf{y} \\ &= 2n^2 d \int_0^1 du_1 \int_0^1 dv_1 \int_0^\infty \int_0^\infty u_1^3 v_1 e^{-nu_1 e^{-z}(1+v_1 e^{-w})} e^{-2z-w} \frac{z^{d-2}}{(d-2)!} \frac{w^{d-2}}{(d-2)!} dz dw. \end{aligned}$$

By the usual calculation starting with the change of variable $z - \ln nu_1 = a$, $w - \ln v_1 = b$, we obtain

$$J_1 \asymp \gamma_d (\ln n)^{d-2} \tag{3.24}$$

where

$$\gamma_d = \frac{d}{((d-2)!)^2} \left(\int_0^1 dv_1 \int_0^1 ds \frac{1}{(1+v_1 s)^2} (\ln \frac{1}{s})^{d-2} v_1 \right).$$

By (3.22)-(3.24) we have

$$I_2 \asymp \gamma_d (\ln n)^{d-2}. \tag{3.25}$$

Now, by (3.3), (3.4), (3.19), (3.21), (3.25), we have Theorem 2.

4 Central limit theorem.

In this section we prove Theorem 3. With the help of a transformation (4.1) which appeared in Baryshnikov [5], we approximate L_n by a space-truncated random variable conditioned on a very likely event V_n, which is defined in (4.8). Then we approximate this conditioned space-truncated random variable by a random variable L''_n generated by a Poisson point process. This Poisson point process approximation idea has been successfully developed in Barbour and Xia [4]. Decomposing L''_n as a sum of locally dependent random variables, we can apply Stein's method to L''_n and get the central limit theorem for L''_n. Since our approximation errors turn out to be small, we can dig out the central limit theorem for L_n, Theorem 3, from the central limit theorem for L''_n.

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be iid uniform on $[0,1]^d$. We apply the following transformation $f : \mathbf{x} = (x_1, \dots, x_d) \mapsto \mathbf{y} = (y_1, \dots, y_d)$;

$$y_i = -\log x_i, \ i = 1, \dots, d. \tag{4.1}$$

Then, $\mathbf{x}$ is minimal if and only if $\mathbf{y}$ is maximal. Furthermore, the distribution of each component of $\mathbf{y}$ is the exponential distribution with mean 1.

Let A_ζ, B_ζ be

$$A_\zeta = \left\{ (y_1, \dots, y_d) \in \mathbf{R}_+^d; \sum_{i=1}^d y_i < \zeta \right\}, \ B_\zeta = \left\{ (y_1, \dots, y_d) \in \mathbf{R}_+^d; \sum_{i=1}^d y_i \geq \zeta \right\}.$$

Then, we would like to choose $\alpha_n < \beta_n$ so that there are not many maximal points in A_{α_n} and there are not many points in B_{β_n}. We let

$$\alpha_n = \ln n - \ln(a \ln\ln n), \ \beta_n = \ln n + b(d-1)\ln\ln n, \tag{4.2}$$

where

$$a > (d-1) + \frac{1}{2}(d-2), \ b > 1 + 3\frac{d-2}{d-1}. \tag{4.3}$$

Then we see that

$$\begin{aligned}
E\sum_{i=1}^n 1(\mathbf{y}_i \text{ maximal}, \ \mathbf{y}_i \in A_{\alpha_n}) &= n\int_{A_{\alpha_n}} e^{-(y_1+\cdots+y_d)}(1-e^{-(y_1+\cdots+y_d)})^{n-1} d\mathbf{y} \\
&\leq n\int_{A_{\alpha_n}} e^{-(y_1+\cdots+y_d)} e^{-(n-1)e^{-(y_1+\cdots+y_d)}} d\mathbf{y} \\
&= n\int_0^{\alpha_n} e^{-s} e^{-(n-1)e^{-s}} \frac{s^{d-1}}{(d-1)!} ds \qquad (4.4) \\
&\leq n\frac{\alpha_n^{d-1}}{(d-1)!}\int_0^{\alpha_n} e^{-s} e^{-ne^{-s}} ds \\
&= n\frac{\alpha_n^{d-1}}{(d-1)!}\int_{e^{-\alpha_n}}^1 e^{-nt} dt \\
&\leq \frac{\alpha_n^{d-1}}{(d-1)!} e^{-ne^{-\alpha_n}} \ \text{(by (4.2)-(4.3))} \\
&= O((\ln n)^{-(a-(d-1))}) \qquad (4.5)
\end{aligned}$$

and

$$\begin{aligned}
E\sum_{i=1}^n 1(\mathbf{y}_i \in B_{\beta_n}) &= n\int_{B_{\beta_n}} e^{-(y_1+\cdots+y_d)} d\mathbf{y} \\
&= n\int_{\beta_n}^\infty e^{-s}\frac{s^{d-1}}{(d-1)!} ds
\end{aligned}$$

$$\asymp \quad ne^{-\beta_n}\frac{\beta_n^{d-1}}{(d-1)!} \text{ (by (4.2)-(4.3))}$$
$$= \quad O((\ln n)^{-(b-1)(d-1)}). \tag{4.6}$$

Define the random variable L'_n as

$$L'_n := \sum_{i=1}^n |\mathbf{x}_i| 1_{G_i} 1(\mathbf{y}_i \in B_{\alpha_n} \cap A_{\beta_n}) | V_n) := L'''_n | V_n) \tag{4.7}$$

where

$$V_n := \cap_{i=1}^n \{\mathbf{x}_i \in A_{\beta_n}\}. \tag{4.8}$$

By the orders given in (4.5) and (4.6) and employing the δ-method (see [9] for the δ-method), the convergence rate of the distribution of L_n should be the same as that of the distribution of L'_n. Furthermore, the distribution of L'_n should be asymptotically equivalent to the distribution of $L''_n := \sum_{\mathbf{y} \in W_n} |f^{-1}(\mathbf{y})| 1(\mathbf{y} \text{ is maximal in } W_n)$, where W_n is a Poisson process on $B_{\alpha_n} \cap A_{\beta_n}$ with intensity $ne^{-(y_1+\cdots+y_d)}/P(V_n)$. So our plan is first to prove the central limit theorem for L''_n using Stein's method. From this central limit theorem we shall obtain the central limit theorem for L'_n and then L_n.

We rewrite L_n as

$$L_n = K_{n,1} + K_{n,2} = J_{n,1} + J_{n,2} + K_{n,2} \tag{4.9}$$

where

$$K_{n,1} = \sum_{i=1}^n |\mathbf{x}_i| 1_{G_i} 1(\mathbf{y}_i \in B_{\alpha_n}), \;\; K_{n,2} = \sum_{i=1}^n |\mathbf{x}_i| 1_{G_i} 1(\mathbf{y}_i \in A_{\alpha_n}), \tag{4.10}$$

$$J_{n,1} = K_{n,1} 1(V_n), \;\; J_{n,2} = K_{n,1} 1(V_n^c). \tag{4.11}$$

Then, since by (4.5), (4.6) and (4.3)

$$P(K_{n,2} \neq 0) \leq E \sum_{i=1}^n 1_{G_i} 1(\mathbf{y}_i \in A_{\alpha_n}) \leq O((\ln n)^{-(a-(d-1))}),$$

$$P(V_n^c) \leq E \sum_{i=1}^n 1(\mathbf{y}_i \in B_{\beta_n}) = O((\ln n)^{-(b-1)(d-1)}),$$

we have

$$\begin{aligned} & d_{TV}(L_n, L'_n) \\ = \;& \sup_A |P(L_n \in A) - P(L'_n \in A)| \\ = \;& \sup_A |P(L_n \in A) - P(L'''_n \in A | V_n)| \end{aligned}$$

$$
\begin{aligned}
&= \sup_A \left| P(L_n \in A) - \frac{P(L_n''' \in A, V_n)}{P(V_n)} \right| \\
&= \sup_A \left| \frac{P(V_n)P(L_n \in A) - P(L_n''' \in A, V_n)}{P(V_n)} \right| \\
&= \sup_A \left| \frac{P(V_n)(P(L_n \in A, V_n) + P(L_n \in A, V_n^c)) - (P(V_n) + P(V_n^c))P(L_n''' \in A, V_n)}{P(V_n)} \right| \\
&\leq \sup_A |P(L_n \in A, V_n) - P(L_n''' \in A, V_n)| + \sup_A P(L_n \in A, V_n^c) + \frac{P(V_n^c)}{P(V_n)} \\
&\leq \sup_A |P(L_n \in A, V_n) - P(L_n''' \in A, V_n)| + P(V_n^c) + \frac{P(V_n^c)}{P(V_n)} \\
&\leq P(K_{n,2} \neq 0) + 2\frac{P(V_n^c)}{P(V_n)} \\
&\leq O((\ln n)^{-(a-(d-1))} + (\ln n)^{-(b-1)(d-1)}). \qquad (4.12)
\end{aligned}
$$

Moreover, with the notation

$$p_n := P(V_n),$$

we have

$$1 - p_n = P(V_n^c) \leq C(\ln n)^{-(b-1)(d-1)}, \qquad (4.13)$$

$$\frac{1}{p_n} - 1 \leq C(\ln n)^{-(b-1)(d-1)}, \qquad (4.14)$$

$$\frac{1}{p_n} + 1 \leq 2 + C(\ln n)^{-(b-1)(d-1)} \leq C. \qquad (4.15)$$

Next, we assert (and then prove) that

$$EJ_{n,2} \leq C(\ln n)^{(d-2)-(b-1)(d-1)}, \qquad (4.16)$$

$$EK_{n,2} \leq C(\ln n)^{-(a-(d-1))}, \qquad (4.17)$$

$$EJ_{n,2}^2 \leq C(\ln n)^{-(b-1)(d-1)+2(d-2)}, \qquad (4.18)$$

$$EK_{n,2}^2 \leq C(\ln n)^{-(a-(d-1))}. \qquad (4.19)$$

Indeed, we let F_i be the event that $\mathbf{y}_i$ lies in B_{β_n}. Since L_{n-1} and F_n are independent and since by (4.6), $P(F_n) \leq Cn^{-1}(\ln n)^{-(b-1)(d-1)}$, by Theorem 1 we have (4.16);

$$
\begin{aligned}
EJ_{n,2} &\leq EL_n(\sum_{i=1}^n 1_{F_i}) \\
&= nEL_n 1_{F_n} \\
&\leq nE(L_{n-1} + d^{1/2})1_{F_n} \\
&= nE(L_{n-1} + d^{1/2})P(F_n) \\
&\leq C(\ln n)^{-(b-1)(d-1)+(d-2)}.
\end{aligned}
$$

By the same argument using Theorem 1 and 2 we have (4.18);

$$\begin{aligned} EJ^2_{n,2} &\leq EL^2_n(\sum_{i=1}^n 1_{F_i}) \\ &= nEL^2_n 1_{F_n} \\ &\leq nE(L_{n-1}+d^{1/2})^2 1_{F_n} \\ &\leq nE(L_{n-1}+d^{1/2})^2 P(F_n) \\ &= n(EL^2_{n-1}+2d^{1/2}EL_{n-1}+d)P(F_n) \\ &\leq C(\ln n)^{-(b-1)(d-1)+2(d-2)}. \end{aligned}$$

Furthermore, (4.17) follows from (4.5). To prove (4.19), we start with a simple observation. Let $Q_{\mathbf{y}}$ be the first orthant of $\mathbf{y}$, that is $Q_{\mathbf{y}} = \{\mathbf{z} : \mathbf{z} \succ \mathbf{y}\}$. Then, with the notation $\|\mathbf{y}\| = y_1 + \cdots + y_d$ the probability that $\mathbf{y}_1$ falls in $Q_{\mathbf{y}}$ is given by

$$P(\mathbf{y}_1 \text{ falls in } Q_{\mathbf{y}}) = \prod_{j=1}^d e^{-y_j} = e^{-\|\mathbf{y}\|}.$$

Now, using the fact that $P(A \cup B) \geq (P(A)+P(B))/2$ for any two events A and B, we have that given $\mathbf{y}_1$ and $\mathbf{y}_2$ the conditional probability that both $\mathbf{y}_1$ and $\mathbf{y}_2$ are maxima is bounded by

$$\left(1-\frac{1}{2}\left(e^{-\|\mathbf{y}_1\|}+e^{-\|\mathbf{y}_2\|}\right)\right)^{n-2} \leq e^{-(n-2)(e^{-\|\mathbf{y}_1\|}+e^{-\|\mathbf{y}_2\|})/2}.$$

Thus, by the same computation as at (4.4) we have (4.19);

$$\begin{aligned} EK^2_{n,2} &= E\left(\sum_{i=1}^n |\mathbf{x}_i| 1(\mathbf{y}_i \text{ is maximal and } \|\mathbf{y}_i\| \leq \alpha_n)\right)^2 \\ &\leq CEK_{n,2} + Cn^2 P(\text{both } \mathbf{y}_1 \text{ and } \mathbf{y}_2 \text{ are maximal falling in } A_{\alpha_n}) \\ &\leq CEK_{n,2} + Cn^2\left(\frac{1}{(d-1)!}\right)^2 \int_0^{\alpha_n}\int_0^{\alpha_n} (xy)^{d-1} e^{-(n-2)(e^{-x}+e^{-y})/2} e^{-x-y}\,dxdy \\ &= CEK_{n,2} + Cn^2\left(\frac{1}{(d-1)!}\int_0^{\alpha_n} x^{d-1} e^{-(n-2)e^{-x}/2} e^{-x}dx\right)^2 \\ &\leq CEK_{n,2} + Cn^2\left(\frac{1}{(d-1)!}\alpha_n^{d-1}\int_0^{\alpha_n} e^{-(n-2)e^{-x}/2} e^{-x}dx\right)^2 \\ &= CEK_{n,2} + Cn^2\left(\frac{1}{(d-1)!}\alpha_n^{d-1}\int_{e^{-\alpha_n}}^1 e^{-(n-2)t/2}dt\right)^2 \\ &\leq C(\ln n)^{-(a-(d-1))} + C(\ln n)^{-2(((n-2)/n)a-(d-1))} \\ &\leq C(\ln n)^{-(a-(d-1))}. \end{aligned}$$

Now, since by (4.7), (4.8), (4.11), $EL'_n = p_n^{-1}EJ_{n,1}$ and $EL'^2_n = p_n^{-1}EJ^2_{n,1}$, by (4.14), Theorem 1, (4.16) and (4.17) we have

$$\begin{aligned} |EL_n - EL'_n| &\leq \Big(\frac{1}{p_n} - 1\Big)EJ_{n,1} + EJ_{n,2} + EK_{n,2} \\ &\leq \Big(\frac{1}{p_n} - 1\Big)EL_n + EJ_{n,2} + EK_{n,2} \\ &\leq C(\ln n)^{(d-2)-(b-1)(d-1)} + C(\ln n)^{-(a-(d-1))}. \end{aligned} \tag{4.20}$$

By (4.15) and Theorem 1,

$$EL_n + EL'_n = EL_n + \frac{1}{p_n}EJ_{n,1} \leq \Big(1 + \frac{1}{p_n}\Big)EL_n \leq C(\ln n)^{(d-2)}. \tag{4.21}$$

By (4.14), Theorems 1 and 2, (4.18) and (4.19),

$$\begin{aligned} &|EL_n^2 - EL'^2_n| \\ \leq\ & \Big(\frac{1}{p_n} - 1\Big)EJ^2_{n,1} + EJ^2_{n,2} + EK^2_{n,2} + 2EK_{n,1}K_{n,2} \\ \leq\ & \Big(\frac{1}{p_n} - 1\Big)EJ^2_{n,1} + EJ^2_{n,2} + EK^2_{n,2} + 2\left(EK^2_{n,1}\right)^{1/2}\left(EK^2_{n,2}\right)^{1/2} \\ \leq\ & \Big(\frac{1}{p_n} - 1\Big)EL^2_n + EJ^2_{n,2} + EK^2_{n,2} + 2\left(EL^2_n\right)^{1/2}\left(EK^2_{n,2}\right)^{1/2} \\ \leq\ & C(\ln n)^{2(d-2)-(b-1)(d-1)} + C(\ln n)^{(d-2)-(1/2)(a-(d-1))}. \end{aligned} \tag{4.22}$$

By (4.20)-(4.22)

$$\begin{aligned} |Var(L_n) - Var(L'_n)| &\leq |EL_n^2 - EL'^2_n| + |(EL_n)^2 - (EL'_n)^2| \\ &= |EL_n^2 - EL'^2_n| + |EL_n - EL'_n||EL_n + EL'_n| \\ &\leq C(\ln n)^{2(d-2)-(b-1)(d-1)} + C(\ln n)^{(d-2)-(1/2)(a-(d-1))} \end{aligned} \tag{4.23}$$

Now, with the notation $N'_n := |\{\mathbf{y}_i : 1 \leq i \leq n\} \cap (B_{\alpha_n} \cap A_{\beta_n})||V_n) := N'''_n|V_n)$ and $N''_n = |W_n \cap (B_{\alpha_n} \cap A_{\beta_n})|$ we have

$$\begin{aligned} &d_{TV}(L'_n, L''_n) \\ =\ & \sup_A |P(L'_n \in A) - P(L''_n \in A)| \\ =\ & \sup_A \left|\sum_{m=0}^{n} P(N'_n = m)P(L'_n \in A|N'_n = m) - \sum_{m=0}^{\infty} P(N''_n = m)P(L''_n \in A|N''_n = m)\right| \\ \leq\ & \sum_{m=0}^{n} |P(N'_n = m) - P(N''_n = m)| + P(N''_n > n) \\ =\ & 2d_{TV}(X_n, Y_n) \end{aligned}$$

where X_n is the binomial distribution with n trials and success rate

$$q_n = \frac{\int_{\alpha_n}^{\beta_n} \frac{x^{d-1}}{(d-1)!} e^{-x} dx}{\int_0^{\beta_n} \frac{x^{d-1}}{(d-1)!} e^{-x} dx} = O(n^{-1}(\ln n)^{d-1} \ln \ln n) \tag{4.24}$$

and where Y_n is the Poisson distribution with mean $\lambda = nq_n$. Since $d_{TV}(X_n, Y_n) \leq (\lambda \vee 1)^{-1} nq_n^2$ and since $\lambda = nq_n$ is large for large n, we have

$$d_{TV}(L'_n, L''_n) \leq 2d_{TV}(X_n, Y_n) \leq 2q_n = O(n^{-1}(\ln n)^{d-1} \ln \ln n). \tag{4.25}$$

Similarly, since $E(L'_n | N'_n = m) = E(L''_n | N''_n = m)$, $E(L''_n | N''_n = m) \leq Cm$ and since for large n, $\sum_{m=0}^n |P(N'_n = m) - P(N''_n = m)| + P(N''_n > n) = 2d_{TV}(X_n, Y_n) \leq 2q_n$, we have for large n that

$$\begin{aligned}
& |EL'_n - EL''_n| \\
= & \left| \sum_{m=0}^{n} P(N'_n = m) E(L'_n | N'_n = m) - \sum_{m=0}^{\infty} P(N''_n = m) E(L''_n | N''_n = m) \right| \\
\leq & \sum_{m=0}^{2nq_n} |P(N'_n = m) - P(N''_n = m)| E(L''_n | N''_n = m) \\
& + \sum_{m=2nq_n+1}^{n} P(N'_n = m) E(L'_n | N'_n = m) + \sum_{m=2nq_n+1}^{\infty} P(N''_n = m) E(L''_n | N''_n = m) \\
\leq & \; C \sum_{m=0}^{2nq_n} |P(N'_n = m) - P(N''_n = m)| 2nq_n \\
& + C \sum_{m=2nq_n+1}^{n} P(N'_n = m) m + C \sum_{m=2nq_n+1}^{\infty} P(N''_n = m) m \\
\leq & \; Cnq_n^2 + C \sum_{m=2nq_n+1}^{n} P(N'_n = m) m + C \sum_{m=2nq_n+1}^{\infty} P(N''_n = m) m.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{m=2nq_n+1}^{n} P(N'_n = m) m &= E(X_n; X_n > 2nq_n) \\
&= E(X_n - EX_n; X_n > 2nq_n) + nq_n P(X_n > 2nq_n) \\
&\leq (nq_n(1-q_n))^{1/2} (P(X_n > 2nq_n))^{1/2} + nq_n P(X_n > 2nq_n) \\
&= O((nq_n)^{1/4} e^{-nq_n/4}),
\end{aligned}$$

and since

$$\sum_{m=2nq_n+1}^{\infty} P(N''_n = m) m = \lambda \sum_{m=2nq_n}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!}$$

$$\begin{aligned} &\leq C\lambda e^{-\lambda}\frac{\lambda^{2nq_n}}{(2nq_n)!} \\ &= O((nq_n)^{1/2}(e/4)^{nq_n}), \end{aligned}$$

by (4.24) we have

$$|EL'_n - EL''_n| \leq Cnq_n^2 \leq Cn^{-1}(\ln n)^{2(d-1)}(\ln\ln n)^2. \tag{4.26}$$

By a similar calculation we also have

$$|EL'_n(L'_n-1) - EL''_n(L''_n-1)| \leq Cn^2q_n^3 \leq Cn^{-1}(\ln n)^{3(d-1)}(\ln\ln n)^3. \tag{4.27}$$

Next, we split $\mathbf{R}^d_+$ into cubes T_i of edge length $l = l(n)$, where we choose $l = l(n)$ very small so that the argument in (4.37) makes sense. Until then, it is safe to think of l as a fixed but small number even though the choice of l depends on n. Let Z_i be the contribution of the Poisson point process falling in the cell T_i to L''_n;

$$Z_i := \sum_{\mathbf{y}\in(W_n\cap T_i)} |f^{-1}(\mathbf{y})| 1(\mathbf{y} \text{ is maximal in } W_n).$$

Then, we can rewrite L''_n as

$$L''_n = \sum_{T_i\cap(B_{\alpha_n}\cap A_{\beta_n})\neq\emptyset} Z_i.$$

Since we decompose L''_n as a sum of locally dependent random variables, we apply Stein's method to L''_n and get the central limit theorem for L''_n. Here is a simple version of Stein's method that we use (this is Theorem 6.31 of Janson, Łuczak, and Rucinski [11]).

Lemma 1. Let X_i be a collection of locally dependent random variables with $E|X_i| < \infty$ and let

$$U_i = \{j;\ X_j \text{ depends on } X_i\},\ V_i = \sum_{j\in U_i} X_j,$$

$$U_{i,j} = \{k;\ X_k \text{ depends on } X_i \text{ or } X_j\}\setminus U_i,\ V_{i,j} = \sum_{k\in U_{i,j}} X_k,\ \text{for } j\in U_i,$$

$$S = \sum_i X_i,\ S_i = S - V_i,\ S_{i,j} = S - V_{i,j}.$$

Suppose that

$$EX_i = 0, \tag{4.28}$$

and

$$ES^2 = \sum_i EX_iV_i = \sum_i \sum_{j\in U_i} EX_iX_j = 1. \tag{4.29}$$

Then, for any function h with $\sup_x |h(x)| + \sup_x |h'(x)| \le 1$ we have

$$|Eh(S) - Eh(N)| \le C \sum_i \sum_{j\in U_i} \sum_{k\in(U_i\cup U_{i,j})} (E|X_iX_jX_k| + E|X_iX_j|E|X_k|)$$

where N is the standard normal random variable.

Proposition 1. The normalized random variable $(L_n'' - EL_n'')/(Var(L_n''))^{1/2}$ converges in distribution to the standard normal with a rate

$$d_1\left(\frac{L_n'' - EL_n''}{(Var(L_n''))^{1/2}}, N(0,1)\right) = O((\ln n)^{-(d-2)/2}(\ln\ln n)^{d+1})$$

where

$$d_1(X,Y) := \sup\{|Eh(X) - Eh(Y)| : \sup_x |h(x)| + \sup_x |h'(x)| \le 1\}. \tag{4.30}$$

Proof. Let X_i be $(Z_i - EZ_i)/(Var(L_n''))^{1/2}$. Then, X_i satisfy (4.28) and (4.29). So by Lemma 1, for any function h with $\sup_x |h(x)| + \sup_x |h'(x)| \le 1$ we have

$$\begin{aligned} &\left|Eh\left(\frac{L_n'' - EL_n''}{(Var(L_n''))^{1/2}}\right) - Eh(N)\right| \\ \le\ & C(Var(L_n''))^{-3/2} \sum_i \sum_{j\in U_i} \sum_{k\in(U_i\cup U_{i,j})} (EZ_iZ_jZ_k \\ &\quad + EZ_iEZ_jZ_k + EZ_jEZ_iZ_k + EZ_kEZ_iZ_j + EZ_iEZ_jEZ_k). \end{aligned} \tag{4.31}$$

We now define the constants

$$Q_n = \max_{i,j\in U_i} \sum_{k\in(U_i\cup U_{i,j})} EN_k,\ \varepsilon_{n,1} = \max_i r_i,\ \varepsilon_{n,2} = (\max_i r_i)(\sum_i r_i) = \varepsilon_{n,1}q_n,$$

where N_i is the number of Poisson points falling in the region T_i and where $r_i = EN_i$.

We now consider the term $EZ_iZ_jZ_k$ in (4.31). If i,j,k are distinct, then

$$EZ_iZ_jZ_k \le CEZ_iEN_jEN_k. \tag{4.32}$$

It is obvious that $E(Z_iZ_j|N_k = m)$ is a decreasing function of m. Thus, $E(Z_iZ_j|N_k)$ and N_k are negatively correlated. So, since $Z_k \le CN_k$, we have

$$EZ_iZ_jZ_k \le CEZ_iZ_jN_k = CE(E(Z_iZ_j|N_k)N_k) \le CEZ_iZ_jEN_k.$$

By the same reasoning, we also have $EZ_iZ_j \le CEZ_iEN_j$. Hence, (4.32) indeed holds.

If two indices are equal among the three i, j, k, then there are three cases we have to consider. In the first case $EZ_iZ_j^2$ is bounded (by the reasoning of (4.32)) by

$$\begin{aligned}
EZ_iZ_j^2 &\le CEZ_iN_j^2 \\
&\le CEZ_iEN_j^2 \\
&= CEZ_i \sum_{m=1}^{\infty} m^2 e^{-r_j} \frac{r_j^m}{m!} \\
&\le CEZ_i \left(EN_j + \sum_{m=2}^{\infty} m^2 e^{-r_j} \frac{r_j^m}{m!} \right) \\
&\le CEZ_i(EN_j + r_j^2) \\
&\le CEZ_i(EN_j + \varepsilon_{n,1} r_j). \qquad (4.33)
\end{aligned}$$

In the second case $EZ_jZ_i^2$ is bounded (by the reasoning of (4.32)) by

$$\begin{aligned}
EZ_jZ_i^2 &\le CEN_jZ_i^2 \\
&\le CEN_jEZ_i^2 \\
&= CEN_j \sum_{m=1}^{\infty} E(Z_i^2|N_i = m) e^{-r_i} \frac{r_i^m}{m!} \\
&\le CEN_j \sum_{m=1}^{\infty} E(Z_i|N_i = m) m e^{-r_i} \frac{r_i^m}{m!} \\
&\le CEN_j \left(E(Z_i|N_i = 1) e^{-r_i} r_i + \sum_{m=2}^{\infty} m^2 e^{-r_i} \frac{r_i^m}{m!} \right) \\
&\le CEN_j(EZ_i + r_i^2) \\
&\le CEN_j(EZ_i + \varepsilon_{n,1} r_i). \qquad (4.34)
\end{aligned}$$

By the same reasoning, we can handle the third case;

$$EZ_kZ_i^2 \le CEN_k(EZ_i + r_i^2) \le CEN_k(EZ_i + \varepsilon_{n,1} r_i). \qquad (4.35)$$

If the three indices are all equal, then by the reasoning of (4.35) we have

$$EZ_i^3 \le EZ_i + Cr_i^2 \le EZ_i + C\varepsilon_{n,1} r_i. \qquad (4.36)$$

Similarly, we take care of all the other four terms in (4.31). Then, by (4.31) we have

$$\begin{aligned}
&\left| Eh\left(\frac{L_n'' - EL_n''}{(Var(L_n''))^{1/2}} \right) - Eh(N) \right| \\
\le\ & C \frac{EL_n''(Q_n^2 + Q_n + 1 + \varepsilon_{n,1}^2 + \varepsilon_{n,1} + Q_n\varepsilon_{n,1} + \varepsilon_{n,2}) + Q_n\varepsilon_{n,2} + \varepsilon_{n,2}}{(Var(L_n''))^{3/2}}.
\end{aligned}$$

By splitting $\mathbf{R}_+^d$ into cubes T_i in a very fine way, i.e., by choosing l very small we can make $\varepsilon_{n,1}$ and $\varepsilon_{n,2}$ arbitrarily small. Hence,

$$\left|Eh\left(\frac{L_n''-EL_n''}{(Var(L_n''))^{1/2}}\right)-Eh(N)\right| \leq C\frac{EL_n''(Q_n^2+Q_n+1)+1}{(Var(L_n''))^{3/2}}. \tag{4.37}$$

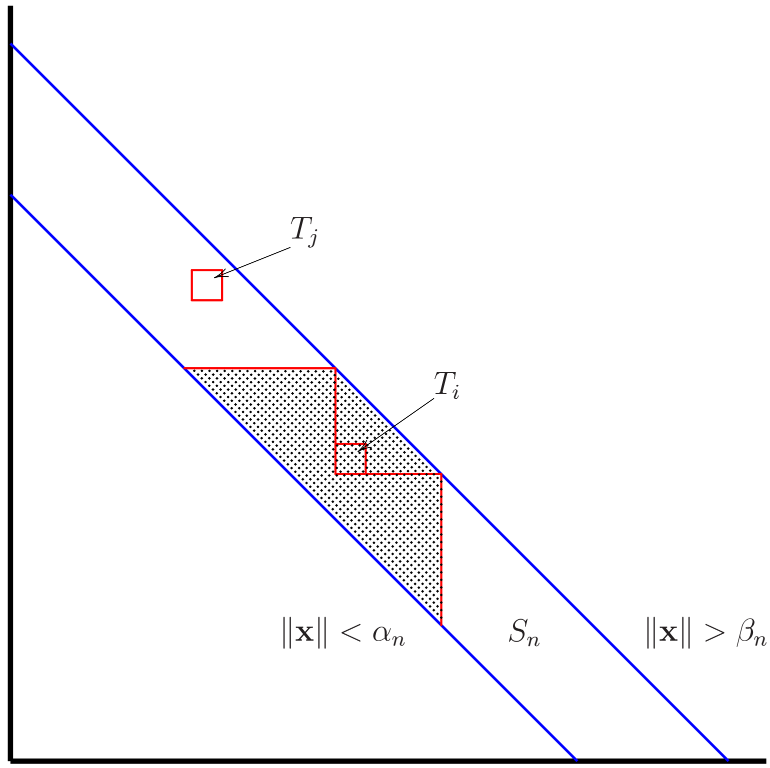

Figure 1: For $S_n := \{\mathbf{x} : \alpha_n < \|\mathbf{x}\| < \beta_n\}$ and for T_i and T_j with $T_i \cap S_n \neq \emptyset$ and $T_j \cap S_n \neq \emptyset$, Z_i and Z_j are independent if T_j and the dark region generated by T_i have no overlapping region.

Now we just need to estimate the following three quantities; EL_n'', Q_n, $Var(L_n'')$. First, we estimate EL_n''; by (4.26) we have

$$\begin{aligned} EL_n'' &\leq EL_n' + Cn^{-1}(\ln n)^{2(d-1)}(\ln\ln n)^2 \text{ (by (4.20))} \\ &\leq EL_n + C(\ln n)^{(d-2)-(b-1)(d-1)} + C(\ln n)^{-(a-(d-1))} \text{ (by Theorem 1)} \\ &\leq C(\ln n)^{d-2}. \end{aligned} \tag{4.38}$$

Now consider Q_n. As we see in Figure 1, Z_j is independent of Z_i if T_j and the dark region generated by T_i have no overlapping region. Thus,

$$Q_n \leq Cne^{-\alpha_n}(\beta_n-\alpha_n)^d \leq C(\ln\ln n)^{d+1}. \tag{4.39}$$

Finally, with the choice b in (4.3) we have the following estimate for $Var(L''_n)$;

$$\begin{aligned} Var(L''_n) &= EL''_n(L''_n - 1) + EL''_n - (EL''_n)^2 \text{ (by (4.27), (4.26), (4.20), (1.2))} \\ &= EL'_n(L'_n - 1) + EL'_n - (EL'_n)^2 + O(n^{-1}(\ln n)^{3(d-1)}(\ln\ln n)^3) \\ &= Var(L'_n) + O(n^{-1}(\ln n)^{3(d-1)}(\ln\ln n)^3) \text{ (by (4.23), (1.3))} \\ &= Var(L_n) + O((\ln n)^{2(d-2)-(1/2)(b-1)(d-1)}) + o((\ln n)^{d-2}) \text{ (by (4.3))} \\ &= Var(L_n) + o((\ln n)^{d-2}). \end{aligned} \tag{4.40}$$

Note that during the estimate for $Var(L''_n)$ we also have

$$|Var(L''_n) - Var(L'_n)| \le Cn^{-1}(\ln n)^{3(d-1)}(\ln\ln n)^3. \tag{4.41}$$

Therefore, by (4.38), (4.39), (4.40), (1.3), (4.37) we have

$$\begin{aligned} \left| Eh\left(\frac{L''_n - EL''_n}{(Var(L''_n))^{1/2}}\right) - Eh(N)\right| &\le C\frac{EL''_n(Q_n^2 + Q_n + 1) + 1}{(Var(L''_n))^{3/2}} \\ &\le C\frac{M_n Q_n^2}{(Var(L''_n))^{3/2}} \\ &= O((\ln n)^{-(d-2)/2}(\ln\ln n)^{d+1}). \end{aligned} \tag{4.42}$$

Therefore, Proposition 1 follows. ∎

To prove Theorem 3, we need the following obvious lemma. We skip its proof.

Lemma 2. Let r_n with $r_n \to 0$ be given. If

(1) $d_{TV}(X_n, Y_n) = O(r_n)$,
(2) $|EX_n - EY_n| = O(r_n(Var(X_n))^{1/2})$,
(3) $|Var(X_n) - Var(Y_n)| = O(r_n(Var(X_n))^{1/2})$,

then,

$$X_n \in CLT(r_n) \text{ if and only if } Y_n \in CLT(r_n).$$

Proposition 2. For any r_n with $r_n \to 0$ and $r_n \ge C(\ln n)^{-(d-2)/2}$ we have

$$L_n \in CLT(r_n) \text{ if and only if } L''_n \in CLT(r_n).$$

Proof. With the choice a and b in (4.3), for any r_n with $r_n \to 0$ and

$$r_n \geq C(\ln n)^{-(d-2)/2},$$

by Lemma 2 with (4.25), (4.26), (4.41), (4.12), (4.20), (4.23), we have the Proposition. ∎

Our final lemma relates the estimates on rate of weak convergence using the d_1 metric (defined at (4.30)) and weak convergence in the sense of $CLT(r_n)$ as defined at (1.7).

Lemma 3. Let $(\xi_n, n \geq 1)$ be a sequence of random variables with finite second moments and let $\bar{\xi}_n := (\xi_n - E\xi_n)/\sqrt{Var\xi_n}$. If $d_1(\bar{\xi}_n, N) = O(r_n)$, where N has a standard normal distribution and where $r_n > 0, r_n \to 0$ as $n \to \infty$, then $\xi_n \in CLT(\sqrt{r_n})$.

Proof. Set $a_n = \sqrt{r_n}$. Given $x \in \mathbf{R}$, and given n, set $y = x + a_n$. Define the bounded, continuous, piecewise linear function h on $\mathbf{R}$ by

$$h(t) = \begin{cases} a_n, & t \leq x \\ y - t & x \leq t \leq y \\ 0 & t \geq y. \end{cases}$$

Then for n large enough so that $a_n < 1$, we have $|h(t)| \leq 1$ for all t and $|h'(t)| \leq 1$ for all t except $t = x$ and $t = y$. So h can be approximated uniformly by continuously differentiable functions g with $|g(t)| \leq 1$ and $|g'(t)| \leq 1$ for all t and hence $|Eh(X) - Eh(Y)| \leq d_1(X, Y)$ for any pair of random variables X and Y. By the choice of h, we have for all X

$$a_n P(X \leq x) \leq Eh(X) \leq a_n P(X \leq y).$$

Hence, if $d_1(\bar{\xi}_n, N) = O(r_n)$, there is a constant C such that

$$\begin{aligned} a_n P(\bar{\xi}_n \leq x) &\leq Eh(\bar{\xi}_n) \\ &\leq Eh(N) + Cr_n \\ &\leq a_n P(N \leq x + a_n) + Cr_n \\ &\leq a_n P(N \leq x) + Ca_n^2 + Cr_n \end{aligned}$$

By the choice of a_n, then we have

$$P(\bar{\xi}_n \leq x) \leq P(N \leq x) + 2C\sqrt{r_n}. \tag{4.43}$$

Here the choice of C can be made independently of n or x. For the other direction, note that there is a constant C (independent of n, y) such that

$$\begin{aligned} a_n P(N \le y) &\le a_n P(N \le x) + C a_n^2 \\ &\le Eh(N) + C a_n^2 \\ &\le Eh(\bar{\xi}_n) + C r_n + C a_n^2 \\ &\le a_n P(\bar{\xi}_n \le y) + C(r_n + a_n^2). \end{aligned}$$

Again, by the choice of a_n we have

$$P(N \le y) \le P(\bar{\xi}_n \le y) + 2C\sqrt{r_n}. \tag{4.44}$$

Combining (4.43) with (4.44) we have $\xi \in CLT(\sqrt{r_n})$. ∎

Now Theorem 3 follows from Propositions 1 and 2 and Lemma 3.

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