

# A class of fully nonlinear equations arising from conformal geometry<sup>1</sup>

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## Abstract

In this paper we study the long time existences for a class of fully nonlinear parabolic equations arising from conformal geometry. Especially we prove that every smooth compact  $n$  dimensional manifold,  $n \geq 3$ , admits a Riemannian metric  $g$  with its Ricci curvature  $\text{Ric}$  and scalar curvature  $R$  satisfying

$$\det(\text{Ric} - R \cdot g) = \text{const..}$$

## 1 Introduction

Let  $(M^n, g)$  be a compact,  $n$ -dimensional connected smooth Riemannian manifold,  $n \geq 3$ , and let the Ricci tensor and the scalar curvature be denoted by  $\text{Ric}$  and  $R$ (or  $R_g$ ), respectively. In [11], Gursky and Viaclovsky introduced following *modified Schouten tensor*

$$A_g^\lambda := \frac{1}{n-2} \left( \text{Ric}_g - \frac{\lambda}{2(n-1)} R_g \cdot g \right),$$

where  $\lambda \in \mathcal{R}$ . When  $\lambda = 1$ ,  $A_g^\lambda$  is just the classical Schouten tensor; while  $\lambda = n-1$ , it is just the Einstein tensor which is important for gravity in General Relativity.

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We are interested in deforming the metric in the conformal class  $[g_0]$  of a fixed background metric  $g_0$  to certain extremal metric with respect to the modified Schouten tensor alone some suitable flows.

Let  $(\lambda_1, \dots, \lambda_n) \in R^n$ . We recall the elementary symmetric functions as functions on  $R^n$

$$\sigma_k(\lambda_1, \dots, \lambda_n) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

and define

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \dots, \lambda_n) \in R^n \mid \sigma_j(\Lambda) > 0, \forall j \leq k\}.$$

We also define  $\Gamma_k^- = -\Gamma_k^+$ .

For a symmetric linear transformation  $A : V \rightarrow V$ , where  $V$  is an  $n$ -dimensional inner product space. The notation  $A \in \Gamma_k^\pm$  means that the eigenvalues of  $A$  lie in the corresponding set. We note that this notation also makes sense for a symmetric tensor on a Riemannian manifold.

The  $\sigma_k$ -scalar curvature of  $g$  with respect to the scale  $\lambda$  can be defined by

$$\sigma_k(A_g^\lambda) := \sigma_k(g^{-1} \cdot A_g^\lambda),$$

where  $g^{-1} \cdot A_g^\lambda$  is defined locally by  $(g^{-1} \cdot A_g^\lambda)_j^i = g^{il}(A_g^\lambda)_{lj}$ . When  $k = 1$ ,  $\sigma_1$ -scalar curvature is just the scalar curvature  $R$  (upto a constant multiple).

The  $\sigma_k$  Yamabe problem is to solve the equation

$$\sigma_k(A_{\tilde{g}}^\lambda) = \text{constant},$$

where  $\tilde{g} = e^{2u}g$  is a conformal metric. Notice that for  $k = 1$  and  $\lambda = 1$ , this is just the Yamabe equation which has been solved by Yamabe, Trudinger, Aubin and Schoen ect.(see [14] for details). For  $\lambda = 1$  and  $A_g \in \Gamma_k^+$ , in [22], Viaclovsky initiated the study of the  $\sigma_k$  Yamabe problem with  $k > 1$ . By use of elliptic approach, A. Li and Y. Y. Li obtained the existence for locally conformally flat manifolds ([15]). Meanwhile, this problem was studied by Guan and Wang in [10] by a fully nonlinear flow. They proved the convergence of the flow in  $C^\infty$  to a solution of this equation for  $k \neq n/2$  on locally conformally flat manifolds. Brendle and Viaclovsky [1] extended their result to the remaining case  $k = n/2$ . Chang, Gursky and Yang succeeded to solved the equation for  $\sigma_2$  in dimension 4 (see [4]).

In another aspect, if  $\lambda < 1$  and  $A_g^\lambda \in \Gamma_k^-$ , in [11], Gursky and Viaclovsky proved that there exists a unique conformal metric  $\tilde{g} = e^{2u}g$  satisfying  $\sigma_k^{1/k}(A_{\tilde{g}}^\lambda) = f(x)$  for any smooth negative function  $f(x) < 0$  on  $M^n$ . As its consequence, they proved that every smooth compact  $n$ -manifold,  $n \geq 3$ , admits a Riemannian metric with  $\text{Ric} < 0$  and its determinant  $\det(\text{Ric}) = \text{constant}$ . By use of a fully nonlinear curvature flow, Li and Sheng [16] proved that any such metric  $g$  can always be deformed to a conformal metric with constant  $\sigma_k$ -scalar curvature at exponential rate.

In this paper, we establish several appropriate curvature flows, and study the deformation of some given metric  $g_0$  in its conformal class  $[g_0]$  under these curvature flows to some extreme metric. Each of our results corresponds to the initial metric  $g_0$  satisfying one of the following conditions:

(C1)  $\lambda > n - 1$  and  $A_{g_0}^\lambda \in \Gamma_k^+$ ;

(C2)  $\lambda \geq n - 1$  and  $A_{g_0}^\lambda \in \Gamma_k^-$ ;

(C3)  $\lambda < n - 1$  and  $A_{g_0}^\lambda \in \Gamma_k^+$ .

For simplification, we denote

$$S_g^\lambda := -A_g^\lambda.$$

Then  $S_g^\lambda \in \Gamma_k^+$  if  $A_g^\lambda \in \Gamma_k^-$ . If the initial metric  $g_0$  satisfies (C1), we consider the following curvature flow

$$\begin{cases} \frac{d}{dt}g &= (\log \sigma_k(A_g^\lambda) - \log f(x)) \cdot g, & f(x) > 0 \\ g(0) &= g_0, & \lambda > n - 1, \end{cases} \quad (1.1)$$

where  $f(x) > 0$  is any given smooth function on  $M^n$ . We prove following

**Theorem 1.1.** *Suppose  $(M, g_0)$  be a compact, connected  $n$ -manifold ( $n \geq 3$ ). Assume that the metric  $g_0$  is smooth with its modified Schouten tensor  $A_{g_0}^\lambda \in \Gamma_k^+$  and  $\lambda > n - 1$ , then flow (1.1) exists for all time  $0 < t < \infty$  and  $g(t) \in C^\infty(M)$  with  $A_{g(t)}^\lambda \in \Gamma_k^+$  for all  $t$ . For any positive integer  $l$ , there exist positive constants  $C$  depending only on  $g_0$ ,  $f(x)$ ,  $k$ ,  $n$  and  $l$  (independent of  $t$ ), such that*

$$\|g\|_{C^l(M)} \leq C, \text{ and } \lim_{t \rightarrow \infty} \|\sigma_k(A_{g(t)}^\lambda) - f(x)\|_{C^l(M)} = 0$$

where the norm are taken with respect to the background metric  $g_0$ . Furthermore,  $\{g(t)\}$  converges smoothly to a unique metric  $g$  at exponential rate as  $t \rightarrow \infty$  and

$$\sigma_k(A_g^\lambda) = f(x).$$

By Theorem 1.1, we have the following

**Corollary 1.1.** *If  $(M, g_0)$  is a compact and connected  $n$ -manifold ( $n \geq 3$ ) and its modified Schouten tensor  $A_{g_0}^\lambda \in \Gamma_k^+$  and  $\lambda > n - 1$ , then there is a smooth metric  $g$  in  $[g_0]$  such that the corresponding  $\sigma_k$ -scalar curvature  $\sigma_k(A_g^\lambda)$  is a constant.*

When  $\lambda = 2(n - 1)$ , if Ricci curvature  $\text{Ric}_{g_0} < 0$ , then  $A_{g_0}^\lambda \in \Gamma_k^+$ . But every smooth compact  $n$ -manifold,  $n \geq 3$ , always admits a Riemannian metric with  $\text{Ric} < 0$  ([2], [18]). Therefore we have

**Corollary 1.2.** *Every smooth compact  $n$  dimensional manifold,  $n \geq 3$ , admits a Riemannian metric  $g$  with its Ricci curvature satisfying*

$$\det(\text{Ric} - R \cdot g) = \text{const..}$$

If the initial metric  $g_0$  satisfies (C2), we propose following curvature flow

$$\begin{cases} \frac{d}{dt}g &= -(\log \sigma_k(S_g^\lambda) - \log r_k(S_g^\lambda)) \cdot g, \\ g(0) &= g_0, \quad \lambda \geq n - 1, \end{cases} \quad (1.2)$$

where  $r_k(S_g^\lambda)$  is given by

$$\log r_k(S_g^\lambda) = \frac{1}{\text{vol}(g)} \int_M \log \sigma_k(S_g^\lambda) d\text{vol}(g).$$

**Theorem 1.2.** *Suppose  $(M, g_0)$  be a compact, connected and locally conformally flat  $n$ -manifold ( $n \geq 3$ ). Assume that the metric  $g_0$  is smooth with its modified Schouten tensor  $A_{g_0}^\lambda \in \Gamma_k^-$  and  $\lambda \geq n - 1$ , then flow (1.2) exists for all time  $0 < t < \infty$  and  $g(t) \in C^\infty(M)$  with  $A_{g(t)}^\lambda \in \Gamma_k^-$  for all  $t$ .*

If the initial metric  $g_0$  satisfies (C3), we consider the following flow which proposed by Guan and Wang ([10])

$$\begin{cases} \frac{d}{dt}g &= -(\log \sigma_k(A_g^\lambda) - \log r_k(A_g^\lambda)) \cdot g, \\ g(0) &= g_0, \quad \lambda < 1, \end{cases} \quad (1.3)$$

where  $r_k(A_g^\lambda)$  is the same as in equation (1.2).

**Theorem 1.3.** *Suppose  $(M, g_0)$  be a compact, connected and locally conformally flat  $n$ -manifold ( $n \geq 3$ ). Assume that the metric  $g_0$  is smooth with its modified Schouten tensor  $A_{g_0}^\lambda \in \Gamma_k^+$  and  $\lambda < 1$ , then flow (1.3) exists for all time  $0 < t < \infty$  and  $g(t) \in C^\infty(M)$  with  $A_{g(t)}^\lambda \in \Gamma_k^+$  for all  $t$ .*

*Remark 1.1.* In Theorem 1.2 and 1.3, we only obtain the long time existence for each flow. These results can be regarded as the continuance of the works in [10] and [16], as well as in [23].

This paper is organized as follows. In section 2, we recall some basic properties of  $\sigma_k$ . In section 3 we reformulate equation (1.1) in terms of other scalar functions and establish a priori estimates for admissible solutions, then prove the long time existence and the exponentially convergence of the flow. In section 4 we establish a priori estimates for equation (1.2) and prove the long time existence. In section 5, by use of the same idea in section 4, we give the outline of the proof of Theorem 1.3.

## 2 Preliminaries

In this section, we recall some basic properties of  $\sigma_k$ , all of which can be found in [8] or [22].

Let  $A : V \rightarrow V$  be a symmetric linear transformation, where  $V$  is an  $n$ -dimensional inner product space. For  $0 \leq q \leq n$ , the  $q$ th Newton transformation associated with  $A$  is

$$T_q(A) = \sigma_q(A) \cdot I - \sigma_{q-1}(A) \cdot A + \cdots + (-1)^q A^q.$$

If  $A_j^i$  are the components of  $A$  with respect to some basis of  $V$ , then

$$T_q(A)_j^i = \frac{1}{q!} \delta_{j_1 \dots j_q}^{i_1 \dots i_q} A_{i_1}^{j_1} \cdots A_{i_q}^{j_q}.$$

Here  $\delta_{j_1 \dots j_q}^{i_1 \dots i_q}$  is the generalized Kronecker delta symbol, and we are using the Einstein summation convention.

We note that if  $A : R \rightarrow \text{Hom}(V, V)$ , then

$$\frac{d}{dt} \sigma_k(A(t)) = T_{k-1}(A(t))_j^i \frac{d}{dt} A(t)_i^j = T_{k-1}(A(t))^{ij} \frac{d}{dt} A(t)_{ij}.$$

We also note the identities

$$\begin{aligned} T_{k-1}(A)_j^i A_i^j &= k \sigma_k(A), \\ \text{tr } T_{k-1}(A) &= T_{k-1}(A)_k^k = (n - k + 1) \sigma_{k-1}(A). \end{aligned}$$

The following Proposition describes some important properties of the cone  $\Gamma_k^\pm$ , and their relation to the Newton transformations:

- Proposition 2.1.** (i) Each set  $\Gamma_k^+$  is an open convex cone.  
(ii) If the eigenvalues of  $A$  are in  $\Gamma_k^+$ , then  $T_{k-1}$  is positive definite.  
(iii)  $\log \sigma_k$  and  $\sigma_k^{1/k}$  are concave.

## 3 Proof of Theorem 1.1

### 3.1 Short Time existence and $C^0$ -estimate

In this section we suppose the initial metric  $g_0$  is smooth with its modified Schouten tensor  $A_{g_0}^\lambda \in \Gamma_k^+$  and  $\lambda > n - 1$ , we consider the curvature flow (1.1). Let  $g(t) = e^{2u(t)} g_0$ , by [11]

$$A_g^\lambda = A_{g_0}^\lambda - \nabla^2 u - \frac{1 - \lambda}{n - 2} (\Delta u) g_0 + du \otimes du - \frac{2 - \lambda}{2} |\nabla u|^2 g_0. \quad (3.1)$$

Here all covariant derivatives are taken with respect to the background  $g_0$ . Then equation (1.1) can be written as

$$\begin{cases} 2\frac{du}{dt} = \log \sigma_k(A_g^\lambda) - \log f(x) \\ u(x, 0) \equiv 0 \end{cases} \quad (3.2)$$

or

$$\begin{cases} 2\frac{du}{dt} = \log \sigma_k(W) - 2ku - \log f(x) \\ u(x, 0) \equiv 0 \end{cases} \quad (3.3)$$

where

$$W = \frac{\lambda - 1}{n - 2}(\Delta u)g_0 - \nabla^2 u + du \otimes du - \frac{2 - \lambda}{2}|\nabla u|^2 g_0 + A_{g_0}^\lambda, \quad (3.4)$$

and

$$\sigma_k(W) := \sigma_k(g_0^{-1}W).$$

**Lemma 3.1.** *We have*

$$\begin{aligned} \frac{d}{dt} \log \sigma_k(A_g^\lambda) &= \frac{1}{2} Q_\lambda^{ij} (\nabla_{g(t)}^2)_{ij} (\log \sigma_k(A_g^\lambda) - \log f(x)) \\ &\quad - k (\log \sigma_k(A_g^\lambda) - \log f(x)) \end{aligned} \quad (3.5)$$

where

$$Q_\lambda^{ij} = \frac{1}{\sigma_k(A_g^\lambda)} \left( \frac{\lambda - 1}{n - 2} \left( \sum_l T_{k-1}^{ll}(A_g^\lambda) \right) g^{ij}(t) - T_{k-1}^{ij}(A_g^\lambda) \right)$$

*Proof.* It is easy to see that

$$\begin{aligned} \frac{d}{dt} \sigma_k(A_g^\lambda) &= T_{k-1}(A_g^\lambda)_j^i \frac{d}{dt} (g^{-1} A_g^\lambda)_i^j \\ &= -k \sigma_k(A_g^\lambda) (\log \sigma_k(A_g^\lambda) - \log f(x)) + \text{tr} (T_{k-1}(A_g^\lambda) g^{-1} \frac{d}{dt} A_g^\lambda) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} (A_g^\lambda) &= -\nabla_{g(t)}^2 \frac{du}{dt} + \frac{\lambda-1}{n-2} \Delta_{g(t)} \left( \frac{du}{dt} \right) g(t) \\ &= -\frac{1}{2} \nabla_{g(t)}^2 (\log \sigma_k(A_g^\lambda) - \log f(x)) \\ &\quad + \frac{\lambda-1}{2(n-2)} \Delta_{g(t)} (\log \sigma_k(A_g^\lambda) - \log f(x)) g(t). \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt} \sigma_k(A_g^\lambda) &= \frac{\lambda-1}{2(n-2)} \sum_i T_{k-1}^{ii}(A_g^\lambda) \Delta_{g(t)} (\log \sigma_k(A_g^\lambda) - \log f(x)) \\ &\quad - \frac{1}{2} \text{tr} (T_{k-1}(A_g^\lambda) \nabla_{g(t)}^2 (\log \sigma_k(A_g^\lambda) - \log f(x))) \\ &\quad - k \sigma_k(A_g^\lambda) (\log \sigma_k(A_g^\lambda) - \log f(x)) \end{aligned}$$

which implies (3.5).  $\square$

Because  $f(x)$  is independent of time, we have

$$\begin{aligned} \frac{d}{dt} \left( \log \sigma_k(A_g^\lambda) - \log f(x) \right) &= \frac{1}{2} Q_\lambda^{ij} (\nabla_{g(t)}^2)_{ij} \left( \log \sigma_k(A_g^\lambda) - \log f(x) \right) \\ &\quad - k \left( \log \sigma_k(A_g^\lambda) - \log f(x) \right). \end{aligned} \quad (3.6)$$

The maximum principle then implies

$$C_2 \leq \log \sigma_k(A_g^\lambda) \leq C_1, \quad e^{C_2} \leq \sigma_k(A_g^\lambda) \leq e^{C_1} \quad (3.7)$$

for some constants  $C_1$  and  $C_2$  depending only on  $f(x)$  and  $g_0$ .

Since  $A_{g(0)}^\lambda \in \Gamma_k^+$ , flow (1.1) (and (3.5)) is parabolic near  $t = 0$ . By the standard implicit function theorem, we have the following short time existence.

**Proposition 3.1.** *For any  $g_0 \in C^2(M)$  with  $A_{g_0}^\lambda \in \Gamma_k^+$ , there exists a positive constant  $T^*$  such that flow (1.1) exists and is parabolic for  $t \in [0, T^*)$ , and for any  $T (< T^*)$ ,*

$$g \in C^{3,\alpha}([0, T] \times M), \quad 0 < \alpha < 1 \quad \text{and} \quad A_{g(t)}^\lambda \in \Gamma_k^+.$$

Here  $T^*$  is the maximal time such that  $A_{g(t)}^\lambda \in \Gamma_k^+$ .

**Proposition 3.2** *Suppose  $A_{g_0}^\lambda \in \Gamma_k^+$  for some  $\lambda \geq n - 1$ . Let  $g(x, t) = e^{2u(x,t)} g_0(x)$  be a solution to equation (3.2) on  $M \times [0, T]$  ( $T < T^*$ ) with  $g(0) = g_0$ . Then there exist constants  $\underline{\delta}$  and  $\bar{\delta}$  depending only on  $g_0, f(x), k, n, \lambda$  (independent of  $T$ ) such that*

$$\underline{\delta} \leq u(x, t) \leq \bar{\delta}. \quad (3.8)$$

*Proof.* By (3.2) and (3.3), we have

$$\begin{aligned} 2ku &= \log \sigma_k(W) - \log \sigma_k(A_g^\lambda) \\ &= \log \sigma_k \left( \frac{\lambda-1}{n-2} (\Delta u) g_0 - \nabla^2 u + du \otimes du - \frac{2-\lambda}{2} |\nabla u|^2 \right. \\ &\quad \left. + A_{g_0}^\lambda \right) - \log \sigma_k(A_g^\lambda). \end{aligned} \quad (3.9)$$

Since  $M$  is compact,  $u(x, t)$  has the maximum at  $(x_0, t_0) \in M \times [0, T]$ . By (3.9)

$$\begin{aligned} 2ku(x_0, t_0) &\leq \log \sigma_k(A_{g_0}^\lambda) - \log \sigma_k(A_g^\lambda)(x_0, t_0) \\ &\leq \log \sigma_k(A_{g_0}^\lambda) - C_2 \leq \bar{\delta}. \end{aligned}$$

Similarly, we also can take an appropriate constant  $\underline{\delta}$  satisfying  $u(x, t) \geq \underline{\delta}$ .  $\square$

### 3.2 $C^1$ -estimate

**Proposition 3.3** *There is a constant  $C$  depending only on  $g_0$ ,  $f(x)$ ,  $k$ ,  $n$  and  $\lambda(\lambda \geq n-1)$  such that for any solution  $u(x, t)$  of flow (3.3),*

$$|\nabla_{g_0} u(x, t)|_{g_0} \leq C$$

for all  $(x, t) \in M^n \times [0, T)$ .

*Proof.* The proof is similar to that in [16], where

$$W = S_{g_0}^\lambda + \nabla^2 u + \frac{1-\lambda}{n-2}(\Delta u)g_0 - du \otimes du + \frac{2-\lambda}{2}|\nabla u|^2 g_0,$$

and  $\lambda < 1$ . We sketch the proof and indicate some minor changes. We first consider the following function

$$H(x, t) = \left(1 + \frac{|\nabla u|^2}{2}\right)e^{\phi(u)},$$

where  $\phi : R \rightarrow R$  is a function of the form

$$\phi(s) = C_1(C_2 + s)^p > 0,$$

and  $\underline{\delta} \leq s \leq \bar{\delta}$ . The constants  $C_1$ ,  $C_2$  and  $p$  depends only on  $\underline{\delta}$  and  $\bar{\delta}$ , such that the function  $\phi(s)$  satisfies  $\phi'(s) > 0$  and  $\phi''(s) - (\phi'(s))^2 - \phi'(s) > 0$ . It is proved in [11] that such function  $\phi(s)$  is always existed.  $\nabla$  and  $|\cdot|$  are taken with respect to  $g_0$ .

Suppose the maximum of  $H$  occurs at  $(x_0, t_0) \in M \times [0, T]$ . We take a normal coordinate system  $(x^1, \dots, x^n)$  with respect to  $g_0$ . Then at  $(p, t_0) \in M \times [0, T]$  we have

$$0 \leq H_t = u_l u_{lt} e^{\phi(u)} + \left(1 + \frac{|\nabla u|^2}{2}\right) e^{\phi(u)} \phi' u_t, \quad (3.10)$$

$$0 = H_i = u_l u_{li} e^{\phi(u)} + \left(1 + \frac{|\nabla u|^2}{2}\right) e^{\phi(u)} \phi' u_i, \quad (3.11)$$

and

$$\begin{aligned} 0 \geq H_{ij} &= u_{ij} u_{li} e^{\phi(u)} + u_l u_{lij} e^{\phi(u)} + u_l u_{li} e^{\phi(u)} \phi' u_j + u_l u_{lj} e^{\phi(u)} \phi' u_i \\ &+ \left(1 + \frac{|\nabla u|^2}{2}\right) e^{\phi(u)} ((\phi')^2 u_i u_j + \phi'' u_i u_j + \phi' u_{ij}). \end{aligned} \quad (3.12)$$

Define

$$Q^{ij}(W) = \frac{1}{\sigma_k(W)} \left( \frac{\lambda-1}{n-2} \sum_l T_{k-1}^{ll}(W) \delta_{ij} - T_{k-1}^{ij}(W) \right), \quad (3.13)$$

where  $W = (w_{ij})$  be a matrix defined by

$$w_{ij} = \frac{\lambda-1}{n-2} \Delta u \delta_{ij} - u_{ij} + u_i u_j - \frac{2-\lambda}{2} |\nabla u|^2 \delta_{ij} + (A_{g_0}^\lambda)_{ij}. \quad (3.14)$$

Obviously,  $(Q^{ij})$  is positive definite.

Let  $v = 1 + \frac{|\nabla u|^2}{2}$ , then by (3.11), (3.12) implies

$$0 \geq \frac{1}{v} Q^{ij} u_l u_{lij} + Q^{ij} (\phi'' - (\phi')^2) u_i u_j + \phi' u_{ij} Q^{ij} \quad (3.15)$$

since  $Q^{ij} u_{li} u_{lj}$  is nonnegative. By (3.14) and  $T_{k-1}^{ij} w_{ij} = k\sigma_k(W)$ , we get

$$\begin{aligned} Q^{ij} u_{ij} &= \frac{1}{\sigma_k(W)} (k\sigma_k(W) - T_{k-1}^{ij} u_i u_j \\ &\quad + \frac{2-\lambda}{2} |\nabla u|^2 \sum_l T_{k-1}^{ll} - (A_{g_0}^\lambda)_{ij} T_{k-1}^{ij}). \end{aligned} \quad (3.16)$$

Since

$$Q^{ij} u_{ijl} u_l = -\frac{1}{\sigma_k(W)} T_{k-1}^{ij} u_{ijl} u_l + \frac{1}{\sigma_k(W)} \frac{\lambda-1}{n-2} \sum_p T_{k-1}^{pp} \sum_q u_{qql} u_l, \quad (3.17)$$

by (3.14) and (3.11), and the fact

$$\frac{1}{\sigma_k(W)} T_{k-1}^{ij} w_{ijl} = (\log \sigma_k(W))_l = (2u_t + 2ku + \log f(x))_l, \quad (3.18)$$

we have

$$\begin{aligned} Q^{ij} u_{ijl} u_l &= (2u_t + 2ku + \log f(x))_l u_l + \frac{1}{\sigma_k(W)} \left( 2\phi' v T_{k-1}^{ij} u_i u_j \right. \\ &\quad \left. + (\lambda-2)\phi' v |\nabla u|^2 \sum_p T_{k-1}^{pp} - (A_{g_0}^\lambda)_{ijl} T_{k-1}^{ij} u_l \right). \end{aligned} \quad (3.19)$$

By the Ricci identity we have  $u_{lij} = u_{ijl} + R_{jlip} u_p$  where  $R_{ijlp}$  is the Riemannian curvature tensor of  $(M, g_0)$ . By (3.15), (3.16) and (3.19), together with above Ricci identity, we have

$$\begin{aligned} 0 &\geq \frac{1}{v} (2u_t + 2ku + \log f(x))_l u_l + \frac{1}{\sigma_k(W)} \phi' T_{k-1}^{ij} u_i u_j \\ &\quad + \frac{1}{\sigma_k(W)} \phi' \frac{(\lambda-2)}{2} |\nabla u|^2 \sum_l T_{k-1}^{ll} + k\phi' + Q^{ij} (\phi'' - (\phi')^2) u_i u_j \\ &\quad - \frac{1}{\sigma_k(W)} \phi' (A_{g_0}^\lambda)_{ij} T_{k-1}^{ij} - \frac{1}{v\sigma_k(W)} (A_{g_0}^\lambda)_{ijl} T_{k-1}^{ij} u_l \\ &\quad + \frac{1}{v} Q^{ij} R_{jlip} u_l u_p. \end{aligned} \quad (3.20)$$

By (3.2) and (3.10), we have

$$\frac{1}{v} u_l u_{lt} = -\phi' u_t = -\frac{1}{2} \phi' \left( \log \sigma_k(A_g^\lambda) - \log f(x) \right) \quad (3.21)$$

which is bounded. By (3.13) we also have

$$\frac{1}{\sigma_k(W)} \phi' T_{k-1}^{ij} u_i u_j = -\phi' Q^{ij} u_i u_j + \frac{1}{\sigma_k(W)} \frac{(\lambda-1)}{(n-2)} \phi' \sum_l T^{ll} |\nabla u|^2. \quad (3.22)$$

Substituting (3.21) and (3.22) into (3.20), we get

$$\begin{aligned}
 0 \geq & \frac{2k|\nabla u|^2}{v} - \phi' \left( \log \sigma_k(A_g^\lambda) - \log f(x) \right) + k\phi' \\
 & + Q^{ij} (\phi'' - (\phi')^2 - \phi') u_i u_j + \frac{1}{\sigma_k(W)} \phi' \left( \frac{(\lambda-1)}{(n-2)} + \frac{(\lambda-2)}{2} \right) \sum_l T^{ll} |\nabla u|^2 \\
 & + \frac{\langle \nabla f(x), \nabla u \rangle}{v f(x)} - \frac{1}{\sigma_k(W)} \phi' (A_{g_0}^\lambda)_{ij} T_{k-1}^{ij} \\
 & - \frac{1}{v \sigma_k(W)} (A_{g_0}^\lambda)_{ijl} T_{k-1}^{ij} u_l + \frac{1}{v} Q^{ij} R_{jlip} u_l u_p.
 \end{aligned} \tag{3.23}$$

It is clear that  $\|R_{ijkl}\|_{g_0}$ ,  $(A_{g_0}^\lambda)_{ijl}$  and  $\phi$  are all bounded. Now the remaining proof is same as [16] (see [11] also).  $\square$

### 3.3 $C^2$ -estimate

**Proposition 3.4** *There is a constant  $C$  depending only on  $g_0$ ,  $f(x)$ ,  $k$ ,  $n$  and  $\lambda(\lambda > n - 1)$  such that for any solution  $u(x, t)$  of flow (3.3),*

$$|\nabla_{g_0}^2 u(x, t)|_{g_0} \leq C$$

for all  $(x, t) \in M^n \times [0, T)$ .

*Proof.* We use the idea of [10] to prove this proposition. If  $k \geq 2$  and  $(w_{ij}) \in \Gamma_k^+$ , we know  $|w_{ij}| \leq C_1 \text{tr}(w_{ij})$ , i.e.

$$|u_{ij}| \leq C_1 (|\Delta u| + |\nabla u|^2 + 1).$$

Because

$$\text{tr}(w_{ij}) = \text{tr} \left( \frac{\lambda-1}{n-2} (\Delta u) g_0 - \nabla^2 u + du \otimes du - \frac{2-\lambda}{2} |\nabla u|^2 g_0 + A_{g_0}^\lambda \right) > 0$$

i.e.  $\Delta u \geq C$ , we only need to prove  $\Delta u$  has upper bound. Consider

$$G := \Delta u + m |\nabla u|^2$$

on  $M \times [0, T]$  for a given  $T \in (0, T^*)$ , here  $m$  is a constant to be fixed later. Assume  $G$  achieves the maximum at  $(x_0, t_0) \in M \times [0, T]$ . Without loss of generality, we may assume that  $G(x_0, t_0) \geq 1$ . At  $(x_0, t_0)$ , we have

$$G_t = \sum_l (u_{llt} + 2m u_{lt} u_l) \geq 0, \tag{3.24}$$

$$G_i = \sum_l (u_{lli} + 2m u_{li} u_l) = 0, \quad \forall i. \tag{3.25}$$

By Ricci identity and (3.25), we have at  $(x_0, t_0)$ ,

$$|\sum_l u_{ll}| \leq cG \quad \forall i, \quad |u_{lij} - u_{ijl}| \leq C, \quad \text{and} \quad |u_{llij} - u_{ijll}| \leq CG.$$

Let  $F = \log \sigma_k(W)$  and  $F^{ij} = \frac{\partial F}{\partial w_{ij}} = \frac{T_{k-1}^{ij}(W)}{\sigma_k(W)}$ . We assume that  $(w_{ij})$  is diagonal at  $x_0$ , so  $(F^{ij})$  and

$$\begin{aligned} Q^{ij} &= \frac{1}{\sigma_k(W)} \left( \frac{\lambda-1}{n-2} \left( \sum_l T_{k-1}^{ll}(W) \right) \delta_{ij} - T_{k-1}^{ij}(W) \right) \\ &= \frac{\lambda-1}{n-2} \left( \sum_l F^{ll} \right) \delta_{ij} - F^{ij} \end{aligned} \quad (3.26)$$

are also diagonal at that point. Since  $(Q^{ij})$  is positive definite, we have

$$\begin{aligned} 0 &\geq \sum_{ij} Q^{ij} G_{ij} = \sum_{i,j,l} Q^{ij} (u_{llij} + 2mu_{li}u_{lj} + 2mu_{lj}u_{li}) \\ &\geq \sum_{ijl} Q^{ij} (u_{ijll} + 2mu_{ijl}u_l) + 2m \sum_{ijl} Q^{ij} u_{li}u_{lj} - c(\sum_i F^{ii})G. \end{aligned} \quad (3.27)$$

In view of (3.14),

$$\begin{aligned} -F^{ij}u_{ijll} &= F^{ij}w_{ijll} - \frac{\lambda-1}{n-2}(\Delta u)_l \sum_p F^{pp} \\ &\quad - F^{ij} \left( u_i u_j - \frac{2-\lambda}{2} |\nabla u|^2 \delta_{ij} \right)_l - F^{ij} \left( A_{g_0}^\lambda \right)_{ijl}. \end{aligned}$$

By the concavity of  $F$ , we have

$$\begin{aligned} Q^{ij}u_{ijll} &= \left( \frac{\lambda-1}{n-2} \left( \sum_l F^{ll} \right) \delta_{ij} - F^{ij} \right) u_{ijll} \\ &= F^{ij}w_{ijll} - F^{ij} \left( u_i u_j - \frac{2-\lambda}{2} |\nabla u|^2 \delta_{ij} \right)_l - F^{ij} \left( A_{g_0}^\lambda \right)_{ijl} \\ &\geq \Delta F - F^{ij} \left( u_i u_j - \frac{2-\lambda}{2} |\nabla u|^2 \delta_{ij} \right)_l - c \left( \sum_i F^{ii} \right) G. \end{aligned} \quad (3.28)$$

Similarly

$$\begin{aligned} -F^{ij}u_{ijl}u_l &= F^{ij}w_{ijl}u_l - \frac{\lambda-1}{n-2}(\Delta u)_l u_l \sum_p F^{pp} - F^{ij} \left( u_i u_j - \frac{2-\lambda}{2} |\nabla u|^2 \delta_{ij} \right)_l u_l \\ &\quad - F^{ij} \left( A_{g_0}^\lambda \right)_{ijl} u_l \end{aligned} \quad (3.29)$$

and

$$\begin{aligned} 2mQ^{ij}u_{ijl}u_l &= 2m \left( \frac{\lambda-1}{n-2} \left( \sum_l F^{ll} \right) \delta_{ij} - F^{ij} \right) u_{ijl}u_l \\ &\geq 2mF_l u_l - 2mF^{ij} \left( u_i u_j - \frac{2-\lambda}{2} |\nabla u|^2 \delta_{ij} \right)_l u_l \\ &\quad - c(\sum_i F^{ii})G \end{aligned} \quad (3.30)$$

Since

$$F = 2\frac{du}{dt} + 2ku + \log f(x),$$

we have

$$\Delta F = \sum_l 2u_{ll} + 2k\Delta u + \Delta \log f(x)$$

and

$$2mF_l u_l = 4mu_{lt}u_l + 4mk|\nabla u|^2 + 2m(\log f(x))_l u_l.$$

Substituting (3.28) and (3.30) into (3.27), we get

$$\begin{aligned} 0 &\geq 2(\Delta u_t + 2mu_{lt}u_l) + 2k\Delta u + 4km|\nabla u|^2 + \Delta \log f(x) \\ &\quad + 2m(\log f(x))_l u_l - 2\sum_{i,j,l} F^{ij} u_{il} u_{jl} + (2 - \lambda) \left( \sum_p F^{pp} \right) u_{lk} u_{lk} \\ &\quad + 2mQ^{ij} u_{il} u_{lj} - c(\sum_i F^{ii}) G. \end{aligned} \quad (3.31)$$

By (3.24), we obtain

$$C \geq \left( 2mQ^{ii} - (\lambda - 2) \sum_p F^{pp} - 2F^{ii} \right) u_{il} u_{il} - C \left( \sum_i F^{ii} \right) G. \quad (3.32)$$

Since  $\lambda > n - 1$ , there exists a constant  $\varepsilon_0$  such that  $\frac{\lambda-1}{n-2} \geq 1 + \varepsilon_0$ , then

$$Q^{ii} = \frac{\lambda - 1}{n - 2} \sum_p F^{pp} - F^{ii} \geq \varepsilon_0 \sum_p F^{pp},$$

If we take  $m$  large enough, we get

$$\begin{aligned} C &\geq \left( \sum_p F^{pp} \right) u_{il} u_{il} - C \left( \sum_i F^{ii} \right) G \\ &\geq \frac{1}{n} \left( \sum_p F^{pp} \right) (\Delta u)^2 - C \left( \sum_i F^{ii} \right) G \\ &\geq \frac{1}{n} \left( \sum_p F^{pp} \right) (G^2 - CG). \end{aligned}$$

Then the Newton-Maclaurin inequality yields  $G < C$ .  $\square$

### 3.4 Long Time Existence and Convergence

By (3.7) and above a priori estimates, we get

**Proposition 3.5** *For  $\lambda > n - 1$ , flow (1.1) exists for all  $t > 0$ , and the equation*

(1.1) is uniform parabolic.

Now by Krylov's theorem [13] and the standard theory of parabolic equations, we have

**Proposition 3.6** *Flow (1.1) with an initial metric in  $C^{4+\alpha}$  ( $0 < \alpha < 1$ ) admits a unique  $C^{4+\alpha, 2+\alpha}$  solution in  $[0, \infty)$ . For any positive integer  $l$ , there exists constant  $C$  depending only on  $g_0, f(x), k, n, \lambda (> n - 1)$  and  $l$  (independent of  $t$ ), such that*

$$\|g\|_{C^l(M)} \leq C,$$

where the norm are taken with respect to the background metric  $g_0$ .

*Proof of Theorem 1.1* By (3.6), we have

$$|\log \sigma_k(A_g^\lambda) - \log f(x)| \leq Ce^{-kt}. \quad (3.33)$$

Then the convergence is directly from propositions 3.5, 3.6 and inequality (3.33).  $\square$

## 4 Proof of Theorem 1.2

Suppose  $(M, g_0)$  be a compact, connected and locally conformally flat  $n$ -manifold ( $n \geq 3$ ), the metric  $g_0$  is smooth with its modified Schouten tensor  $A_{g_0}^\lambda \in \Gamma_k^-$  and  $\lambda \geq n - 1$ . We now consider the curvature flow (1.2). Let  $g(t) = e^{-2u(t)}g_0$ . By (3.1),

$$S_g^\lambda = -A_g^\lambda = S_{g_0}^\lambda - \nabla^2 u - \frac{1-\lambda}{n-2}(\Delta u)g_0 - du \otimes du + \frac{2-\lambda}{2}|\nabla u|^2 g_0, \quad (4.1)$$

where the covariant derivatives are taken with respect to the background metric  $g_0$ . Then equation (1.2) can be rewritten as

$$\begin{cases} 2\frac{du}{dt} = \log \sigma_k(S_g^\lambda) - \log r_k \\ u(x, 0) \equiv 0 \end{cases} \quad (4.2)$$

or

$$\begin{cases} 2\frac{du}{dt} = \log \sigma_k(W) + 2ku - \log r_k \\ u(x, 0) \equiv 0 \end{cases} \quad (4.3)$$

where

$$W = \frac{\lambda-1}{n-2}(\Delta u)g_0 - \nabla^2 u - du \otimes du + \frac{2-\lambda}{2}|\nabla u|^2 g_0 + S_{g_0}^\lambda. \quad (4.4)$$

In the same way as Lemma 3.1, we have

**Lemma 4.1.**

$$\frac{d}{dt} \left( \log \sigma_k(S_g^\lambda) \right) = \frac{1}{2} Q_\lambda^{ij} (\nabla_{g(t)}^2)_{ij} \left( \log \sigma_k(S_g^\lambda) - \log r_k \right) + k \left( \log \sigma_k(S_g^\lambda) - \log r_k \right)$$

where

$$Q_\lambda^{ij} = \frac{1}{\sigma_k(S_g^\lambda)} \left( \frac{\lambda-1}{n-2} \left( \sum_l T_{k-1}^{ll}(S_g^\lambda) \right) g^{ij}(t) - T_{k-1}^{ij}(S_g^\lambda) \right). \quad (4.5)$$

By the standard implicit function theorem we have the following short time existence.

**Proposition 4.1.** *For any  $g_0 \in C^2(M)$  with  $A_{g_0}^\lambda \in \Gamma_k^-$ , there exists a positive constant  $T^*$  such that flow (1.2) exists and is parabolic for  $t \in [0, T^*)$ , and for any  $T(< T^*)$ ,*

$$g \in C^{3,\alpha}([0, T] \times M), \quad 0 < \alpha < 1 \text{ and } A_{g(t)}^\lambda \in \Gamma_k^-.$$

Here  $T^*$  is the maximal time such that  $A_{g(t)}^\lambda \in \Gamma_k^-$ .

Since  $A_{g(t)}^\lambda \in \Gamma_k^-$ , we have

$$\sigma_1(A_{g(t)}^\lambda) = \text{tr}(A_{g(t)}^\lambda) < 0$$

i.e.

$$\frac{1}{n-2} \left( \frac{2(n-1) - n\lambda}{2(n-1)} \right) R < 0,$$

when  $\lambda \geq n-1$ . This means that  $R > 0$ . Since  $(M, g_0)$  is locally conformally flat  $n$ -manifold ( $n \geq 3$ ), by [7], we have

$$|\nabla u|_{g_0} \leq C, \quad \text{on } M \times [0, T^*), \quad (4.6)$$

where  $\nabla$  is taken with respect to  $g_0$ ,  $C$  is independent of  $T^*$ .

For  $C^2$  estimate, we have

**Proposition 4.2** *There is a constant  $C$  depending only on  $g_0$ ,  $k$ ,  $n$  and  $\lambda(\lambda \geq n-1)$  such that for any solution  $u(x, t)$  of flow (4.3),*

$$|\nabla_{g_0}^2 u(x, t)|_{g_0} \leq C$$

for all  $(x, t) \in M^n \times [0, T)$ .

*Proof.* We assume  $k \geq 2$  and  $\lambda \geq n-1$ . let

$$G = \Delta u + m|\nabla u|^2,$$

$$Q^{ij} = \frac{1}{\sigma_k(W)} \left( \frac{\lambda-1}{n-2} \sum_l T_{k-1}^{ll}(W) \delta_{ij} - T_{k-1}^{ij} \right).$$

Assume  $G$  achieves the maximum at  $(x_0, t_0) \in M \times [0, T]$ , as in the proof of proposition 3.4, we have at  $(x_0, t_0)$

$$\begin{aligned} 0 &\geq Q^{ij} G_{ij} \geq -4kG + 2 \sum_{i,j,l} F^{ij} u_{il} u_{jl} \\ &\quad + (\lambda-2) (\sum_p F^{pp}) \sum_{k,l} u_{kl}^2 + 2m \sum_{i,j,l} Q^{ij} u_{li} u_{lj} - c(\sum_i F^{ii})G, \end{aligned} \quad (4.7)$$

where  $F = \log \sigma_k(W)$  and  $F^{ij} = \frac{\partial F}{\partial w_{ij}} = \frac{T_{k-1}^{ij}(W)}{\sigma_k(W)}$ . Let  $(w_{ij})$  is diagonal at  $(x_0, t_0)$ , then  $(F^{ij})$  and  $(Q^{ij})$  is also diagonal at the point.

We now consider two cases.

*Case 1.*  $\lambda > n - 1$ .

Then

$$Q^{ij} = \frac{\lambda - 1}{n - 2} \left( \sum_p F^{pp} \right) \delta_{ij} - F^{ij} \geq \varepsilon_0 \left( \sum_p F^{pp} \right) \delta_{ij}, \quad (4.8)$$

where  $\varepsilon_0 = \frac{\lambda - (n-1)}{n-2} > 0$ . Consequently,

$$\begin{aligned} 2mQ^{ij}u_{li}u_{lj} &\geq 2m\varepsilon_0 \left( \sum_p F^{pp} \right) \sum_{l,i} u_{li}^2 \\ &\geq \frac{2m}{n} \varepsilon_0 \left( \sum_p F^{pp} \right) (\Delta u)^2 \\ &\geq \frac{2m}{n} \varepsilon_0 \left( \sum_p F^{pp} \right) G^2 - c \left( \sum_i F^{ii} \right) G. \end{aligned} \quad (4.9)$$

Using (4.9) in (4.7), we can easily obtain  $G < C$  if we choose  $m > 0$  large enough.

*Case 2.*  $\lambda = n - 1$ .

In this case,

$$Q^{ij} = \left( \left( \sum_p F^{pp} \right) \delta_{ij} - F^{ij} \right).$$

Then

$$2m \sum_{i,j,l} Q^{ij} u_{li} u_{lj} + 2 \sum_{i,j,l} F^{ij} u_{li} u_{lj} = (2m - 2) \sum_{i,j,l} Q^{ij} u_{li} u_{lj} + 2 \sum_p F^{pp} \sum_{k,l} u_{kl}^2. \quad (4.10)$$

But

$$\begin{aligned} \sum_{i,j,l} Q^{ij} u_{li} u_{lj} &= \sum_{i,l} \left( \sum_{j \neq i} F^{jj} u_{il}^2 \right) \\ &\geq \sum_i \left( \sum_{j \neq i} F^{jj} u_{ii}^2 \right) \\ &= \sum_i F^{ii} \left( \sum_{j \neq i} u_{jj}^2 \right) \end{aligned}$$

By (4.4), we have

$$w_{ii} = \sum_{j \neq i} u_{jj} - (u^i)^2 + \frac{3-n}{2} |\nabla u|^2 + (S_{g_0}^\lambda)_{ii}.$$

Then by (4.6)

$$\sum_i F^{ii} \left( \sum_{j \neq i} u_{jj}^2 \right) \geq \frac{1}{n-1} \sum_i F^{ii} w_{ii}^2 - C \sum_i F^{ii} G.$$

It is shown in [10] that

$$\sum_i F^{ii} w_{ii}^2 \geq \frac{k}{n} \sigma_1(W) \geq \frac{k}{n} (n-1) \Delta u - C.$$

Then (4.10) becomes

$$\begin{aligned} & 2m \sum_{i,j,l} Q^{ij} u_{li} u_{lj} + 2 \sum_{i,j,l} F^{ij} u_{li} u_{lj} \\ \geq & \frac{k(2m-2)}{n} G - (2m-2)C \sum_i F^{ii} G + 2(\sum_p F^{pp}) \sum_{k,l} u_{kl}^2 - C. \end{aligned} \quad (4.11)$$

Substituting (4.11) into (4.7) and taking  $m$  large enough, we have

$$C \sum_i F^{ii} G^2 \leq (n-1) (\sum_p F^{pp}) \sum_{k,l} u_{kl}^2 \leq C \sum_i F^{ii} G$$

which implies  $G \leq C$ .  $\square$

By the method of [10], we can prove

**Proposition 4.3.** *Suppose  $g(t)$  is a solution of equation (1.2) with the initial metric  $g_0$  on the time interval  $[0, T^*)$ , then there is a positive constant  $c > 0$  depending on  $g_0, k, n$  (independent of  $t$ ), such that  $\forall t \in [0, T^*)$ ,*

$$\sigma_k(g(t)) \geq ce^{-\frac{t}{c}}.$$

Therefore, flow (1.2) exists for all  $t > 0$ . A simply modification as in [10], we also have

**Proposition 4.4.** *There is a constant  $C_0 > 0$  independent of  $t$  such that*

$$\sigma_k(g)(t) \geq C_0, \quad \text{for } t \in [0, +\infty).$$

*Proof of Theorem 1.2* By (4.6) and propositions 4.2 and 4.4, we know flow (1.2) is uniformly parabolic. Then by Krylov's theorem [13] and the standard theory of parabolic equations, we obtain Theorem 1.2.  $\square$

## 5 Proof of Theorem 1.3

In this section, we suppose  $(M, g_0)$  be a compact, connected and locally conformally flat  $n$ -manifold ( $n \geq 3$ ), the metric  $g_0$  is smooth with its modified Schouten tensor  $A_{g_0}^\lambda \in \Gamma_k^+$  and  $\lambda < 1$ . We consider the curvature flow (1.3). To prove Theorem 1.3

we argue as section 4. Here we only indicate the outline of the proof. At first we Let  $g(t) = e^{-2u(t)}g_0$ . By (3.1),

$$A_g^\lambda = A_{g_0}^\lambda + \nabla^2 u + \frac{1-\lambda}{n-2}(\Delta u)g_0 + du \otimes du - \frac{2-\lambda}{2}|\nabla u|^2 g_0, \quad (5.1)$$

where the covariant derivatives are taken with respect to the background metric  $g_0$ . Then equation (1.3) can be rewritten as

$$\begin{cases} 2\frac{du}{dt} = \log \sigma_k(A_g^\lambda) - \log r_k \\ u(x, 0) \equiv 0 \end{cases} \quad (5.2)$$

or

$$\begin{cases} 2\frac{du}{dt} = \log \sigma_k(W) + 2ku - \log r_k \\ u(x, 0) \equiv 0 \end{cases} \quad (5.3)$$

where

$$W = \frac{1-\lambda}{n-2}(\Delta u)g_0 + \nabla^2 u + du \otimes du - \frac{2-\lambda}{2}|\nabla u|^2 g_0 + A_{g_0}^\lambda. \quad (5.4)$$

We can derive following evolution equation

$$\frac{d}{dt} \left( \log \sigma_k(A_g^\lambda) \right) = \frac{1}{2} Q_\lambda^{ij} (\nabla_{g(t)}^2)_{ij} \left( \log \sigma_k(A_g^\lambda) - \log r_k \right) + k \left( \log \sigma_k(A_g^\lambda) - \log r_k \right)$$

where

$$Q_\lambda^{ij} = \frac{1}{\sigma_k(A_g^\lambda)} \left( \frac{1-\lambda}{n-2} \left( \sum_l T_{k-1}^{ll}(A_g^\lambda) \right) g^{ij}(t) + T_{k-1}^{ij}(A_g^\lambda) \right). \quad (5.5)$$

Then the short time existence of the flow (1.3) follows from the implicit function theorem.

Since  $A_{g(t)}^\lambda \in \Gamma_k^+$ , we have

$$\sigma_1(A_{g(t)}^\lambda) = \text{tr}(A_{g(t)}^\lambda) > 0.$$

Therefore  $R > 0$  when  $\lambda < 1$ . Since we assume  $(M, g_0)$  is locally conformally flat  $n$ -manifold ( $n \geq 3$ ), by [7], we have the following  $C^1$ -estimate

$$|\nabla u|_{g_0} \leq C, \quad \text{on } M \times [0, T^*), \quad (5.6)$$

where  $\nabla$  is taken with respect to  $g_0$ ,  $C$  is independent of  $T^*$ .

For  $C^2$ -estimate, we consider the same function  $G$  as the previous sections

$$G = \Delta u + m|\nabla u|^2.$$

We denote

$$Q^{ij} = \frac{1}{\sigma_k(W)} \left( \frac{1-\lambda}{n-2} \sum_l T_{k-1}^{ll}(W) \delta_{ij} + T_{k-1}^{ij} \right).$$

Assume  $G$  achieves the maximum at  $(x_0, t_0) \in M \times [0, T]$ , as in the proof of proposition 3.4, we have (3.24) and (3.25) at  $(x_0, t_0)$ . Let  $F = \log \sigma_k(W)$  and  $F^{ij} = \frac{\partial F}{\partial w_{ij}} = \frac{T_{k-1}^{ij}(W)}{\sigma_k(W)}$ . We assume that  $(w_{ij})$  is diagonal at  $x_0$ , so  $(F^{ij})$  and

$$Q^{ij} = \frac{1-\lambda}{n-2} \left( \sum_l F^{ll} \right) \delta_{ij} + F^{ij} \quad (5.7)$$

are also diagonal at that point. Then at  $(x_0, t_0)$  we have

$$0 \geq \sum_{ij} Q^{ij} G_{ij} = \sum_{i,j,l} Q^{ij} (u_{llij} + 2mu_{li}u_{lj} + 2mu_{il}u_{li}).$$

By Ricci identities and straightforward computation, we have

$$\begin{aligned} 0 \geq & -4kG + (2-\lambda)(\sum_i F^{ii}) \sum_{j,l} u_{jl}^2 + 2m \sum_{i,l} Q^{ii} u_{il}^2 \\ & - 2 \sum_{i,l} F^{ii} u_{il}^2 - C(\sum_i F^{ii})G. \end{aligned} \quad (5.8)$$

The term

$$\sum_{i,l} Q^{ii} u_{il}^2 \geq \sum_{i,l} F^{ii} u_{il}^2 + \left( \frac{1-\lambda}{n-2} \right) (\sum_i F^{ii}) u_{11}^2$$

where we assume  $|u_{11}| \geq |u_{ii}|$  for all  $i = 1, \dots, n$ , and  $\Delta u = \sum_i u_{ii} \leq n|u_{11}|$ . Now we choose  $m$  large enough, so that (5.8) becomes

$$\begin{aligned} 0 & \geq -4kG + 2m \left( \frac{1-\lambda}{n-2} \right) (\sum_i F^{ii}) u_{11}^2 + (2-\lambda)(\sum_i F^{ii}) \sum_{j,l} u_{jl}^2 - C(\sum_i F^{ii})G \\ & \geq -4kG + \left( \frac{2m(1-\lambda)}{n(n-2)} \right) (\sum_i F^{ii}) G^2 - C(\sum_i F^{ii})G \\ & \quad + \left( \frac{2-\lambda}{n} \right) (\sum_i F^{ii}) G^2 - C(\sum_i F^{ii})G \\ & \geq C(\sum_i F^{ii}) G^2 - C(\sum_i F^{ii})G. \end{aligned}$$

Consequently we have  $G \leq C$ . Thus we obtain the  $C^2$ -estimate.

With obvious modifications as [10], we can prove

$$\sigma_k(g)(t) \geq C_0, \quad \text{for } t \in [0, +\infty)$$

where the constant  $C_0$  is independent of  $t$ . Then flow (1.3) is uniformly parabolic. By Krylov's theorem [13] and the standard theory of parabolic equations, Theorem 1.3 follows.

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