

ON THE LOCAL EXISTENCE OF ONE CALABI-TYPE FLOW

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As a subsequent paper of [Z], we mainly discuss here the details about the local existence of one negative gradient flow for one L^2 -integral of Ricci curvature on any compact manifold, which is actually one fourth order degenerate parabolic equation. Meanwhile, the further discussions about the phenomenon of blowing up for the singularities of the flow are given, according to the earlier results in [Z].

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1 INTRODUCTION

As in [Z], let's consider here the compact and smooth Riemannian n -manifold (M, g_0) with the given Riemannian metric g_0 . Denote r_g the Ricci curvature of metric g . Consider the following Riemannian functional \mathcal{L} ,

$$\mathcal{L} = \int_M |r_g|^2 d\mu_g.$$

Then the Euler-Lagrange equation([B]) is

$$\nabla_g^* \nabla_g r_g + \nabla_g ds_g + \frac{1}{2}(\Delta_g s_g)g + \frac{1}{2}|r_g|^2 g - 2\overset{\circ}{R}_g r_g = 0, \quad (1.1)$$

where s_g denotes the scalar curvature of metric g , ∇_g denotes the covariant derivative with respect to g , ∇_g^* is the formal adjoint of ∇_g (see [B]), $\Delta_g \triangleq \nabla_g^* \nabla_g$, R_g denotes the $(4, 0)$ Riemannian curvature. For any symmetric $(0, 2)$ tensor field h , put

$$\overset{\circ}{R}(h)(a, b) = \sum_{i,j=1}^m h(R_g(a, e_i)b, e_i),$$

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where $\{e_i\}$ is the orthonormal basis of $TM_x, x \in M, a, b \in TM_x$.

Consider now the negative gradient flow of \mathcal{L} on the manifold M , that is the following initial value problem of fourth order nonlinear equation:

$$\begin{cases} \frac{\partial g}{\partial t} = -(\nabla_g^* \nabla_g r_g + \nabla_g ds_g + \frac{1}{2}(\Delta_g s_g)g + \frac{1}{2}|r_g|^2 g - 2\overset{\circ}{R}_g r_g), \\ g(0) = g_0. \end{cases} \quad (1.2)$$

It is not difficult to see that (1.2) is at heart a fourth order nonstrictly parabolic equation, due to invariance under the group of diffeomorphism, which makes it highly degenerate. The geometric equation of higher order is known to get more consideration only recently. In the article of Chrusciel([Chr]), the global existence of a fourth order flow of metrics on a two dimensional Riemannian manifold is applied to construct solutions of Einstein Vacuum equation, called Robison-Trautman metrics. Meanwhile in the articles of [Chn] and [Cha], the global existence and its applications of Calabi flow in Einstein metric and Kaehler manifold is well studied. Recently, Kuwert and Shaetzle([KS]) study the global existence and regularity of the Willmore flow, Mantegazza([M]) studies the global existence and singularities of the more higher order gradient flow of the functional about the hypersurface based on the work of Huisken([Hu]).

For this type of equation, Richard Hamilton supports an original proof of short time existence of Ricci flow([H1]) by using the Nash-Moser inverse function theorem. Soon after, De Tuck ([De]) simplified the proof of the short time existence by the trick of so called "breaking the symmetry" .

Note however that there are some difference between this flow and the heat flow or Ricci flow, besides the fact that the first flow is a fourth order nonstrictly parabolic equation . We are forced to pass to deal with the operator which is quasilinear and complexly mixed with so many kinds of curvature that the evolution equation of the curvature is more complicated to handle than ever before.

Using the trick of DeTurck after some detail analysis and calculations about the principal signature of some related operators for metric, we get here one of the aim of this paper about the more detail proof of following local existence theorem of flow (1.2).

Theorem 1.1. *Given any smooth metric g_0 on any compact manifold M , there exists only one unique smooth solution $g(t)$ of equation (1.2) for at least a short time.*

For the long time existence problem, it is still an open problem which need to be discussed in the proceeding paper later. But at least we have known its asymptotic behavior from our earlier results (Theorem 1.3 in [Z]) that if its solution has a global existence and converges smoothly to a unique smooth metric g_∞ as $t \rightarrow \infty$, the limit metric g_∞ must be flat. In particular, it means that there exists a flat metric on this 3-manifold.

Recalled from our earlier results and discussions in [Z] about the occurrences of the singularities in finite time during the evolution of (1.2), we know that the flow will blow up when at least one of the two cases happened, that is either the curvature will become infinity or the injective radius $inj_{(M,g(t))}$ has no lower bound during the time period $[0, T)$. As our another new result here, we can now replace the second case about the estimate of the injective radius by a more intuitive estimation about the diameter as follows.

Theorem 1.2. *On any compact and oriented 3-manifold M without boundary, the flow (1.2) has an unique solution on a maximal time interval $t \in [0, T)$. If $T < \infty$, there exists one sequence $\{t_i\}$ in $[0, T)$ which tends to T and satisfies at least one of the following two cases .*

- (a) $\max_M |r_{g(t_i)}| \rightarrow \infty$, as $t_i \rightarrow T$;
- (b) the diameter $\text{diam}_{(M, g(t_i))} \rightarrow \infty$, as $t_i \rightarrow T$.

To finish the proof of the above results, we need first to study the principal symbols of some related operators in §2. Then in §3, one new class of the diffeomorphism of M is introduced to get another new equivalent flow. After the necessary principal symbol calculations of this new flow, we prove that this flow is strictly parabolic which means the local existence of flow (1.2). Finally some further discussions about the singularities of flow (1.2) are given at end.

2 BASIC RESULTS

As using in ([B]), we will denote here the trace of any (2,0) tensor field T by $tr_g(T)$ with respect to metric g , that is $tr_g(T) = g(g, T)$.

For the covariant derivative operator ∇_g with respect to metric g , the formal adjoint ∇_g^* of ∇_g is defined as the opposite of the trace tr_g of the following $\otimes^s TM$ -valued 2-form for any tensor field $T \in \Omega^1 M \otimes T^{(r,s)} M$,

$$(X, Y) \rightarrow (\nabla_X T)(Y, X_1, \dots, X_r),$$

where X_1, \dots, X_r and X, Y are vector fields on M . Then we define also $\Delta_g T = \nabla_g^* \nabla_g T$.

Now in order to prove Theorem 1.1 we make the following calculations of the principal symbol of the linearization of $grad\mathcal{L}$ with respect to metric g according to [B].

Denotes the linearization of $grad\mathcal{L}$ as $\mathcal{D}grad\mathcal{L}$. It is easy to see that $\mathcal{D}grad\mathcal{L}$ is a fourth differential operator of g . According to the definition of principle symbol of differential operator (see [B]), it is easy to see that for any variant (2,0) symmetric tensor h of metric g and $\xi \in T_x^* M$ at one point x , the principal symbol of the linearized operator of $grad\mathcal{L}$ satisfies

$$\begin{aligned} & \sigma_\xi(\mathcal{D}grad\mathcal{L}, g)h \\ &= \sigma_\xi(\nabla_g^* \nabla_g) \sigma_\xi(r'_g, g)h + \sigma_\xi(\nabla_g d) \sigma_\xi(s'_g, g)h + \frac{1}{2}(\sigma_\xi(\Delta_g) \sigma_\xi(s'_g, g)h)g \end{aligned} \quad (2.3)$$

where r'_g, s'_g denotes the linearized operator or derivative operator of r_g and s_g with respect to metric g respectively. $\sigma_\xi(\nabla_g^* \nabla_g)$, $\sigma_\xi(\nabla_g d)$ and $\sigma_\xi(\Delta_g)$ denotes the principal symbol of the linear differential operator $\nabla_g^* \nabla_g$, $\nabla_g d$ and Δ_g respectively for fixed metric g . Meanwhile $\sigma_\xi(r'_g, g)$ and $\sigma_\xi(s'_g, g)$ denotes the principle symbol of operator r'_g and s'_g with respect to g respectively.

According to the definition, at point $x \in M$, choosing suitable orthonormal frame $\{e_i\}$ such that $\xi = |\xi|e^1$, $g_{ij} = \delta_{ij}$ and $tr_g h = \sum_{i=1}^n h_{ii}$, we have the following calculations about the principal symbol of differential operator $\mathcal{D}grad\mathcal{L}$.

Theorem 2.1.

$$\sigma_\xi(\mathcal{D}grad\mathcal{L}, g)h = \frac{|\xi|^4}{2} \left(\sum_{i,j \geq 2} h_{ij} e^i \otimes e^j + \sum_{i=2}^n (tr_g h + h_{ii} - h_{11}) e^i \otimes e^i \right). \quad (2.4)$$

Proof First according to the formula (1.1), we have at any fixed point $x \in M$,

$$\begin{aligned} & \sigma_\xi(\mathcal{D}grad\mathcal{L}, g)h \\ &= \sigma_\xi(\nabla_g^* \nabla_g) \sigma_\xi(r'_g)h + \sigma_\xi(\nabla_g d) \sigma_\xi(s'_g)h + \frac{1}{2}(\sigma_\xi \Delta_g \sigma_\xi s'_g(h))g. \end{aligned}$$

Then according to the following lemma 2.2-2.5, we get

$$\begin{aligned} & \sigma_\xi(\mathcal{D}grad\mathcal{L}, g)h \\ &= |\xi|^2 \left(\frac{1}{2} |\xi|^2 h - \frac{1}{2} (\xi \otimes i_{\xi^*} h + i_{\xi^*} h \otimes \xi) + \frac{1}{2} tr_g h \xi \otimes \xi \right) \\ & \quad - (|\xi|^2 tr_g h - h(\xi^*, \xi^*)) \xi \otimes \xi + \frac{|\xi|^2}{2} (|\xi|^2 tr_g h - h(\xi^*, \xi^*))g \end{aligned}$$

Choosing the suitable orthonormal frame $\{e_i\}$ at fixed point $x \in M$, such that $\xi^* = |\xi|^2 e_1$ and $g_{ij} = \delta_{ij}$, $tr_g h = \sum_{i=1}^n h_{ii}$, then the formula (2.4) can be gotten. \square

To finish the proof of Theorem 2.1, we need the following lemmas about the calculations of principal symbol of the related operators, where all the subindex g denoting the operator of metric g will be omitted conveniently in the process of the calculation.

Lemma 2.2. *Given fixed metric g , we have for any $(0, 2)$ tensor field h and $\xi \in T_x M$*

$$\sigma_\xi(\nabla_g^* \nabla_g)h = |\xi|^2 h. \quad (2.5)$$

Proof Under the local orthonormal frame $\{E_i\}$, by the definitions of ∇_g^* (see [B]) we have

$$\begin{aligned} & \nabla_g^* \nabla_g h(X, Y) \\ &= - \sum_i \nabla_{E_i} \nabla h(E_i, X, Y) \\ &= - \sum_i E_i(\nabla h(E_i, X, Y)) - \nabla h(\nabla_{E_i} E_i, X, Y) - \nabla h(E_i, \nabla_{E_i} X, Y) - \nabla(E_i, X, \nabla_{E_i} Y) \\ &= - \sum_i E_i E_i h(X, Y) + \text{lower order terms.} \end{aligned}$$

for any two vector fields X and Y . Therefore we get (2.5). \square

Lemma 2.3. *Given fixed metric g , we have for any $(0, 2)$ tensor field h and $\xi \in T_x M$*

$$\sigma_\xi \delta_g^* \delta_g h = \frac{1}{2} (\xi \otimes i_{\xi^*} h + i_{\xi^*} h \otimes \xi). \quad (2.6)$$

Proof Suppose $\{E_i\}$, X and Y as in Lemma 2.3, then according to the definition of δ_g

and δ_g^* in [B],

$$\begin{aligned}
\delta_g^* \delta_g h(X, Y) &= \frac{1}{2}((\nabla_X \delta h)(Y) + (\nabla_Y \delta h)(X)) \\
&= \frac{1}{2}(X(\delta h(Y)) - \delta h(\nabla_X Y) + Y(\delta h(X)) - \delta h(\nabla_Y X)) \\
&= -\frac{1}{2}(X(\sum_i \nabla_{E_i} h(E_i, Y)) - \sum_i \nabla_{E_i} h(E_i, \nabla_X Y) \\
&\quad + Y(\sum_i \nabla_{E_i} h(E_i, X)) - \sum_i \nabla_{E_i} h(E_i, \nabla_Y X)) \\
&= -\frac{1}{2}(X \sum_i E_i(h(E_i, Y)) + Y(\sum_i E_i(h(E_i, X)))) + \text{lower order terms}.
\end{aligned}$$

So that

$$\sigma_\xi \delta_g^* \delta_g h = \frac{1}{2}(\xi \otimes h(\xi^*, \cdot) + h(\xi^*, \cdot) \otimes \xi).$$

This is just the formula (2.6) we get. \square

Lemma 2.4. *Under the same assumptions as in above lemmas,*

$$\sigma_\xi(r'_g)h = \frac{1}{2}|\xi|^2 h - \frac{1}{2}(\xi \otimes i_{\xi^*} h + i_{\xi^*} h \otimes \xi) + \frac{tr_g h}{2} \xi \otimes \xi. \quad (2.7)$$

Proof From [B] we know that

$$r'_g h = \frac{1}{2} \Delta_L h - \delta_g^* \delta_g h - \frac{1}{2} \nabla_g d(tr_g h),$$

where

$$\Delta_L h = \nabla^* \nabla h + r_g \circ h + h \circ r_g - 2\overset{\circ}{R}_g h.$$

Therefore, from Lemma 2.2 and Lemma 2.3 we get

$$\begin{aligned}
\sigma_\xi(r'_g)h &= \frac{1}{2} \sigma_\xi(\Delta_L)h - \sigma_\xi(\delta_g^* \delta_g)h - \frac{1}{2} \sigma_\xi(\nabla_g d(tr_g h)) \\
&= \frac{1}{2} |\xi|^2 h - \frac{1}{2} (\xi \otimes i_{\xi^*} h + i_{\xi^*} h \otimes \xi) + \frac{tr_g h}{2} \xi \otimes \xi. \quad \square
\end{aligned}$$

Meanwhile from the calculations in [B], we have also the following principal symbol of s'_g .

Lemma 2.5. *Under the same assumption as above,*

$$\sigma_\xi(s'_g)h = |\xi|^2 tr_g h - h(\xi^*, \xi^*).$$

Define $N \triangleq \sigma_\xi(\mathcal{D}grad \mathcal{L}, g)$, then any $h \in \ker N$ will imply

$$0 = (Nh)_{ij} = \begin{cases} 0, & i = 1 \text{ or } j = 1, \\ \frac{|\xi|^4}{2} \left(\sum_{\substack{k=2 \\ k \neq i}}^n h_{kk} + 2h_{ii} \right), & i = j \geq 2, \\ \frac{|\xi|^4}{2} h_{ij}, & i \neq j, \quad i, j \geq 2. \end{cases}$$

So that as $\xi \neq 0$, $h \in \ker N$, we have

$$\begin{cases} 2h_{22} + h_{33} + \cdots + h_{nn} = 0 \\ h_{22} + 2h_{33} + \cdots + h_{nn} = 0 \\ \cdots \\ h_{22} + h_{33} + \cdots + 2h_{nn} = 0 \end{cases}$$

which means $h_{22} = h_{33} = \cdots = h_{nn} = 0$, that is

$$\ker N = \{h \in \mathcal{L}^2 | h_{ij} = 0, i, j \geq 2\}, \quad \dim \ker N = n.$$

Therefore it is not difficult to get the following results according to Theorem 2.1 and the invariance of functional \mathcal{L} under the diffeomorphism.

Corollary 2.6.

- (1) $\delta_g(\text{grad}\mathcal{L}) = 0$;
- (2) $\text{Im}(\sigma_\xi(\mathcal{D}\text{grad}\mathcal{L}, g)) \subseteq \ker(\sigma_\xi(\delta_g))$;
- (3) $\ker(\sigma_\xi)$ is the invariant subspace of $\sigma_\xi(\mathcal{D}\text{grad}\mathcal{L}, g)$ which is positive on this subspace.

Proof As \mathcal{L} is invariant under the diffeomorphism, (1) can be proved according to the tangent space of the moduli space of all Riemannian metric on M (see [B]). Meanwhile as $\sigma_\xi(\delta_g)h = -i|\xi| \sum_{i=1}^n (h_{1i}e^1 \otimes e^i + h_{i1}e^i \otimes e^1)$, it is easy to prove (3) according to Theorem 2.1. \square

Proposition 2.7. (Hamilton[1]) *Let $\partial f/\partial t = E(f)$ be a second order evolution equation with integrability condition $L(f)$. Suppose that*

- (A) $L(f)E(f) = Q(f)$ has degree 1 or $Q(f) = 0$,
- (B) all the eigenvalues of the eigenspaces of $\sigma DE(f)(\xi)$ in $\text{Null } \sigma L(f)\xi$ have strictly positive real parts.

Then the initial value problem $f = f_0$ at $t = 0$ has a unique smooth solution for a short time $0 \leq t \leq \epsilon$ where ϵ may depend on f_0 .

Compared corollary 2.6 with the above Hamilton's local existence Proposition(Theorem 5.1 in [H1]) on Ricci flow, we find that the sufficient conditions for the local existence theorem of Hamilton's flow are also satisfied in the case of flow (1.2), except the fact that this flow is fourth order other than the second order of Hamilton's flow. Therefore a natural question rises, whether the local existence theorem of Hamilton is also available to the higher order cases. The further study of this question will be kept on this topic in the subsequent papers later.

3 SHORT-TIME EXISTENCE AND UNIQUENESS

From the discussion in §2, we can see that the negative gradient flow (1.2) is not a strictly parabolic equation. Therefore the normal method can not be used directly to this equation. Here we use the "De Turck" trick to solve it.

Similar as the discussion in [H2], we first given any fixed metric h_0 on M . For any map $F : (M, g) \rightarrow (M, h_0)$, let $\{x^i\}, 1 \leq i \leq n$, $\{y^\alpha\}, 1 \leq \alpha \leq n$ is the local coordinate system of $x \in M$ and $F(x) \in M$ respectively. Γ_{ij}^k and $\Delta_{\alpha\beta}^\gamma$ is the Levi-Civita connection coefficients of metric g and h_0 respectively. Meanwhile $\Delta_{\alpha\beta}^\gamma$ induces the connection $F^*\Delta_{\beta l}^\alpha$ on the pull-back bundle F^*TM of TM by F over M , that is

$$F^*\Delta_{\beta l}^\alpha = \Delta_{\beta\gamma}^\alpha \frac{\partial y^\gamma}{\partial x^l}.$$

The derivative ∇F is just a section of the bundle $E(TM, F^*TM)$ of linear maps of TM into F^*TM . The second order derivative $\nabla^2 F$ is the covariant derivative of ∇F using the induced connection in the bundle $E(TM, F^*TM)$ coming from the connections on M and F^*TM as locally given by

$$\nabla_{ij}^2 F^\alpha = \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial y^\alpha}{\partial x^k} + F^* \Delta_{\beta\gamma}^\alpha \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j}.$$

The Laplacian $\Delta_{g, h_0} F$ is then the trace of $\nabla^2 F$, that is locally

$$\Delta_{g, h_0} F^\alpha = g^{ij} \nabla_{ij}^2 F^\alpha,$$

and the bi-Laplacian $\Delta_{g, h_0}^2 F$ are just the section of $F^*(TM)$, which means that $\Delta_{g, h_0}^2 F(x)$ is just one tangent vector of $T_{F(x)}M$, so we define the vector field $V(F(x))$ as $V(F(x)) = -\Delta_{g, h_0}^2 F(x)$.

Suppose now map F be one diffeomorphism of M . Let $\tilde{g} = G^*g$, G be the inverse of F . Then $F : (M, g) \rightarrow (M, \tilde{g})$ is a isometric map, and we have

$$V(F(x)) = (-\Delta_{g, h_0}^2 F)(x) = (-\Delta_{\tilde{g}, h_0}^2 id)(F(x)). \quad (3.1)$$

Lemma 3.1. *Suppose $F(t, \cdot) : (M, g(t)) \rightarrow (M, h_0)$ be a flow of diffeomorphism of M induced by a vector field V , that is*

$$\frac{\partial F}{\partial t} = V(F(x)), \quad F(0) = F_0.$$

where $g(t)$ be any flow of metric on M , then we have

$$\frac{\partial \tilde{g}}{\partial t} = -\mathcal{L}_V \tilde{g} + G^*(t, \cdot) \frac{\partial g}{\partial t},$$

where $G(t, \cdot)$ is the inverse of $F(t, \cdot)$, $\tilde{g}(t) = G(t, \cdot)^*g(t)$, \mathcal{L}_V denotes the Lie derivative under the vector field V .

Proof. For any t_0 and $y \in M$,

$$\begin{aligned} \frac{\partial \tilde{g}}{\partial t}(t_0, y) &= \frac{\partial}{\partial t}[G(t, \cdot)^*g_t](t_0, y) \\ &= G(t_0, \cdot)^* \frac{\partial g}{\partial t}(t_0, y) + \frac{\partial}{\partial t}[G(t, \cdot)^*g_{t_0}](t_0, y) \\ &= G^*(t_0, \cdot) \frac{\partial g}{\partial t}(t_0, y) + \frac{\partial}{\partial t}[(F(t_0, \cdot) \circ G(t, \cdot))^*G(t_0, \cdot)^*g_{t_0}](t_0, y). \end{aligned}$$

As $\frac{\partial}{\partial t}(F(t_0, \cdot) \circ G(t, \cdot))(y) = -V(y)$, therefore from the definition of Lie derivative we get

$$\frac{\partial \tilde{g}(t_0, y)}{\partial t} = -\mathcal{L}_V \tilde{g} + G(t_0, \cdot)^* \frac{\partial g}{\partial t}(t_0, y). \quad \square$$

Now taking $g(t)$ from the flow of (1.2) and vector field V of (3.1), we get the following flow of metric $\tilde{g}(t)$,

$$\begin{cases} \frac{\partial}{\partial t} \tilde{g} = -\mathcal{L}_V \tilde{g} - \text{grad} \mathcal{L}(\tilde{g}), & \tilde{g}(0) = G^*(0, \cdot) g_0; \\ V = -\Delta_{\tilde{g}, h}^2 \text{ id.} \end{cases} \quad (3.2)$$

Lemma 3.2. *The two flow equations (1.2) and (3.2) are equivalent, which means that the solution $g(t)$ of (1.2) is in one-to-one correspondence to the solution of (3.2).*

Proof. For any solution $g(t)$ of (1.2), it is obviously from the lemma 3.1 that $\tilde{g}(t)$ is the solution of (3.2), where $\tilde{g}(t) = G(t, \cdot)^* g(t)$, $G(t, \cdot) = F(t, \cdot)^{-1}$, and $F(t, \cdot)$ satisfies the ordinary equations defined by vector field V in Lemma 3.1. Conversely for any solution \tilde{g} of flow (3.2), define vector field $\tilde{V} = -V$, and denotes the corresponding diffeomorphism as \tilde{F} , then obviously $\tilde{G} = \tilde{F}^{-1} = F$. Applying Lemma 3.1 for vector field \tilde{V} again, we see that the metric $g(t) = \tilde{G}^* \tilde{g}(t) = F^* \tilde{g}(t)$ satisfies that

$$\begin{aligned} \frac{\partial}{\partial t} g &= -\mathcal{L}_{\tilde{V}} g + F^* \frac{\partial \tilde{g}}{\partial t} \\ &= \mathcal{L}_V g + F^* (-\mathcal{L}_V \tilde{g} - \text{grad} \mathcal{L}(\tilde{g})) \\ &= -\text{grad} \mathcal{L}(g). \end{aligned}$$

Therefore we get the solution $g(t)$ of flow (1.2). Meanwhile from the proof aboved it is easy to see that any different solutions $g_1(t), g_2(t)$ of (1.2) will correspond to two different solutions $\tilde{g}_1(t)$ and $\tilde{g}_2(t)$ respectively and inversely true also. Therefore this completes the proof. \square

Now consider the metric flow of (3.2) further.

Lemma 3.3. *For the flow of (3.2),*

$$\begin{aligned} &\sigma_\xi(\mathcal{D}(P(\tilde{g}), \tilde{g})h) \\ &= -\frac{|\xi|^4}{2} \left(\sum_{\substack{i, j \geq 2 \\ i \neq j}}^n h_{ij} e^i \otimes e^j + \sum_{i=2}^n (\text{tr}_{\tilde{g}} h + h_{ii} - h_{11}) e^i \otimes e^i + (4h_{11} - \text{tr}_{\tilde{g}} h) e^1 \otimes e^1 \right. \\ &\quad \left. + 2 \sum_{j=1}^n (h_{1j} e^1 \otimes e^j + h_{j1} e^j \otimes e^1) \right), \end{aligned}$$

where $P(\tilde{g}) \triangleq -\mathcal{L}_V \tilde{g} - \text{grad} \mathcal{L}(\tilde{g})$, \mathcal{D} denotes the linearized operator, vector field V is defined in (3.1). Here we choose one suitable local orthonormal frame $\{e_i\}$ at $x \in M$, such that $\xi^* = |\xi| e_1$ and $\tilde{g}_{ij}(x) = \delta_{ij}$.

Proof. Note that

$$\begin{aligned} & \sigma_\xi(\mathcal{D}(P(\tilde{g})), \tilde{g})h \\ &= -\sigma_\xi(\mathcal{D}(\mathcal{L}_V \tilde{g}), \tilde{g})h - \sigma_\xi(\mathcal{D}(\text{grad}\mathcal{L}(\tilde{g})), \tilde{g})h \end{aligned} \quad (3.3)$$

As $\mathcal{L}_V \tilde{g} = 2\delta_{\tilde{g}}^* V^*$ (see [B]), where V^* is the dual of V . By the definition of symbol and formula $\delta_{\tilde{g}}^* \alpha = \frac{1}{2} \mathcal{L}_{\alpha^*} \tilde{g}$, where α is 1-form on M , α^* is the dual of α , we get

$$\begin{aligned} \sigma_\xi(\mathcal{D}(\mathcal{L}_V \tilde{g}), \tilde{g})h &= \sigma_\xi(2\mathcal{D}(\delta_{\tilde{g}}^* V^*), \tilde{g})h \\ &= 2\sigma_\xi(\delta_{\tilde{g}}^*) \cdot \sigma_\xi(\mathcal{D}(V), \tilde{g})^* h \end{aligned} \quad (3.4)$$

Meanwhile,

$$\sigma_\xi(\delta_{\tilde{g}}^*)\eta = \frac{i}{2}(\xi \otimes \eta + \eta \otimes \xi),$$

According to the definition of $\nabla^2 F$, $\Delta_{\tilde{g}, h}$, $\Delta_{\tilde{g}, h}^2 F$ and the property of principal symbol,

$$\begin{aligned} & \sigma_\xi(\mathcal{D}(V), \tilde{g})^* h \\ &= -|\xi|^2 \sigma_\xi(\mathcal{D}(\Delta_{\tilde{g}, h_0} id), \tilde{g})^* h. \end{aligned} \quad (3.5)$$

Locally,

$$\begin{aligned} (\mathcal{D}(\Delta_{\tilde{g}, h_0} id)h)^i &= -\frac{d}{dt}\Big|_{t=0}(\tilde{g}_t^{jk}(\tilde{\Gamma}_{jk}^i - \Delta_{jk}^i)) \\ &= -\frac{1}{2}\delta^{jk}\delta^{il}(\tilde{\nabla}_j h_{kl} + \tilde{\nabla}_k h_{jl} - \tilde{\nabla}_l h_{jk}) \\ &= -\frac{1}{2}(\tilde{\nabla}_j h_{ij} + \tilde{\nabla}_j h_{ji} - \tilde{\nabla}_i h_{jj}) \end{aligned}$$

where $\tilde{\nabla}$ and $\tilde{\Gamma}_{ij}^k$ denotes the covariant derivative and the corresponding Levi-Civita connecton coefficients with respect to \tilde{g} respectively, so that

$$(\mathcal{D}(\Delta_{\tilde{g}, h_0} id)h)^* = \tilde{\nabla}_{\tilde{g}}^* h + \frac{1}{2}d(\text{tr}_{\tilde{g}} h).$$

Therefore

$$\sigma_\xi(\mathcal{D}(\Delta_{\tilde{g}, h_0} id), \tilde{g})^* h = i(i_{\xi^*} h - \frac{1}{2} \langle \tilde{g}, h \rangle \xi). \quad (3.6)$$

Then according to (3.3)-(3.6), we get

$$\begin{aligned} & \sigma_\xi(\mathcal{D}(\mathcal{L}_V \tilde{g}), \tilde{g})h \\ &= |\xi|^2(\xi^* \otimes i_{\xi^*} h + i_{\xi^*} h \otimes \xi - \frac{1}{2} \langle \tilde{g}, h \rangle \xi \otimes \xi) \\ &= \frac{1}{2}|\xi|^4(2 \sum_{j=1}^n (h_{1j} e^1 \otimes e^j + h_{j1} e^j \otimes e^1) + (4h_{11} - \text{tr}_{\tilde{g}} h) e^1 \otimes e^1). \end{aligned} \quad (3.7)$$

Thus combine Theorem 2.1 with equalities (3.3) and (3.7), we get finally the proof of the lemma. \square

Using Lemma 3.2 and Lemma 3.3, we get now finally the proof of Theorem 1.1.

Proof of Theorem 1.1 From Lemma 3.3, we know that the flow of (3.2) is strictly parabolic, therefore according to the normal theory of nonlinear parabolic equation, we know that flow (3.2) has one unique and smooth solution $\tilde{g}(t)$ in some time interval $[0, T)$. According to Lemma 3.2, then the flow of (1.2) has the unique smooth solution $g(t)$ for any initial metric g_0 in $[0, T)$. This completes the proof. \square

4 THE SINGULARITY OF THE FLOW IN FINITE TIME

In this section we will discuss again the singularity behavior according to our earlier results in [Z].

Recalled from [Z], we knew that the volume will increase along the flow of (1.2). Actually according to the fact from the calculations of Anderson(see [An], page 205) that on any 3-manifold

$$|R_g|^2 = 4|r_g|^2 - s_g^2, \quad (4.1)$$

we have

$$tr_g(\dot{g}) = -tr_g(grad\mathcal{L}) = -\frac{3}{2} \Delta_g s_g + \frac{1}{2}|r_g|^2. \quad (4.2)$$

Then we got the following result about the speed of the volume (see Lemma 2.1 in [Z]).

Lemma 4.1. *If the flow (1.2) exists during the finite period of $t \in [0, T)$ on a compact and oriented 3-manifold M without boundary, then the volume of (M, g_t) will be bounded .*

According to results in [Z], we have the following estimations(see Theorem 1.2 in [Z]) about the singularity behavior in any finite time period $[0, T)$.

Lemma 4.2. *On any compact and oriented 3-manifold M without boundary, the negative flow (1.2) has a unique solution on a maximal time interval $t \in [0, T)$. If $T < \infty$, there exists one sequence $\{t_i\}$ in $[0, T)$ which tends to T and satisfies at least one of the following two cases*

- (a) $\max_M |r_{g(t_i)}| \rightarrow \infty$, as $t_i \rightarrow T$;
- (b) $inj_{M, g(t_i)} \rightarrow 0$, as $t_i \rightarrow T$.

Combine the above two lemmas with one earlier work of J. Cheeger(see [Ch])about the finiteness of Riemannian manifold, we can now get the following proof of Theorem 1.2.

Proof of Theorem 1.2. Similar as in the proof of Lemma 4.2 in [Z] by the continuation method for the flow (1.2), we also suppose that neither condition (a) nor (b) is satisfied, which implies that along the flow (1.2) in time period $(0, T)$, $|r(g(t))|$ and the diameter of M under the metric $g(t)$ are both upperbounded. Recalled from a earlier result in [Ch], Cheeger got a basic lower bounded estimate on the length of the shortest closed geodesic $l(\gamma)$ in compact Riemannian manifold M , namely when there are three constants λ , v , and D , such that the sectional curvature $K_M \geq -\lambda$, the volume $vol_M \geq v$ and the diameter $diam_M \leq D$, then $l(\lambda) \geq c(\lambda, v, D)$, where c is a constant depending only on the constants λ, v and D . Now from

(4.1) or the fact that the Ricci curvature can actually determine all the other curvatures in the case of three dimensional Riemannian manifold, we find that in this case the sectional curvature will also satisfy the above lower bounded conditions of Cheeger's estimation along the flow of (1.2) in any finite existence time $[0, T)$. Meanwhile, from Lemma 4.1 we can also know that the volume of the flow is also lower bounded up to T . Thus these two bounded conditions will imply a lower bound of $l(\lambda)$. According to Klingenberg's estimate (see [K]) on the injectivity radius $inj_{(M,g)}$, one can then obtains a lower bound on $inj_{(M,g)}$ in terms of the above corresponding constants λ , v , and D along the flow in any finite time interval $[0, T)$. Therefore combined the proof of Lemma 4.2, we will find that the solution of the flow will not blow up as t tends to T , which will means that the solution $g(t)$ will tends to a newer metric $g(T)$ as t tends to T , and therefore T will not be the last time of the existence of flow (1.2). Thus we can get finally the contradiction. \square

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