HESSIAN EQUATIONS WITH INFINITE DIRICHLET BOUNDARY VALUE

XIUQING CHEN AND HUAIYU JIAN

Abstract In this paper, we will show the existence and non-existence of Hessian equations with infinite Dirichlet boundary value conditions.

Kewwords Hessian equation, k-convex solution, singular boundary value, existence/non-existence, viscous solution.

1. Introduction

Let n and $1 \leq k \leq n$ be positive integers. Recall that the k-th elementary symmetric function for $\lambda \in \mathbb{R}^n$ is defined by

$$S_k(\lambda) = S_k(\lambda_1, ..., \lambda_n) = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1} \lambda_{i_2} ... \lambda_{i_k}$$

and the k-th elementary symmetric function over the space $M_s(\mathbb{R}^n)$ of all the $n \times n$ real symmetric matrices is given by

$$S_k(\mathbf{A}) = S_k(\lambda_1, ..., \lambda_n), \ \ \forall \ \mathbf{A} \in M_s(\mathbf{R^n}),$$

where $(\lambda_1, ..., \lambda_n)$ are the eigenvalues of the matrix $\mathbf{A}$.

Let Ω be a domain in $\mathbb{R}^n$ and ψ a positive function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$. In this paper , we deal with the Hessian equation

$$S_k(D^2 u) = \psi(x, u, Du) \ \text{ in } \Omega \tag{1.1}$$

with the singular boundary value condition

$$u(x) = +\infty \ \text{ on } \partial\Omega. \tag{1.2}$$

The research was supported by the National 973-Project from MOST and Trans-Century Training Programme Foundation for the Talents by the MOE.
The work was partially done while the author was visiting the Institute for Mathematical Sciences, National University of Singapore in 2004. The visit was supported by the Institute.

Correspondence should be address to Huaiyu Jian (email: hjian@math.tsinghua.edu.cn).

Here $D^2u = (u_{ij})_{n\times n}$ is the Hessian matrix of u and (1.2) means that $u(x) \to +\infty$ as $d(x) = d(x, \partial\Omega) \to 0$, where $d(x, \partial\Omega)$ denotes the distance of the point $x \in \Omega$ from $\partial\Omega$.

Obvious examples of Hessian in (1.1) are Laplace operator, $k = 1$, and the Monge-Ampère operator, $k = n$. General Hessian operators were studied by many authors. See, for example, [1-5] for Hessian in Ω and [6-9] for Hessian on the sphere.

A natural class of functions for the solutions to (1.1)-(1.2) is k-convex functions. Recall that a function $u \in C^2(\Omega)$ is called k-convex (or strictly k-convex) if $(\lambda_1, ..., \lambda_n) \in \overline{\Gamma_k}(or(\lambda_1, ..., \lambda_n) \in \Gamma_k)$ for every $x \in \Omega$, where $\lambda_1, ..., \lambda_n$ are the eigenvalue of $D^2u(x)$ and Γ_k is the connected component $\{\lambda \in \mathbb{R}^n : S_k(\lambda) > 0\}$ containing the positive cone

$$\Gamma^+ = \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n : \lambda_i > 0, i = 1, 2, ..., n\}.$$

It follows from [1] that

$$\Gamma^+ = \Gamma_n \subset ... \subset \Gamma_{k+1} \subset \Gamma_k \subset ... \subset \Gamma_1 \tag{1.3}$$

and $S_k(D^2u)$ turns to be elliptic in the class of k-convex functions.

The problem (1.1)-(1.2) was studied in [10,11] for $k = 1$, Laplace operator, and in [12-15] for $k = n$, Monge-Ampère operator, and $\psi = \psi(x, u)$ independent of Du. The results of Matero [15] were extended by Salani [16] to some Hessian equation; while the results of Cheng and Yau [12,13] were generalized recently by Guan and Jian [17], in which various existence and non-existence results were shown for rather general $\psi = \psi(x, u, Du)$ and the optimal growth condition of $\psi(x, z, \mathbf{p})$ was given for the existence of (1.1)-(1.2) in the case $k = n$. The aim of this paper is to extend the main results of [17] to the case $k \in \{1, 2, ..., n-1\}$. The difficulty here arises when one tries to construct barriers which is necessary for the existence or non-existence of problem (1.1)-(1.2). The methods and results of this paper are different from those of [16].

From now on,we assume

$$k \in \{1, 2, ..., n-1\}. \tag{1.4}$$

Our main results are stated as follows.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. If there are constants $M, \gamma, q \geq 0, \gamma + q \leq k$, such that*

$$0 \leq \psi(x, z, \mathbf{p}) \leq M(1 + (z^+)^q)(1 + |\mathbf{p}|^\gamma), \ \forall\ (x, z, \mathbf{p}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{1.5}$$

where $z^+ = max\{z, 0\}$, then there exists no k-convex solution to (1.1)-(1.2) in $C^2(\Omega)$.

Theorem 1.2. *If there are constants* $\alpha > 1$ *and* $M > 0$ *such that*

$$\psi(x, z, \mathbf{p}) \geq M(1 + |\mathbf{p}|^k)^\alpha, \ \ \forall\, (x, z, \mathbf{p}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{1.6}$$

and Ω *is a domain containing some ball of radius* a *where*

$$a > \Big[\frac{(k+1)(n-1)!}{M(\alpha-1)k!(n-k-1)!}\Big]^{\frac{1}{k}},$$

then there exists no k-convex solution to (1.1)-(1.2) in $C^2(\Omega)$*.*

We will deal with the existence of problem (1.1)-(1.2) in viscosity sense. For the details of viscosity solutions to Hessian equations like (1.1), as for the notion of k-convexity in viscosity sense, we refer to [5].

Theorem 1.3. *Let* Ω *be a bounded convex domain in* $\mathbb{R}^n$ *and* $\psi \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ *satisfy*

$$\psi(x, z, \mathbf{p}) > 0, \ \psi_z(x, z, \mathbf{p}) > 0, \ \ \forall\, (x, z, \mathbf{p}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \tag{1.7}$$

Suppose that there exist $q > k$ *and* $M > 0$ *such that for all* $(x, z, \mathbf{p}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$\psi(x, z, \mathbf{p}) \geq M(z^+)^q \tag{1.8}$$

and

$$\psi(x, z, \mathbf{p}) \leq \phi(z)(1 + |\mathbf{p}|^k), \tag{1.9}$$

where ϕ *is a smooth positive function satisfying*

$$\sup_{z \leq 0} e^{-\varepsilon z}\phi(z) < +\infty \tag{1.10}$$

for some $\varepsilon > 0$*, and for each fixed* $(x, z) \in \overline{\Omega} \times \mathbb{R}$,

$$\psi^{\frac{1}{k}}(x, z, \mathbf{p}) \text{ is convex in } \mathbf{p}, \ \inf_{\mathbf{p} \in R^n} \psi > 0, \ \sup_{\mathbf{p} \in R^n} \frac{|(\psi^{\frac{1}{k}})_x|}{1 + |\mathbf{p}|} < +\infty. \tag{1.11}$$

Then there exists a k-convex viscosity solution $u \in C^{0,1}_{loc}(\Omega)$ *to (1.1)-(1.2). Moreover, there exist functions* $\underline{h}$, $\overline{h} \in C(\mathbb{R}^+)$ *with* $\underline{h}(r)$, $\overline{h}(r) \to +\infty$ *as* $r \to 0$*, such that*

$$\underline{h}(d(x)) \leq u(x) \leq \overline{h}(d(x)), \forall x \in \Omega. \tag{1.12}$$

2. A comparison principle and uniqueness

This section is similar to section 2 in [17]. For the sake of convenience, we will give the details here.

Suppose that $u, v \in C^2(\Omega)$ are k-convex satisfying

$$S_k(D^2u) \geq \psi(x, u, Du) \text{ and } S_k(D^2v) \leq \phi(x, v, Dv) \text{ in } \Omega, \tag{2.1}$$

where $\phi, \psi \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, such that

$$\psi(x, z, \mathbf{p}) \geq \phi(x, z, \mathbf{p}), \ \ \forall\ (x, z, \mathbf{p}) \in (\Omega \times \mathbb{R} \times \mathbb{R}^n). \tag{2.2}$$

We will see if $u \leq v$ in Ω. Let $M_s^k(\mathbb{R}^n)$ be the subset of $M_s(\mathbb{R}^n)$, with the eigenvalues $\lambda_1, ..., \lambda_n$ satisfying $(\lambda_1, ..., \lambda_n) \in \Gamma_k$. Recall that

$$\Big(\frac{\partial S_k^{\frac{1}{k}}(\mathbf{w})}{\partial \mathbf{w}_{ij}}\Big)_{n\times n} > 0 \text{ (or } \geq 0), \ \ \forall\ \mathbf{w} \in M_s^k(\mathbb{R}^n) \text{ (or } \overline{M_s^k(\mathbb{R}^n)}) \tag{2.3}$$

(See [1, 8]). Hence for any $\mathbf{w}_1, \mathbf{w}_2 \in M_s^k(\mathbb{R}^n)$ (or $\overline{M_s^k(\mathbb{R}^n)}$),

$$S_k(\mathbf{w}_1) > S_k(\mathbf{w}_2) \text{ (or } S_k(\mathbf{w}_1) \geq S_k(\mathbf{w}_2)) \tag{2.4}$$

if $\mathbf{w}_1 - \mathbf{w}_2$ is positive definite (or semi-definite).

Lemma 2.1. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is a bounded domain, and* $u, v \in C(\overline{\Omega})$ *satisfy* $u \leq v$ *on* $\partial\Omega$. *If either* $\psi_z(x, z, \mathbf{p}) > 0$ *or* $\phi_z(x, z, \mathbf{p}) > 0$ *for any* $(x, z, \mathbf{p}) \in (\Omega \times \mathbb{R} \times \mathbb{R}^n)$, *then* $u \leq v$ *in* Ω.

Proof. Suppose the contrary that for some $y \in \Omega$,

$$u(y) - v(y) = \max_{\Omega}(u - v) > 0.$$

Then $S_k(D^2v(y)) \geq S_k(D^2u(y))$, as the Hessian $D^2(v-u)$ is positive semi-definite at y and u, v are k-convex. On the other hand ,we use (2.1)-(2.4) and the facts $u(y) > v(y)$ and $Du(y) = Dv(y)$ to obtain

$$S_k(D^2v(y)) \leq \phi(y, v(y), Dv(y)) < \psi(y, u(y), Du(y)) \leq S_k(D^2u(y)).$$

This is a contradiction. □

Remark 2.2. The assumption we have used is $\psi_z(y, u(y), Du(y)) > 0$ (or $\phi_z(y, u(y), Du(y)) > 0$) at the point y instead of $\psi_z(x, z, \mathbf{p}) > 0$ (or $\phi_z(x, z, \mathbf{p}) > 0$) for all $(x, z, \mathbf{p}) \in (\Omega \times \mathbb{R} \times \mathbb{R}^n)$.

Theorem 2.3. *Assume* $u=+\infty, v=+\infty$ *on* $\partial\Omega$, u *is* k*-convex and* v *is strictly* k*-convex in* Ω. *Suppose the domain* Ω *is bounded and star-shaped with respect to a point* x_0 *and* ψ *satisfies*

$$x\cdot D_x\psi(x,z,\mathbf{p})\le 0 \text{ and } \mathbf{p}\cdot D_{\mathbf{p}}\psi(x,z,\mathbf{p})\ge 0,\ \ \forall\,(x,z,\mathbf{p})\in\Omega\times\mathbb{R}\times\mathbb{R}^n. \tag{2.5}$$

If, in addition, either there is a $q>k$ *such that*

$$\psi(x,\lambda z^+,\mathbf{p})\ge\lambda^q\psi(x,z,\mathbf{p}),\ \ \forall\,\lambda\ge 1,\forall\,(x,z,\mathbf{p})\in(\Omega\times\mathbb{R}\times\mathbb{R}^n) \tag{2.6}$$

or there is a $\varepsilon>0$ *such that*

$$\psi_z(x,z,\mathbf{p})\ge\varepsilon\psi(x,z,\mathbf{p}),\ \ \forall\,(x,z,\mathbf{p})\in(\Omega\times\mathbb{R}\times\mathbb{R}^n), \tag{2.7}$$

then $u\le v$ *in* Ω. *Particularly, problem (1.1)-(1.2) has at most one strictly* k*-convex solution in* $C^2(\Omega)$.

Proof. Without loss of generality we may assume $x_0=0$. For $\lambda\in(0,1)$, let

$$u_\lambda(x)=\lambda^\alpha u(\lambda x)-a,\ \ x\in\Omega_\lambda$$

where $\Omega_\lambda=\{x\in\mathbb{R}^n:\lambda x\in\Omega\}$ and

$$\begin{cases} a=0,\ \alpha=\frac{2k}{q-k}, & \text{if (2.6) holds,}\\ a=-\frac{2k}{\varepsilon}\ln\lambda,\ \alpha=0, & \text{if (2.7) holds.}\end{cases}$$

Using (2.1) and (2.5), we have

$$\begin{aligned} S_k(D^2u_\lambda(x)) &= \lambda^{k(2+\alpha)}S_k(D^2u(\lambda x))\\ &\ge \lambda^{k(2+\alpha)}\psi\big(\lambda x,u(\lambda x),Du(\lambda x)\big)\\ &= \lambda^{k(2+\alpha)}\psi\big(\lambda x,\lambda^{-\alpha}(u_\lambda(x)+a),\lambda^{-(1+\alpha)}Du_\lambda(x)\big)\\ &= \lambda^{k(2+\alpha)}\int_0^1\partial_t\psi\big(x+t(\lambda x-x),\lambda^{-\alpha}(u_\lambda(x)+a),\\ &\quad Du_\lambda(x)+t(\lambda^{-(1+\alpha)}Du_\lambda(x)-Du_\lambda(x))\big)dt\\ &\quad+\lambda^{k(2+\alpha)}\psi\big(x,\lambda^{-\alpha}(u_\lambda(x)+a),Du_\lambda(x)\big)\\ &\ge \lambda^{k(2+\alpha)}\psi\big(x,\lambda^{-\alpha}(u_\lambda(x)+a),Du_\lambda(x)\big). \end{aligned} \tag{2.8}$$

Note that (2.6) implies $\psi\le 0$ where $z\le 0$. Then if $u_\lambda(x)\le 0$, the k-convexity implies that

$$S_k(D^2u_\lambda(x))\ge 0\ge\psi\big(x,u_\lambda(x),Du_\lambda(x)\big).$$

We conclude that

$$S_k(D^2u_\lambda(x))\ge\psi\big(x,u_\lambda(x),Du_\lambda(x)\big),\forall x\in\Omega. \tag{2.9}$$

In fact, if $u_\lambda(x)) \geq 0$ and (2.6) holds, we use (2.8) and (2.6) to obtain

$$S_k(D^2u_\lambda(x)) \geq \lambda^{k(2+\alpha)-\alpha q}\psi\big(x, u_\lambda(x), Du_\lambda(x)\big) = \psi\big(x, u_\lambda(x), Du_\lambda(x)\big).$$

Note that (2.7) implies

$$\psi(x, z_1, \mathbf{p}) \geq e^{\varepsilon(z_1-z_2)}\psi(x, z_2, \mathbf{p}).$$

So, if $u_\lambda(x)) \geq 0$ and (2.7) holds, we use (2.8) to obtain

$$S_k(D^2u_\lambda(x)) \geq \lambda^{2k}e^{\varepsilon a}\psi\big(x, u_\lambda(x), Du_\lambda(x)\big) = \psi\big(x, u_\lambda(x), Du_\lambda(x)\big).$$

Therefore, (2.9) holds in any case. We claim that

$$v \geq u_\lambda \text{ on } \Omega \text{ for all } \lambda \in (0, 1). \tag{2.10}$$

Suppose the contrary that $u_\lambda(x_0) > v(x_0)$ for some $x_0 \in \Omega$. Since $\overline{\Omega} \subset \Omega_\lambda$ and $v - u_\lambda = +\infty$ on $\partial\Omega$, we have a $y \in \Omega$, such that

$$u_\lambda(y) - v(y) = \max_\Omega(u_\lambda - v) > 0.$$

Hence, by(2.5),(2.2),(2.1) and the strict k-convexity of v,

$$\begin{aligned}\psi(y, u_\lambda(y), Du_\lambda(y)) &\geq \psi(y, v(y), Du_\lambda(y))\\ &= \psi(y, v(y), Dv(y))\\ &\geq \phi(y, v(y), Dv(y))\\ &\geq S_k(D^2v)\\ &> 0\end{aligned}$$

which, together with (2.6) or (2.7), implies $\psi_z(y, u_\lambda(y), Du_\lambda(y)) > 0$. (Note that if (2.6) holds, then (2.2) implies $\psi \leq 0$ where $z \leq 0$. So $v(y) > 0$ and $u_\lambda(y) > 0$ then (2.6) implies $\psi_z(y, u_\lambda(y), Du_\lambda(y)) > 0$). Consequently, we obtain a contradiction as in the proof of Lemma 2.1(See Remark 2.2). This proves (2.10). Letting $\lambda \to 1^-$ in (2.10), we obtain the desired result. □

3. BARRIERS AND NON-EXISTENCE

In this section, we construct some barriers that will be used in the proof of our main results. In particular, we will prove theorems 1.1 and 1.2.

Let $u(x) = u(|x|)$ be a radially symmetric function. A straightforward calculation gives

$$S_k(D^2(u)) = A_k r^{1-n}[r^{n-k}(u')^k]', \quad r = |x|, \tag{3.1}$$

where $A_k = \frac{(n-1)!}{k!(n-k)!}$ (See[16;p285]). Hence equation (1.1) is written as

$$A_k[(r^{\frac{n}{k}-1}u')^k]' = r^{n-1}\psi(x, u, \frac{xu'}{r}). \tag{3.2}$$

Lemma 3.1. *Let* $\eta \in C(\mathbb{R})$ *satisfy* $\eta(z) > 0,\ \eta'(z) \geq 0$ *for all* $z \in \mathbb{R}$. *Then, for any* $a > 0$, *there exists a strictly convex radially symmetric function* $v \in C^2(B_a(0))$ *satisfying*

$$\begin{cases} S_k(D^2v) \geq e^v\eta(v)(1 + |Dv|^k) \text{ in } B_a(0), \\ v(0) \leq 0,\ v = +\infty \ \text{ on } \partial B_a(0). \end{cases} \tag{3.3}$$

Proof. Consider the initial value problem

$$\begin{cases} \varphi' = [exp(A_k^{-1}r^ke^{\varphi}\eta(\varphi)) - 1]^{\frac{1}{k}}, \ \ r > 0 \\ \varphi(0) = 0. \end{cases} \tag{3.4}$$

Let $[0, T)$ be the maximal interval on which the solution to (3.4) exists. We conclude that T is finite. In fact, it follows from (3.4) and $\eta' \geq 0$ that

$$\varphi'(r) \geq r[A_k^{-1}e^{\varphi}\eta(\varphi)]^{\frac{1}{k}} \geq r[A_k^{-1}e^{\varphi(r)}\eta(0)]^{\frac{1}{k}},\ 0 < r < T.$$

Since $\varphi(0) = 0$, we have

$$k \geq k(1 - e^{\frac{-\varphi(\rho)}{k}}) = \int_0^{\rho} \varphi'(r)e^{\frac{-\varphi(r)}{k}}dr \geq (\frac{\eta(0)}{A_k})^{\frac{1}{k}} \int_0^{\rho} rdr = \frac{1}{2}(\frac{\eta(0)}{A_k})^{\frac{1}{k}}\rho^2$$

for any $\rho \in (0, T)$. This proves $T < \infty$. Furthermore, we see that $\varphi \in C^2[0, T)$ and $\varphi(T) = +\infty$ as $\varphi'(\rho) > 0$ for $\rho > 0$. It follows from (3.4) that

$$\ln(1 + (\varphi')^k) = A_k^{-1}r^ke^{\varphi}\eta(\varphi),$$

whose differentiation yields

$$\frac{k(\varphi')^{k-1}\varphi''}{1 + (\varphi')^k} \geq kA_k^{-1}r^{k-1}e^{\varphi}\eta(\varphi). \tag{3.5}$$

This, particularly, implies $\varphi'' > 0$ in $(0, T)$.

For given $a > 0$, define v by

$$v(x) = \varphi\big(\frac{T|x|}{a}\big) - 2k\big(-\ln\frac{T}{a}\big)^+,\ x \in B_a(0).$$

Then $v \in C^2(B_a(0)),\ v(0) \leq 0,\ v = +\infty$ on $\partial B_a(0)$, and it is strictly convex, since $\varphi \in C^2[0,T)$ and $\varphi'' > 0$. Using (3.1) and (3.5), we compute for $x \in B_a(0)$ that

$$\begin{aligned}
S_k(D^2 v(x)) &= \big(\frac{T}{a}\big)^{2k} A_k\big(\frac{T}{a}|x|\big)^{1-n}\big[k\big(\frac{T}{a}|x|\big)^{n-k}\big(\varphi'\big(\frac{T}{a}|x|\big)\big)^{k-1}\varphi''\big(\frac{T}{a}|x|\big) \\
&\quad + (n-k)\big(\frac{T}{a}|x|\big)^{n-k-1}\big(\varphi'\big(\frac{T}{a}|x|\big)\big)^{k}\big] \\
&\geq \big(\frac{T}{a}\big)^{2k} A_k\big(\frac{T}{a}|x|\big)^{1-k} k\big(\varphi'\big(\frac{T}{a}|x|\big)\big)^{k-1}\varphi''\big(\frac{T}{a}|x|\big) \\
&\geq \big(\frac{T}{a}\big)^{2k} e^{\varphi\left(\frac{T}{a}|x|\right)}\eta\big(\varphi\big(\frac{T}{a}|x|\big)\big)\big[1 + \big(\varphi'\big(\frac{T}{a}|x|\big)\big)^{k}\big] \\
&\geq \big(\frac{T}{a}\big)^{2k} e^{v(x)+2k\left(-\ln\frac{T}{a}\right)^{+}}\eta\big(v(x) + 2k\big(-\ln\frac{T}{a}\big)^{+}\big)\big[1 + \big(\frac{T}{a}\big)^{-k}|Dv(x)|^{k}\big] \\
&\geq e^{v(x)}\eta\big(v(x)\big)\big(1 + |Dv(x)|^{k}\big).
\end{aligned}$$

□

Notation 3.2. *Given a and η, we will use $v^{a,\eta}(x) = v^{a,\eta}(|x|)$ to denote the function $v \in C^2(B_a(0))$ obtained as in Lemma 3.1.*

Lemma 3.3. *Let Ω be a domain contained in a ball $B_a(x_0)$ and $u \in C^2(\Omega)$ a k-convex solution of (1.1)-(1.2). If there exists a function $\eta \in C^1(\mathbb{R})$, $\eta > 0$ and $\eta' \geq 0$ in $\mathbb{R}$, such that*

$$\psi(x, z, \mathbf{p}) \leq e^z \eta(z)(1 + |\mathbf{p}|^k), \ \ \forall\, (x, z, \mathbf{p}) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n,$$

then $u(x) \geq v^{a,\eta}(x - x_0)$ for all $x \in \Omega$.

Proof. Without loss of generality, we assume $x_0 = 0$. For any $r > a$, since $u - v^{r,\eta} = +\infty$ on $\partial\Omega$, by Lemma 2.1 and Lemma 3.1, we have $u \geq v^{r,\eta}$ in Ω. Letting $r \to a^+$, we obtain $u \geq v^{a,\eta}$. □

Next for $q > k$, consider the function

$$w(x) = (1 - |x|^2)^{\frac{k+1}{k-q}} = (1 - r^2)^{\frac{k+1}{k-q}} = w(r), \ \ r = |x|. \tag{3.6}$$

Observing that $w' > 0$ and $w'' > 0$, by (3.1) we have for any $r \in [0, 1)$ that

$$\begin{aligned}
S_k(D^2 w(x)) &= A_k[(n-k)r^{-k}(w')^k + kr^{1-k}(w')^{k-1}w''] \\
&\leq C_1(n,k,q)[(1-r^2)^{(\frac{q+1}{k-q})k} + (1-r^2)^{(\frac{q+1}{k-q})(k-1)}(1-r^2)^{\frac{q+1}{k-q}-1}] \\
&= C_2(n,k,q)(1-r^2)^{(\frac{q+1}{k-q})k-1}[(1-r^2) + 1] \\
&\leq 2C_2(n,k,q)(1-r^2)^{(\frac{k+1}{k-q})q} \\
&= 2C_2(n,k,q)w^q(x).
\end{aligned}$$

Hence, we have a positive constant $B = B(n, k, q)$ such that

$$S_k(D^2 w) \le B w^q \text{ in } B_1(0). \tag{3.7}$$

By rescaling, we define

$$w^{a,M}(x) = \lambda w(\frac{x}{a}), \ \ x \in B_a(0), \ \ \lambda = \big(\frac{B}{a^{2k}M}\big)^{\frac{1}{q-k}}. \tag{3.8}$$

Lemma 3.4. *For any $a, M > 0$ and $q > k$, $w^{a,M} \in C^\infty(B_a(0))$, $w^{a,M} = +\infty$ on $\partial B_a(0)$ and*

$$S_k\big(D^2 w^{a,M}\big) \le M(w^{a,M})^q \ \text{ in } B_a(0).$$

Proof. By a direct calculation, (3.7) and (3.8), we have

$$S_k\big(D^2 w^{a,M}(x)\big) = \frac{\lambda^k}{a^{2k}} S_k\big(D^2 w(\frac{x}{a})\big) \le \frac{\lambda^k B}{a^{2k}} w^q\big(\frac{x}{a}\big) = M\big(w^{a,M}(x)\big)^q \ \text{ in } B_a(x).$$

□

Lemma 3.5. *Let $u \in C^2(\Omega)$ be a k-convex solution of (1.1). If Ω contains a ball $B_a(x_0)$ and ψ satisfies (1.8) for some $q > k$ and $M > 0$, then $u(x) \le w^{a,M}(x - x_0)$ in $B_a(x_0)$.*

Proof. It is immediate from Lemmas 3.5 and 2.1. □

Note that for any domain Ω and any $x \in \Omega$, the ball $B_{d(x)}(x) \subset \Omega$, where $d(x)$ is the distance function to $\partial\Omega$. Then Lemma 3.5 implies

Corollary 3.6. *Let $u \in C^2(\Omega)$ be a k-convex solution of (1.1). If ψ satisfies (1.8) for some $q > k$ and $M > 0$, then*

$$u(x) \le \bar{h}(d(x)), \ \ \forall\, x \in \Omega$$

where $\bar{h} \in C^\infty(\mathbb{R}^+)$ is given by

$$\bar{h}(r) = w^{r,M}(0), \ \ r > 0. \tag{3.9}$$

Lemma 3.7. *Assume $\gamma, q \ge 0$, $\gamma + q \le k$ and $M > 0$. Then there exists a strictly convex radially symmetric positive function $\tilde{u} \in C^\infty(\mathbb{R}^n)$ satisfying*

$$S_k(D^2\tilde{u}(x)) \ge M\big(1 + (\tilde{u}(x))^q\big)\big(1 + |D\tilde{u}(x)|^\gamma\big), \ \ \forall\, x \in \mathbb{R}^n. \tag{3.10}$$

Proof. We want only to combine the following three conclusions. First, for any $p', q' \ge 0, p' + q' \le n$, and $M' > 0$, by Lemma 3.7 in [17], one has a strictly convex radially symmetric positive function $\tilde{u} \in C^\infty(\mathbb{R}^n)$ satisfying

$$S_n(D^2\tilde{u}(x)) \ge M'\big(1 + (\tilde{u}(x))^{p'}\big)\big(1 + |D\tilde{u}(x)|^{q'}\big), \ \ \forall\, x \in \mathbb{R}^n. \tag{3.11}$$

Second, for each $k \in \{1, 2, ..., n-1\}$ and $\lambda \in \Gamma_{k+1}$, it follow from [5] or [8] that

$$\frac{(k+1)!(n-k-1)!}{n!} S_{k+1}(\lambda) \leq \big[\frac{k!(n-k)!}{n!} S_k(\lambda)\big]^{\frac{k+1}{k}}.$$

Hence, there is a positive constant $C_1 = C(n, k)$ such that

$$C_1 S_n^{\frac{k}{n}}(\lambda) \leq S_k(\lambda), \ \ \forall\, \lambda \in \Gamma_n. \tag{3.12}$$

Finally, we can choose a positive constant $C_2 = C(n, k)$ such that

$$(1+t)^{\frac{k}{n}} \geq C_2(1 + t^{\frac{k}{n}}), \ \ \forall\, t \geq 0. \tag{3.13}$$

Combing this with (3.11) and (3.12), we obtain (3.10). □

Proof of Theorem 1.1. Let $u \in C^2(\Omega)$ be a k-convex solution of (1.1)-(1.2). We will induce a contradiction.

Let $\tilde{u}$ be the same function as in Lemma 3.7, where γ, q and M are as in (1.5). Observing that $u - C\tilde{u} = +\infty$ on $\partial\Omega$ for any $C > 0$, we can choose $C > 1$ and a $y \in \Omega$ such that

$$u(y) - C\tilde{u}(y) = \min_{\Omega}(u - C\tilde{u}) < 0.$$

Hence $Du(y) = CD\tilde{u}(y)$ and $\big(D^2u(y) - CD^2\tilde{u}(y)\big)$ is a positive semi-definite matrix. However, it follows from (1.1) and (1.5) that

$$\begin{aligned} S_k(D^2u(y)) &\leq M(1 + \big(u^+(y))^q\big)\big(1 + |Du(y)|^\gamma\big) \\ &< M\big(1 + (C\tilde{u}(y))^q\big)\big(1 + C|D\tilde{u}(y)|^\gamma\big) \\ &< C^k M\big(1 + (\tilde{u}(y))^q\big)\big(1 + |D\tilde{u}(y)|^\gamma\big) \\ &\leq C^k S_k(D^2\tilde{u}(y)) \\ &= S_k(CD^2\tilde{u}(y)), \end{aligned}$$

a contradiction. □

In order to prove Theorem 1.2, we need the following

Lemma 3.8. *For any $\alpha > 1$ and $a > 0$, there exists a strictly convex radially symmetric function $\bar{u} \in C^2(B_a(0))$ satisfying*

$$S_k(D^2\bar{u}) \leq \frac{(k+1)(n-1)!}{a^k(\alpha-1)k!(n-k-1)!}\big(1 + |D\bar{u}|^k\big)^\alpha \ \ \textit{in } B_a(0)$$

and

$$\frac{\partial\bar{u}}{\partial\nu} = +\infty \ \ \textit{on } \partial B_a(0),$$

where ν is the unit normal to $\partial B_a(0)$.

Proof. Let $\beta = \frac{1}{\alpha-1}$ and

$$\varphi(r) = \begin{cases} \int_0^r [\frac{(1-t^{k+1})^{-\beta}-1}{t}]^{\frac{1}{k}} dr, & r \in (0,1), \\ 0, & r = 0. \end{cases}$$

It is easy to verify that

$$\varphi \in C^2[0,1), \ \ \varphi(0) = \varphi'(0) = 0; \ \ 1 + r(\varphi'(r))^k = (1-r^{k+1})^{-\beta}, \ \ \forall\, 0 \le r < 1. \tag{3.14}$$

and

$$\begin{aligned} (\varphi'(r))^k + kr(\varphi'(r))^{k-1}\varphi''(r) &= \frac{k+1}{\alpha-1} r^k \big(1 + r(\varphi'(r))^k\big)^\alpha \\ &= (k+1)\beta r^k (1-r^{k+1})^{-\beta-1}, \ \ \forall\, r \in (0,1). \end{aligned} \tag{3.15}$$

We claim that

$$\varphi' \ge 0 \ \text{ and } \ \varphi'' > 0 \ \text{ in } [0,1). \tag{3.16}$$

In fact, a direct differentiation yields

$$\varphi'(r) = \big[\frac{(1-r^{k+1})^{-\beta}-1}{r}\big]^{\frac{1}{k}}, \ \ \forall\, r \in (0,1).$$

Then

$$\lim_{r\to 0^+} \varphi'(r) = 0, \ \ \lim_{r \to 1^-} \varphi'(r) = +\infty, \ \ \varphi' \ge 0 \ \text{ in } [0,1) \tag{3.17}$$

and

$$\begin{aligned} \varphi''(0) &= \lim_{r\to 0^+} \frac{\big[(1-r^{k+1})^{-\beta}-1\big]^{\frac{1}{k}}}{r^{\frac{1}{k}+1}} \ \ (\text{letting } t = r^{k+1}) \\ &= \lim_{t \to 0^+} \big[\frac{(1-t)^{-\beta}-1}{t}\big]^{\frac{1}{k}} \\ &= \beta^{\frac{1}{k}} > 0. \end{aligned}$$

For $r \in (0,1)$, it follows from (3.15) that $\varphi''(r)$ has the same sign as $g_1(r^{k+1})$ where

$$g_1(t) = \beta(k+1)t(1-t)^{-\beta-1} - (1-t)^{-\beta} + 1, \ \ \forall\, t \in (0,1).$$

Let $1 - t = s$. We see that $g_1(t) = g_2(s)$ where

$$g_2(s) = \beta(k+1)(1-s)s^{-\beta-1} - s^{-\beta} + 1, \ \ \forall\, s \in (0,1).$$

Furthermore, $g_2(s)$ has the same sign as $g_3(s)$ where

$$g_3(s) = \beta(k+1) - \beta(k+1)s - s + s^{\beta+1}, \ \ \forall\, s \in (0,1).$$

Since $g_3'(s) < 0$ for $s \in (0,1)$ and $g_3(1) = 0$, we see that $g_3(s) > 0$ for all $s \in (0,1)$. Therefore, we obtain (3.16).

Now, let $\bar{u}(x) = a\varphi(a^{-1}|x|)$, $x \in B_a(0)$. Then $\bar{u} \in C^2(B_a(0))$ and is strictly convex by (3.16), satisfying $\frac{\partial \bar{u}}{\partial \nu} = +\infty$ on $\partial B_a(0)$ by (3.17). Moreover by (3.1),(3.15) and (3.16), we have

$$\begin{aligned} S_k(D\bar{u}(x)) &= \frac{A_k}{a^k}\big(\frac{|x|}{a}\big)^{-k}\big[(n-k)(\varphi'(\frac{|x|}{a}))^k + k\frac{|x|}{a}(\varphi'(\frac{|x|}{a}))^{k-1}\varphi''(\frac{|x|}{a})\big] \\ &\le \frac{(n-k)(k+1)A_k}{a^k(\alpha-1)}\big[1+\frac{|x|}{a}(\varphi'(\frac{|x|}{a}))^k\big]^\alpha \\ &\le \frac{(n-k)(k+1)A_k}{a^k(\alpha-1)}\big[1+|D\bar{u}(x)|^k\big]^\alpha \\ &= \frac{(k+1)(n-1)!}{a^k(\alpha-1)k!(n-k-1)!}\big[1+|D\bar{u}(x)|^k\big]^\alpha \text{ in } B_a(0). \end{aligned}$$

□

Proof of Theorem 1.2. We may assume $\Omega \supset \overline{B_a(0)}$. Suppose the contrary that there was a k-convex solution $u \in C^2(\Omega)$ to (1.1)-(1.2). We will derive a contradiction. Let a, α and M be the same as in Theorem 1.2. Choose a function $\bar{u}$ as in Lemma 3.8. Then we have

$$S_k(D^2\bar{u}(x)) < M\big(1+|D\bar{u}(x)|^k\big)^\alpha \text{ in } B_a(0), \tag{3.18}$$

and

$$\frac{\partial}{\partial\nu}(\bar{u}-u) = +\infty \text{ on } \partial B_a(0).$$

It follows from the last equation that

$$\bar{u}(y) - u(y) = \min_{B_a(0)}(\bar{u}-u)$$

for some $y \in B_a(0)$. Using (1.1),(1.6) and (3.18), and repeating the same arguments as in the proof of Theorem 1.2, we obtain a contradiction immediately. □

4. Proof of theorem 1.3

We divide the proof into two steps.

Step 1. Assume Ω is a bounded strictly convex smooth domain. We will find a solution of (1.1)-(1.2) as required as in Theorem 1.3 by the limit of solutions, u_m, of the following Dirichlet problem

$$\begin{cases} S_k(D^2u) = \psi(x,u,Du) & \text{in } \Omega \\ u = m & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where $m = 1, 2, 3, ...$. By assumption (1.10), we may find a positive nondecreasing function $\eta \in C^\infty(\mathbb{R}^n)$ such that

$$\max_{y \le z} \phi(y) \le e^{\varepsilon z}\eta(z), \;\; \forall\, z \in \mathbb{R}. \tag{4.2}$$

Without loss of generality, we assume $\varepsilon = 1$ as this may be achieved by rescaling. Since Ω is bounded, we may choose $a > 0$ such that $\Omega \subset B_a(0)$ and $v^{a,\eta} \le 1$ on $\partial\Omega$ (See (3.3) and Notation 3.2). Using (4.2), (1.9), (1.10) and applying (1.7) and Lemma 2.1, we obtain that for any k-convex solution $u_m \in C^2(\overline{\Omega})$ to (4.1),

$$v^{a,\eta} \le u_m(x) \le m, \;\; \forall\, x \in \Omega, \;\; \forall\, m \ge 1. \tag{4.3}$$

Let

$$C_m^0 = \max\{m, \sup_{\Omega} |v^{a,\eta}|\}, \;\; \forall\, m \ge 1.$$

In order to show the existence of (4.1), we want to use results of Lions [14] as well as of Guan [2]. First, by a result of [14], there exists, for each m and any constant $C_m > 0$, a strictly convex function $\underline{u}_m \in C^2(\overline{\Omega})$ satisfying

$$\begin{cases} \det(D^2\underline{u}_m) \ge C_m(1 + |D\underline{u}_m|^n) & \text{in } \Omega \\ \underline{u}_m = m & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Using this, (3.12) and (3.13), and choosing a suitable C_m, we see that

$$\begin{cases} S_k(D^2\underline{u}_m) \ge \phi(C_m)(1 + |D\underline{u}_m|^k) & \text{in } \Omega \\ \underline{u}_m = m & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

which means that $\underline{u}_m$ is a subsolution of (4.1) for each m. This fact, together with (1.7), (1.9), (1.11), Lemma 2.1 and Theorem 1.2 of Guan [2], implies that problem (4.1) has a unique k-convex solution $u_m \in C^\infty(\overline{\Omega})$ for each m. Moreover, we have

$$u_m(x) \le u_{m+1}(x), \;\; \forall\, x \in \Omega, \;\; \forall\, m \ge 1 \tag{4.6}$$

again by Lemma 2.1. We claim that there exists $a > 0$ depending only on Ω and an decreasing sequence $a_m \to a(m \to \infty)$ such that

$$v^{a_m,\eta}(a - d(x)) \le u_m(x) \le \bar{h}(d(x)), \;\; \forall\, x \in \Omega, \; \forall m \ge 1. \tag{4.7}$$

In fact, the second inequality in (4.7) follows directly from Corollary (3.6). To show the first one, we use the strict convexity of Ω to find the smallest positive number a, such that for any $\bar{x} \in \partial\Omega$, there is a ball $B_a(x_0) \supset \Omega$ satisfying $\overline{\Omega} \cap \partial B_a(x_0) = \{\bar{x}\}$. Choose $a_1 > a_2 > ... > a_m > a_{m+1} > ...$, $a_m \to a(m \to \infty)$, such that $v^{a_m,\eta}(a) = m$

for each $m \geq 1$. For any $y \in \Omega$, choose $\bar{y} \in \partial\Omega$ and a ball $B_a(x_0)$ such that $d(y) = |y - \bar{y}|$, $\Omega \subset B_a(x_0)$, $\overline{\Omega} \cap \partial B_a(x_0) = \{\bar{y}\}$. Observing that

$$v^{a_m,\eta}(x - x_0) \leq v^{a_m,\eta}(a) = m \leq u_m(x), \ \ \forall\, x \in \partial\Omega,$$

we use Lemma 2.1 to obtain

$$v^{a_m,\eta}(x - x_0) \leq u_m(x), \ \ \forall\, x \in \Omega.$$

In particular,

$$v^{a_m,\eta}(a - d(y)) = v^{a_m,\eta}(y - x_0) \leq u_m(y).$$

This proves the first inequality in (4.7), since $y \in \Omega$ is arbitrary.

Now by (4.6) and (4.7), we see that for each $x \in \Omega$, the limit

$$u(x) = \lim_{m\to\infty} u_m(x)$$

exists and it satisfies

$$v^{a,\eta}(a - d(x)) \leq u(x) \leq \bar{h}(d(x)), \ \ \forall\, x \in \Omega. \tag{4.8}$$

Moreover, by Theorem 3.1 of [18], the convergence is uniform in every compact set $K \subset \Omega$ and $u \in C^{0,1}_{loc}(\Omega)$. By the stability theorem of viscosity solutions under the uniform convergence, we see that u is a viscosity k-convex solution of (1.1)-(1.2), which satisfies (1.12) by (4.8). □

Step 2. Suppose now that Ω is a bounded convex domain. We will complete the proof of Theorem 1.3 . In this case , we choose a sequence of strictly convex smooth domains

$$\Omega_1 \subset \Omega_2 \subset ... \subset \Omega_m \subset \Omega_{m+1} \subset ... \subset \Omega, \tag{4.9}$$

such that

$$\Omega = \bigcup_{m=1}^{\infty} \Omega_m.$$

For each $m \geq 1$, by the result of Step 1, we choose a k-convex viscosity solution $u_m \in C^{0,1}_{loc}(\Omega_m)$ to the problem

$$\begin{cases} S_k(D^2u) = \psi(x, u, Du) & \text{in } \Omega_m, \\ u = +\infty & \text{on } \partial\Omega_m. \end{cases}$$

By (4.8), (4.9) and Lemma 2.1, we may assume

$$\begin{aligned} v^{a_{m+1},\eta}(a_{m+1} - d_{m+1}(x)) &\leq u_{m+1}(x) \\ &\leq u_m(x) \leq \bar{h}(d_m(x)), \ \ \forall\, x \in \Omega_m, \ \ \forall\, m \geq 1, \end{aligned} \tag{4.10}$$

where $d_m(x) = dist(x, \partial\Omega_m)$ and a_m is the smallest positive number, such that for any $\bar{x} \in \partial\Omega_m$, there is a ball $B_{a_m}(x_0) \supset \Omega$ satisfying $\overline{\Omega} \cap \partial B_{a_m}(x_0) = \{\bar{x}\}$. Note that

(4.9) and the boundedness od Ω imply $\{a_m\}$ is a bounded nondecreasing sequence. Letting $a = \lim_{m\to\infty} a_m$ and using (4.10), we see that the limit function

$$u(x) = \lim_{m\to\infty} u_m(x)$$

exists for each $x \in \Omega$ and it satisfies (4.8) again. Repeating the arguments from (4.8) to the end in Step 1, we have completed the proof of Theorem 1.3 . $\square$

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Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
E-mail address: `chen-xq03@mails.tsinghua.edu.cn`

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
E-mail address: `hjian@math.tsinghua.edu.cn`