

**SUBDIVISION SCHEMES FOR NON-CONVEX
COMPACT SETS WITH A NEW DEFINITION OF
SET AVERAGE ***

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Abstract

Set valued functions possess a lot of application future. In this paper we improve the definition of set average defined in Artstein [1]. Based on the new definition the spline subdivision schemes as well as the interpolatory subdivision schemes for general compact sets are introduced. Furthermore the convergence properties of the subdivision schemes are discussed. Comparison results based on the new definition with the old one are included to demonstrate that the new one possesses more physical meanings for most cases of the solid modelling.

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1. Introduction

Set valued functions possess a lot of application future. Here we consider univariate set-valued functions, especially the set-valued splines and the interpolatory set-valued functions. The interest in developing subdivision schemes for general compact sets is motivated by the problems of the solid modelling and reconstructions, especially the reconstruction of the 3D object from a set of their 2D parallel cross sections. For example in medicine, the reconstruction problems arise in planning of cosmetic surgery, tumor treatment (where the exact size, shape and location of a tumor must be calculated in order to design radiation treatment), dental surgery, internal medicine and so on. The typical devices used to collect data include CAT, PET, NMR and Ultrasonic scanners. For a review on these topics, please refer to Schumaker [10].

In this paper, we introduce the spline subdivision schemes and the interpolatory subdivision schemes respectively for compact sets with the new definition of set average and give the convergence properties of these schemes. The advantages of the new definition of set average are showed in section 5 to demonstrate that the new one possesses more physical meaning.

In our approach every n -dimensional object is regarded as a univariate set-valued function with $n - 1$ dimensional compact sets as its images, determined by paralleled cross sections. The set-valued function is then approximated from the given samples (cross sections). The approximating procedure used here is an extension of the subdivision schemes especially the spline subdivision schemes and the interpolatory subdivision schemes for compact sets.

When the given samples $\{f_\alpha^0\}$ are scalar values, the m 'th degree spline subdivision schemes are given by the recursive procedures as follows: $f_\alpha^k = \sum_{\beta \in Z} a_{\alpha-2\beta}^{[m]} f_\beta^{k-1}$, $\alpha \in Z$, $k = 1, 2, 3, \dots$ with the mask $a_\alpha^{[m]} = \frac{1}{2^m} \binom{m+1}{\alpha}$ for $i \in \{0, 1, \dots, m+1\}$ and $a_\alpha^{[m]} = 0$ for $\alpha \in Z \setminus \{0, 1, \dots, m+1\}$. Let $f^k = \{f_\alpha^k \mid \alpha \in Z\} \subset \mathfrak{R}$, it is well known that the values $\{f^k\}$ can be obtained by a sequence of binary averaging:

Algorithm 1:

Let

$$f_{2\alpha}^{k,0} = f_{\alpha}^{k-1}, \quad f_{2\alpha+1}^{k,0} = \frac{1}{2}(f_{\alpha}^{k-1} + f_{\alpha+1}^{k-1}), \quad \alpha \in Z.$$

For $1 \leq j \leq m-1$ let

$$f_{\alpha+\frac{1}{2}}^{k,j} = \frac{1}{2}(f_{\alpha}^{k,j-1} + f_{\alpha+1}^{k,j-1}), \quad \alpha \in I_j,$$

where

$$I_j = \begin{cases} Z & j \text{ odd} \\ \frac{1}{2}Z \setminus Z & j \text{ even} \end{cases}$$

The final values at the level k are

$$\begin{aligned} f_{\alpha}^k &= f_{\alpha}^{k,m-1} && \text{for } m \text{ is odd, } \alpha \in Z \\ f_{\alpha}^k &= f_{\alpha-\frac{1}{2}}^{k,m-1} && \text{for } m \text{ is even, } \alpha \in Z \end{aligned}$$

For example, one yields the Chaikin's algorithm for $m=2$:

$$f_{2i}^k = \frac{1}{4}f_{i-1}^{k-1} + \frac{3}{4}f_i^{k-1} = \frac{1}{2}[\frac{1}{2}(f_{i-1}^{k-1} + f_i^{k-1}) + f_i^{k-1}],$$

$$f_{2i-1}^k = \frac{3}{4}f_{i-1}^{k-1} + \frac{1}{4}f_i^{k-1} = \frac{1}{2}[f_{i-1}^{k-1} + \frac{1}{2}(f_{i-1}^{k-1} + f_i^{k-1})].$$

At each level k , for the data $(2^{-k}\alpha, f_{\alpha}^k)$ a piecewise linear interpolation function is defined to be

$$f^k(t) = (1 - \frac{t - 2^{-k}\alpha}{2^{-k}})f_{\alpha}^k + (\frac{t - 2^{-k}\alpha}{2^{-k}})f_{\alpha+1}^k$$

for $\alpha 2^{-k} \leq t \leq (\alpha+1)2^{-k}$. We mentioned that, the values of $f^k(t)$ are weighted average of two consecutive elements in f^k .

Therefore if we want to generalize the spline subdivision schemes to the ones whose sampling data are compact sets, following problems should be considered:

1. Definition of weighted set average of two compact sets.
2. Convergence properties of the subdivision schemes.
3. The physical meaning of the final set-valued function.

Dyn etc. have done a lot of work in this topic. For the case that the initial sampling data are all convex compact sets, they use the Minkowski average of two compact sets in spline subdivision

schemes (see Dyn et al [7]) where the t-weighted Minkowski average of two compact sets is defined to be

$$(1-t)A + tB = \{(1-t)a + tb : a \in A, b \in B\}$$

The limit set-valued spline function is proved to be the linear combinations of a uniform B-spline functions with the initial sampling sets as the coefficients and possesses the shape preserving properties which are the monotonicity and convexity preserving similar to those of scalar ones (details see Dyn et al [7]).

For the case that the initial sampling data are non-convex compact sets, it is shown in Dyn et al [7] that the spline subdivision schemes based on the Minkowski average will make the result too 'big'. To resolve the problem Dyn etc. used the metric average introduced in Artstein [1] (see Dyn et al [8]) instead of the Minkowski average. The t-weighted metric average of compact sets is defined to be

$$A \oplus_t B = \{(1-t)\{a\} + t \prod_B(a) : a \in A\} \cup \{(1-t) \prod_A(b) + t\{b\} : b \in B\}$$

where $\prod_B(a)$ is the set of the points in B which are nearest to a , details will be introduced in the next sections too. Dyn et al [8] reveals that such scheme converges to a limit set-valued spline function too, and the results possess more physical meaning.

However the metric average still makes the limit set-valued spline function too 'big' in some sense. For example when we want to reconstruct a bifurcated blood vessel, the metric average will result to three pieces of caecum but not a connected bifurcated blood vessel (see figure 1 and figure 3 in section 4). This result is not very satisfied for a lot of application purpose. Therefore we will introduce a new definition of set average to resolve this kind of problems.

The paper is organized as follow: in section 2 the new definition of set average are introduced and some of its properties are discussed. In section 3, we discuss the spline subdivision schemes based on the new definition of set average and give its convergence properties, furthermore the related content for the interpolatory subdivision schemes are introduced in section 4. The comparison

of the result based on the new set average with the result based on metric average are showed in section 5 to demonstrate the advantages of the new definition.

2. The Improved New Definition of Set Average

At first, we introduce some notations and definitions used in this paper. Denote the set of all compact nonempty subsets of \mathfrak{R}^n by $\kappa(\mathfrak{R}^n)$, the set of all compact convex nonempty subsets of \mathfrak{R}^n by $C(\mathfrak{R}^n)$ and the set of finite unions of nonempty intervals of \mathfrak{R} by $\kappa_F(\mathfrak{R})$. $|x|$ is the Euclidean norm of $x \in \mathfrak{R}^n$, $\text{co}A$ is the convex hull of the set $A \in \kappa(\mathfrak{R}^n)$, the difference of A and B is $A - B = \{a \in A, a \notin B\}$, where $A, B \in \kappa(\mathfrak{R}^n)$.

Definition 2.1. Let $A, B \in \kappa(\mathfrak{R}^n)$, $\lambda, \mu \in \mathfrak{R}$, so the t -weighted Minkowski average of two sets A and B is $(1 - t)A + tB = \{(1 - t)a + tb : a \in A, b \in B\}$.

Definition 2.2. The Hausdorff distance between A and B in $\kappa(\mathfrak{R}^n)$ is defined to be

$$h(A, B) = \max\{\sup_{x \in A}\{dist(x, B)\}, \sup_{y \in B}\{dist(y, A)\}\},$$

where $dist(x, B) = \min\{|x - y| : y \in B\}$, the $dist(y, A)$ can be defined analogously.

Definition 2.3. Let $A, B \in \kappa(\mathfrak{R}^n)$, $0 \leq t \leq 1$, $\prod_B(a) = \{b \in B : |a - b| = dist(a, B)\}$, the t -weighted metric average of A and B is

$$A \oplus_t B = \{(1-t)\{a\} + t \prod_B(a) : a \in A\} \cup \{(1-t) \prod_A(b) + t\{b\} : b \in B\} \quad (2.1)$$

where the linear combination in (2.1) is in the Minkowski sense.

Proposition 2.1. Let $A, B \in \kappa(\mathfrak{R}^n)$, $0 \leq t < s \leq 1$, then

$$h(A \oplus_t B, A \oplus_s B) = |t - s| h(A, B).$$

The above proposition 2.1 and the definition can be found in Artstein [1] and the proposition 2.1 is also right for the case $t, s \in \mathfrak{R}$. We now give the new definition of set average as follows:

Definition 2.4. Let $A, B \in \kappa(\mathfrak{R}^n)$, $0 \leq t \leq 1$, the t -weighted new set average of A and B is defined to be

$$A \uplus_t B = (coA \oplus_t coB) - ((coA - A) \oplus_t (coB - B)). \quad (2.2)$$

Remark 2.1. This definition is in fact well defined for all $t \in \mathfrak{R}$, too.

Now we show some properties of the new definition of set average.

Proposition 2.2. Let $A, B \in \kappa(\mathfrak{R}^n)$, $0 \leq t \leq 1$, we have

- (1) End sets interpolation: $A \uplus_0 B = A$, $A \uplus_1 B = B$.
- (2) Symmetry: $A \uplus_t B = B \uplus_{(1-t)} A$.
- (3) Self identity: $A \uplus_t A = A$.
- (4) $A \cap B \subseteq A \uplus_t B \subseteq (1-t)A + tB \subseteq co(A \cup B)$.
- (5) Monotonicity: If $0 \leq t < s \leq 1$, $A \subset B$, $B \in C(\mathfrak{R}^n)$, then $A \subseteq A \uplus_t B \subseteq A \uplus_s B \subseteq B$.

Proof: The properties (1) (2) (3) (4) are trivial from the definition. From property (4) we have

$$A \subseteq A \uplus_t B \subseteq co(A \cup B) = B$$

$$A \subseteq A \uplus_s B \subseteq co(A \cup B) = B.$$

Then it remains to prove

$$A \uplus_t B \subseteq A \uplus_s B. \quad (2.3)$$

Denote $M(A, t, B) = \bigcup_{a \in A} ((1-t)\{a\} + t \prod_B(a))$,

we have

$$A \oplus_t B = M(A, t, B) \cup M(B, 1-t, A).$$

For $B \in C(\mathfrak{R}^n)$, $A \uplus_t B = coA \oplus_t B$. To prove (2.3), we noticed that $coA \subseteq B$, therefore

$$\begin{aligned} coA \oplus_t B &= M(coA, t, B) \cup M(B, 1-t, coA) \\ &= coA \cup M(B, 1-t, coA) \\ &= M(B, 1-t, coA). \end{aligned}$$

To any $b \in B$, $c \in \prod_{coA}(b)$, the whole segment $[c, (1-t)c + tb]$ is a subset of $M(B, 1-t, c)$. Since $s \geq t$, then

$$[c, (1-t)c + tb] \subseteq [c, (1-s)c + sb],$$

therefore

$$M(B, 1-t, coA) \subseteq M(B, 1-s, coA),$$

thus

$$coA \oplus_t B \subseteq coA \oplus_s B$$

the desired conclusion follows. \square

Remark 2.2. $A, B \in \kappa(\mathfrak{R}^n)$, if $A \subset B$, from proposition 3 we know $A \subset A \uplus_t B$. However $A \uplus_t B \subset B$ is true only if B is convex which is proved in proposition 4.

If $A, B \in \kappa_F(\mathfrak{R})$, we can get more properties for $A \uplus_t B$:

Proposition 2.3. (1) If $C, D \in C(\mathfrak{R})$, then $C \uplus_t D = C \oplus_t D = (1-t)C + tD$. This means our set average is equivalent to the Minkowski average and metric average for one dimensional convex sets.

(2) For $A_1, A_2, B \in \kappa_F(\mathfrak{R})$, if there is a t such that $A_1 \uplus_t B = A_2 \uplus_t B$, then $A_1 = A_2$.

Proof: Property 1 is trivial from the definition.

To prove property 2, it is enough to verify that for any $B, C \in \kappa_F(\mathfrak{R})$ the solution of the set equation $X \uplus_t B = C$ is unique. When either $t = 1$ or $t = 0$, the conclusion is easy to prove, then we only have to consider $t \in (0, 1)$.

Denote $B = \bigcup_{i=1}^M [b_i^l, b_i^r]$, $C = \bigcup_{j=1}^N [c_j^l, c_j^r]$ where $M, N \in \mathbb{Z}_+$ and $b_i^l \leq b_i^r$, $c_j^l \leq c_j^r$ for all $1 \leq i \leq M$ and $1 \leq j \leq N$, then

$$coB = [b_1^l, b_M^r], \quad coC = [c_1^l, c_N^r]$$

$$coB - B = \bigcup_{i=1}^{M-1} (b_i^r, b_{i+1}^l),$$

$$coC - C = \bigcup_{j=1}^{N-1} (c_j^r, c_{j+1}^l).$$

Denote $A_1 = [a_1, a_2]$ where

$$a_1 = \frac{1}{1-t}c_1^l - \frac{t}{1-t}b_1^l$$

$$a_2 = \frac{1}{1-t}c_N^r - \frac{t}{1-t}b_M^r.$$

It follows that $A_1 \oplus_t coB = coC$. From Dyn et al [8], we have the unique set A_2 satisfies:

$$A_2 \oplus_t (coB - B) = coC - C$$

Denote $A = A_1 - A_2$, it is clearly that $A \uplus_t B = C$, therefore property 2 follows. \square

Remark 2.3. For $A, B \in C(\mathfrak{R}^n)$ the property above will not be true for $n \geq 2$ and $A \uplus_t B$ will not be always convex, We give an example to show this as follows.

Example 2.1. When $n = 2$, $A, B \in C(\mathfrak{R}^n)$, Let $A = \{(x, y) \mid x = 0, 0 \leq y \leq 1\}$, $B = \{(x, y) \mid y = 0, 0 \leq x \leq 1\}$, then for $t \in (0, 1)$,

$$A \uplus_t B = \{(x, y) \mid x = 0, 0 \leq y \leq 1-t\} \cup \{(x, y) \mid y = 0, 0 \leq x \leq t\}$$

It is clear that $A \uplus_t B$ is non-convex and $A \uplus_t B \neq (1-t)A + tB$

We have mainly two ways to construct curves with the subdivision schemes: spline subdivision schemes and the interpolatory subdivision schemes. In the next section we consider at first the spline subdivision schemes and left the discussion of interpolatory subdivision schemes to section 4.

3. The Spline Subdivision Schemes based on new Definition of Set Average

We know that if the initial data are scalar values $\{f_i^0\}$, the uniform B-spline subdivision schemes can be given as

$$f_i^{k+1} = \sum_{j \in Z} a_{i-2j}^{[m]} f_j^k \quad (3.4)$$

with the spline weights (or masks)

$$a_i^{[m]} = \binom{m+1}{i} / 2^m$$

for $i \in \{1, \dots, m+1\}$ and $a_i^{[m]} = 0$ for $i \in Z \setminus \{0, 1, \dots, m+1\}$.

Note that Chaikin's algorithm is the special case of $m = 2$. It follows from Cavaretta et al [3], Dyn [5] that the scheme (3.4) is uniformly convergent to a limit function $f^\infty(\cdot)$ which is of the form

$$f^\infty(t) = \sum_{j \in Z} f_j^0 B_m(t - i) \quad (3.5)$$

where the $B_m(\cdot)$ is the B-spline of degree m with the integer knots and the support $[0, m+1]$.

Now we want to generalize the scheme (3.4) for initial data sets $\{F_\alpha^0\}_{\alpha \in Z}$ where $F_\alpha^0 \in \kappa(\mathfrak{R}^n)$, $\alpha \in Z$. Define the subdivision scheme to get $\{\{F_\alpha^k\}_{\alpha \in Z}\}_{k \in Z_+}$ recursively by

Algorithm 2:

$$\begin{aligned} F_{2\alpha}^{k,0} &= F_\alpha^{k-1} \\ F_{2\alpha+1}^{k,0} &= F_\alpha^{k-1} \uplus_{\frac{1}{2}} F_{\alpha+1}^{k-1}, \quad \alpha \in Z \end{aligned} \quad (3.6)$$

Then for $1 \leq j \leq m-1$, define the intermediate new set average by

$$F_{\alpha+\frac{1}{2}}^{k,j} = F_\alpha^{k,j-1} \uplus_{\frac{1}{2}} F_{\alpha+1}^{k,j-1}, \quad \alpha \in I_j \quad (3.7)$$

where

$$I_j = \begin{cases} Z & j \text{ odd} \\ \frac{1}{2}Z \setminus Z & j \text{ even} \end{cases} \quad (3.8)$$

the entries of the set sequence at level k are defined to be

$$\begin{aligned} F_\alpha^k &= F_\alpha^{k,m-1} && \text{for } m \text{ odd} \\ F_\alpha^k &= F_{\alpha-\frac{1}{2}}^{k,m-1} && \text{for } m \text{ even,} \end{aligned} \quad (3.9)$$

for $\alpha \in Z$. Furthermore a piecewise linear interpolation function is defined to be

$$F^k(t) = F_\alpha^k \uplus_{2^k t - \alpha} F_{\alpha+1}^k, \quad \alpha 2^{-k} \leq t \leq (\alpha+1)2^{-k} \quad (3.10)$$

To verify the convergence of $\{F^k(t)\}$, we introduce a new metric such that

$$\begin{aligned} d(A \uplus_t B, A \uplus_s B) &= \frac{1}{2}h(\text{co}A \oplus_t \text{co}B, \text{co}A \oplus_s \text{co}B) \quad (3.11) \\ &+ \frac{1}{2}h(A \oplus_t B, A \oplus_s B) \end{aligned}$$

For any $A, B \in \kappa(\mathfrak{R}^n)$, it is clear that $A = A \uplus_0 B$ and $B = A \uplus_1 B$, therefore

$$\begin{aligned} d(A, B) &= d(A \uplus_0 B, A \uplus_1 B) \\ &= \frac{1}{2}h(\text{co}A \oplus_0 \text{co}B, \text{co}A \oplus_1 \text{co}B) \\ &\quad + \frac{1}{2}h(A \oplus_0 B, A \oplus_1 B) \\ &= \frac{1}{2}h(\text{co}A, \text{co}B) + \frac{1}{2}h(A, B) \end{aligned}$$

Proposition 3.4. For any $A, B, C \in \kappa(\mathfrak{R}^n)$, $t \in \mathfrak{R}$, $s \in \mathfrak{R}$

- (1) $d(A, B) = d(B, A)$.
- (2) $d(A, B) \geq 0$, $d(A, B) = 0$ if and only if $A = B$.
- (3) $d(A, C) \leq d(A, B) + d(B, C)$. (Schwarz's inequality)
- (4) $d(A \uplus_t B, A \uplus_s B) = |t - s|d(A, B)$.

Therefore the definition of the formula (3.11) is well defined as a metric.

Proof: (1) Since the definition is symmetric Property 1 is clear.

(2) From the definition $d(A, B) = \frac{1}{2}h(\text{co}A, \text{co}B) + \frac{1}{2}h(A, B) \geq 0$

$$\begin{aligned} d(A, B) = 0 &\iff h(\text{co}A, \text{co}B) = 0, \quad h(A, B) = 0 \\ &\iff \text{co}A = \text{co}B, \quad A = B \end{aligned}$$

therefore $d(A, B) = 0$ if and only if $A = B$.

(3) From the properties of Hausdorff distance,

$$\begin{aligned} d(A, C) &= \frac{1}{2}h(\text{co}A, \text{co}C) + \frac{1}{2}h(A, C) \\ &\leq \frac{1}{2}h(\text{co}A, \text{co}B) + \frac{1}{2}h(\text{co}B, \text{co}C) \\ &\quad + \frac{1}{2}h(A, B) + \frac{1}{2}h(B, C) \\ &= d(A, B) + d(B, C). \end{aligned}$$

(4) From the definition,

$$d(A \uplus_t B, A \uplus_s B) = \frac{1}{2}h(\text{co}A \oplus_t \text{co}B, \text{co}A \oplus_s \text{co}B) + \frac{1}{2}h(A \oplus_t B, A \oplus_s B).$$

From the property of Hausdorff distance

$$h(\text{co}A \oplus_t \text{co}B, \text{co}A \oplus_s \text{co}B) = |t - s|h(\text{co}A, \text{co}B)$$

$$h(A \oplus_t B, A \oplus_s B) = |t - s|h(A, B).$$

Therefore

$$\begin{aligned} d(A \uplus_t B, A \uplus_s B) &= |t - s| \frac{1}{2} (h(\text{co}A, \text{co}B) + h(\text{co}A - A, \text{co}B - B)) \\ &= |t - s| d(A, B). \end{aligned}$$

Then the property 4 follows. \square

Lemma 3.1. *Let $F^k = \{F_\alpha^k\}$ be defined as above and let $d^k = \sup_{\alpha \in Z} \{d(F_\alpha^k, F_{\alpha+1}^k)\}$, then*

$$d^k \leq 2^{-k} d^0 \quad (3.12)$$

Proof: Denote $d^{k,j} = \sup_{\alpha \in I_{j-1}} \{d(F_\alpha^{k,j}, F_{\alpha+1}^{k,j})\}$,

follows from (3.6) and the properties of $d(\cdot, \cdot)$ that

$$d(F_{2\alpha}^{k,0}, F_{2\alpha+1}^{k,0}) \leq \frac{1}{2} d^{k-1}, \alpha \in Z,$$

$$d(F_{2\alpha}^{k,0}, F_{2\alpha-1}^{k,0}) \leq \frac{1}{2} d^{k-1}, \alpha \in Z,$$

therefore $d^{k,0} \leq \frac{1}{2} d^{k-1}$.

$$\begin{aligned} d(F_{\alpha-\frac{1}{2}}^{k,j}, F_{\alpha+\frac{1}{2}}^{k,j}) &\leq d(F_{\alpha-\frac{1}{2}}^{k,j}, F_\alpha^{k,j}) + d(F_\alpha^{k,j}, F_{\alpha+\frac{1}{2}}^{k,j}) \\ &\leq \frac{1}{2} d(F_{\alpha-1}^{k,j-1}, F_\alpha^{k,j-1}) + \frac{1}{2} d(F_\alpha^{k,j-1}, F_{\alpha+1}^{k,j-1}) \\ &\leq d^{k,j-1}, \end{aligned}$$

therefore $d^{k,j} \leq d^{k,j-1}$.

Then it follows that

$$d^{k,j} \leq d^{k,0}. \quad (3.13)$$

Therefore

$$d^k = d^{k,m-1} \leq d^{k,0} \leq \frac{1}{2}d^{k-1}.$$

Hence 3.12 holds. \square

Lemma 3.2. *Let $\{F^k(\cdot)\}_{k \in Z_+}$ be defined as above,*

$$d(F^{k+1}(t), F^k(t)) \leq Cd^k, \quad k \in Z_+ \quad (3.14)$$

where $C = \frac{3}{4} + \frac{m}{4}$, and m is the degree of the spline.

Proof: At first, we verify the inequality

$$d(F_{2\alpha}^{k+1}, F_\alpha^k) \leq \frac{m-1}{4}d^k, \quad k \in Z_+ \quad (3.15)$$

we will only prove (3.15) for odd m , the case for even m can be got analogously. By (3.6), (3.7), (3.9) and the Schwarz's inequality, we get

$$\begin{aligned} d(F_\alpha^k, F_{2\alpha}^{k+1}) &= d(F_{2\alpha}^{k+1,0}, F_{2\alpha}^{k+1,m-1}) \\ &\leq d(F_{2\alpha}^{k+1,0}, F_{2\alpha+\frac{1}{2}}^{k+1,1}) + \cdots + d(F_{2\alpha+\frac{1}{2}}^{k+1,m-2}, F_{2\alpha}^{k+1,m-1}) \\ &\leq \sum_{j=0}^{m-2} \frac{1}{2}d^{k+1,j} \leq \frac{(m-1)}{2}d^{k+1,0} \leq \frac{(m-1)}{4}d^k. \end{aligned}$$

Let

$$\alpha 2^{-k} \leq t \leq (\alpha + \frac{1}{2})2^{-k}.$$

Then

$$\begin{aligned} d(F_\alpha^k, F^k(t)) &\leq \frac{1}{2}d^k \\ d(F^{k+1}(t), F_{2\alpha}^{k+1}) &\leq d^{k+1}. \end{aligned}$$

Hence

$$\begin{aligned} &d(F^{k+1}(t), F^k(t)) \\ &\leq d(F^{k+1}(t), F_{2\alpha}^{k+1}) + d(F_{2\alpha}^{k+1}, F_\alpha^k) + d(F_\alpha^k, F^k(t)) \\ &\leq d^{k+1} + \frac{m-1}{4}d^k + \frac{1}{2}d^k \\ &\leq Cd^k. \end{aligned}$$

For $(\alpha + \frac{1}{2})2^{-k} \leq t \leq (\alpha + 1)2^{-k}$, we can get similar bound by using $F_{2(\alpha+1)}^{k+1}$ to instead of $F_{2\alpha}^{k+1}$. \square

Definition 3.5. $A \in \kappa(\mathfrak{R}^n)$, the parallel body A_δ is defined to be

$$A_\delta = \{x \in \mathfrak{R}^n : \text{dist}(x, A) \leq \delta\}$$

where $\delta \geq 0$.

Here we give an equivalent definition of the Hausdorff distance.

Definition 3.6. Let $A, B \in \kappa(\mathfrak{R}^n)$, the Hausdorff distance of A and B is

$$h(A, B) = \inf\{\delta \geq 0 : A \subset B_\delta, B \subset A_\delta\}$$

Proposition 3.5. The new metric $d(\cdot, \cdot)$ is equivalent to the Hausdorff metric $h(\cdot, \cdot)$.

Proof: To arbitrary $A, B \in \kappa(\mathfrak{R}^n)$, we have

$$d(A, B) \geq \frac{1}{2}h(A, B)$$

Assume $h(A, B) = \alpha$, from the above definition,

$$A \subset B_\alpha, \quad B \subset A_\alpha,$$

therefore we have

$$\text{co}A \subset (\text{co}B)_\alpha, \quad \text{co}B \subset (\text{co}A)_\alpha.$$

In fact, to any $a \in \text{co}A$ there is $t \in [0, 1]$, $a_1, a_2 \in A$ s.t.
 $a = (1-t)a_1 + ta_2$.

Because $A \subset B_\alpha$, there is $b_1, b_2 \in B$ s.t.

$$\text{dist}(a_1, b_1) \leq \alpha, \quad \text{dist}(a_2, b_2) \leq \alpha,$$

Thus

$$\begin{aligned} & \text{dist}((1-t)a_1 + ta_2, (1-t)b_1 + tb_2) \\ & \leq \text{dist}((1-t)a_1 + ta_2, (1-t)a_1 + tb_2) + \\ & \quad \text{dist}((1-t)a_1 + tb_2, (1-t)b_1 + tb_2) \\ & = t\text{dist}(a_2, b_2) + (1-t)\text{dist}(a_1, b_1) \\ & \leq \alpha \end{aligned}$$

$(1-t)b_1 + tb_2 \in coB$ thus $coA \subset (coB)_\alpha$. Similarly we have $coB \subset (coA)_\alpha$. So $h(coA, coB) \leq \alpha = h(A, B)$.

Therefore

$$h(A, B) \geq d(A, B) \geq \frac{1}{2}h(A, B),$$

the conclusion follows.

Lemma 3.3. $\kappa(\mathfrak{R}^n)$ is a closed metric space under the new metric $d(\cdot, \cdot)$.

Proof: Because $\kappa(\mathfrak{R}^n)$ is a closed metric space under the Hausdorff metric (see Sun [11]). From the property above the conclusion is trivial since the Hausdorff metric is equivalent to our new metric. \square

Theorem 3.1. The set-valued function sequence $\{F^k(\cdot)\}_{k \in \mathbb{Z}_+}$ converge uniformly to a set-valued function $F^\infty(\cdot)$, which is Lipschitz continuous under the new metric $d(\cdot, \cdot)$.

Proof: From the lemmas above, we have

$$d(F^k(t + \delta), F^k(t)) \leq \delta 2^k d^k \leq \delta d^0$$

Hence the $\{F^k(t)\}$ are uniformly Lipschitz continuous under the metric $d(\cdot, \cdot)$.

By Schwarz's inequality it follows that

$$d(F^{k+M}(t), F^k(t)) \leq \sum_{i=k}^{k+M-1} d(F^{i+1}(t), F^i(t))$$

by the lemmas above

$$d(F^{k+M}(t), F^k(t)) \leq C \sum_{i=k}^{k+M-1} d^i \leq C \frac{d^0}{2^{k-1}}. \quad (3.16)$$

From the lemma 3.3 the set-valued function sequence $\{F^k(t)\}_{k \in \mathbb{Z}_+}$ tends to a set $F^\infty(t)$ for any t . Then the desired result follows. \square

Theorem 3.2. The set-valued function $G(\cdot) : \mathfrak{R} \rightarrow \kappa(\mathfrak{R}^n)$ is Lipschitz continuous under the metric $d(\cdot, \cdot)$ and the Lipschitz constant

is L . If the initial data of sets is given by $F_\alpha^0 = G(a + \alpha h)$, $\alpha \in Z$ for any $a \in [0, h)$, then

$$d(F^k(t), G(t)) \leq Dh, \quad \text{and}$$

$$d(F^\infty(t), G(t)) \leq Dh,$$

where $F^k(\cdot)$ is defined by (3.10) and D depend only on L .

Proof: From the continuous condition we have $d^0 \leq Lh$.

$$\begin{aligned} d(F^k(t), G(t)) &\leq d(F^k(t), F^0(t)) + d(F^0(t), F_\alpha^0) + d(F_\alpha^0, G(t)) \\ &\leq 2Cd^0 + d^0 + Lh \\ &\leq (2CL + 2L)h \\ &= Dh \end{aligned}$$

where C is defined in lemma 3.2, $D = 2CL + 2L$. Therefore we get our result. \square

Definition 3.7. A mapping $F : R \rightarrow \kappa(\mathfrak{R}^n)$ is called monotone increasing if $t_1 \leq t_2$ implies $F(t_1) \subset F(t_2)$.

Theorem 3.3. Let $\{F_\alpha^0\}_{\alpha \in Z}$ be an initial sequence of compact convex sets, if $\{F_\alpha^0\}_{\alpha \in Z}$ is monotone increasing, i.e. $F_\alpha^0 \subset F_{\alpha+1}^0$, $\alpha \in Z$, then the map $F^\infty(t)$ is monotone increasing in the sense of definition above.

Proof: We use the induction on the level. Assume $\{F_\alpha^{k-1}\}$ is monotone increasing, from formula (3.6)

$$F_{2\alpha+1}^{k,0} \supset F_{2\alpha}^{k,0}, \quad F_{2\alpha+2}^{k,0} \supset F_{2\alpha+1}^{k,0}$$

From formula(3.7)

$$F_\alpha^{k,j-1} \subset F_{\alpha+\frac{1}{2}}^{k,j} \subset F_{\alpha+1}^{k,j-1}, \quad \alpha \in I_j$$

When m is odd

$$F_\alpha^k = F_\alpha^{k,m-1} \subset F_{\alpha+1}^{k,m-1} = F_{\alpha+1}^k$$

Therefore $\{F_\alpha^k\}$ is monotone increasing. Similar conclusion can be got when m is even.

To any $t \in [0, 1]$ there is α s.t. $\alpha 2^{-k} \leq t \leq (\alpha+1)2^{-k}$. To any $h > 0$, when $\alpha 2^{-k} \leq t+h \leq (\alpha+1)2^{-k}$ we have $F^k(t) \subset F^k(t+h)$ from the monotone property of our new set average. When $t+h > (\alpha+1)2^{-k}$ the same conclusion is implied similarly.

Thus $F^k(t) \subset F^k(t+h)$ for all t and $h > 0$, therefore the set-valued map $F^k(t)$ is monotone increasing. The last conclusion holds also for the $F^\infty(t) = \lim_{k \rightarrow \infty} F^k(t)$. \square

4. The interpolatory Subdivision Schemes based on new Definition of Set Average

In section 3 we have discussed the spline subdivision schemes for the compact sets and have gotten a good approximation to the initial sets. For the application purpose, more interesting set-valued function is the interpolatory function to the sampling data. Then we can get a 3D- object from some of its 2D sectional pictures. Here we consider interpolatory subdivision scheme based on 4-point method as an example. To general case of the interpolatory subdivision schemes, we can get an analogously result. From Dyn et al [4] we can have a interpolatory subdivision scheme for scalar data $\{f_i^0\}_i$

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \quad \text{for } -1 \leq i \leq 2^k n + 1 \\ f_{2i+1}^{k+1} &= \left(\frac{1}{2} + \omega\right)(f_i^k + f_{i+1}^k) - \omega(f_{i-1}^k + f_{i+2}^k), \text{ for } -1 \leq i \leq 2^k n \end{aligned}$$

Now we discuss the parallel results for the initial data of compact sets $\{F_i^0\}_{i=-2}^{n+2}$. The 4- points scheme for compact sets is defined to be

$$\begin{aligned} F_{2i}^{k+1} &= F_i^k, \text{ for } -1 \leq i \leq 2^k n + 1 \\ F_{2i+1}^{k+1} &= (F_i^k \uplus_{\frac{1}{2}} F_{i+1}^k) \uplus_{(-2\omega)} (f_{i-1}^k \uplus_{\frac{1}{2}} f_{i+2}^k), \text{ for } -1 \leq i \leq 2^k n \end{aligned}$$

where $\omega > 0, k = 0, 1, \dots$,

Theorem 4.4. *Given $\{F_i^0\}_{i=-2}^{n+2}$, let the sets $F_i^k, -2 \leq i \leq 2^k n + 2, k \geq 0$, be defined recursively as above. The respected piecewise linear interpolation is denoted to be $F^k(t)$. Then for $\omega < \frac{1}{12}$ there exists continuous set-valued function $F(t)$ such that $d(F^k(t), F(t)) \rightarrow 0$ when $k \rightarrow +\infty$ and $F(2^{-k}i) = F_i^k, 0 \leq i \leq 2^k n, k \geq 0$.*

Proof: Let $F^k(\cdot)$ be the linear interpolation functions to the values at the level k . The maximal difference between the functions $F^{k+1}(\cdot)$ and $F^k(\cdot)$ on any interval $[2^{-k}i, 2^{-k}(i+1)]$ is

$$\begin{aligned}
& d(F_{2i+1}^{k+1}, F_i^k \uplus_{\frac{1}{2}} F_{i+1}^k) \\
&= d((F_i^k \uplus_{\frac{1}{2}} F_{i+1}^k) \uplus_{(-2\omega)} (F_{i-1}^k \uplus_{\frac{1}{2}} F_{i+2}^k), F_i^k \uplus_{\frac{1}{2}} F_{i+1}^k) \\
&= d((F_i^k \uplus_{\frac{1}{2}} F_{i+1}^k) \uplus_{(-2\omega)} (F_{i-1}^k \uplus_{\frac{1}{2}} F_{i+2}^k), \\
&\quad (F_i^k \uplus_{\frac{1}{2}} F_{i+1}^k) \uplus_0 (F_{i-1}^k \uplus_{\frac{1}{2}} F_{i+2}^k)) \\
&\leq | -2\omega | d(F_i^k \uplus_{\frac{1}{2}} F_{i+1}^k, F_{i-1}^k \uplus_{\frac{1}{2}} F_{i+2}^k) \\
&\leq d(F_i^k \uplus_{\frac{1}{2}} F_{i+1}^k, F_{i-1}^k \uplus_{\frac{1}{2}} F_{i+2}^k) \\
&\leq d(F_i^k \uplus_{\frac{1}{2}} F_{i+1}^k, F_i^k) + d(F_{i-1}^k, F_{i-1}^k \uplus_{\frac{1}{2}} F_{i+2}^k) \\
&\leq \frac{1}{2}d(F_i^k, F_{i+1}^k) + d(F_i^k, F_{i-1}^k) + \frac{1}{2}d(F_{i-1}^k, F_{i+2}^k) \\
&\quad + \frac{1}{2}d(F_i^k, F_{i+1}^k) + \frac{1}{2}d(F_{i+1}^k, F_{i+2}^k) \\
&= \frac{3}{2}d(F_{i-1}^k, F_i^k) + d(F_i^k, F_{i+1}^k) + \frac{1}{2}d(F_{i+1}^k, F_{i+2}^k)
\end{aligned}$$

therefore

$$\begin{aligned}
d(F^{k+1}(\cdot), F^k(\cdot)) &\leq 6\omega \cdot \max\{d(F_i^k, F_{i+1}^k)\} \\
&\leq 6\omega d^k
\end{aligned}$$

Because

$$\begin{aligned}
& d(F_{2i+1}^k, F_{2i}^k) \\
&= d((F_i^{k-1} \uplus_{\frac{1}{2}} F_{i+1}^{k-1}) \uplus_{(-2\omega)} (F_{i-1}^{k-1} \uplus_{\frac{1}{2}} F_{i+2}^{k-1}), F_i^{k-1}) \\
&\leq d((F_i^{k-1} \uplus_{\frac{1}{2}} F_{i+1}^{k-1}) \uplus_{(-2\omega)} (F_{i-1}^{k-1} \uplus_{\frac{1}{2}} F_{i+2}^{k-1}), F_i^{k-1} \uplus_{\frac{1}{2}} F_{i+1}^{k-1}) \\
&\quad + d((F_i^{k-1} \uplus_{\frac{1}{2}} F_{i+1}^{k-1}), F_i^{k-1}) \\
&\leq 2\omega d(F_i^{k-1} \uplus_{\frac{1}{2}} F_{i+1}^{k-1}, F_{i-1}^{k-1} \uplus_{\frac{1}{2}} F_{i+2}^{k-1}) + \frac{1}{2}d(F_i^{k-1}, F_{i+1}^{k-1}) \\
&\leq 2\omega \left[\frac{3}{2}d(F_{i-1}^{k-1}, F_i^{k-1}) + d(F_i^{k-1}, F_{i+1}^{k-1}) + \frac{1}{2}d(F_{i+1}^{k-1}, F_{i+2}^{k-1}) \right] \\
&\quad + \frac{1}{2}d(F_i^{k-1}, F_{i+1}^{k-1})
\end{aligned}$$

Analogously we have

$$\begin{aligned}
& d(F_{2i+1}^k, F_{2i+2}^k) \\
&\leq 2\omega \left[\frac{3}{2}d(F_{i-1}^{k-1}, F_i^{k-1}) + d(F_{i-1}^{k-1}, F_{i+1}^{k-1}) + \frac{1}{2}d(F_{i+1}^{k-1}, F_{i+2}^{k-1}) \right] \\
&\quad + \frac{1}{2}d(F_i^{k-1}, F_{i+1}^{k-1})
\end{aligned}$$

Thus we have

$$d^k \leq (6\omega + \frac{1}{2})d^{k-1}$$

therefore

$$\max_k \{d(F^{k+1}(\cdot), F^k(\cdot))\} \leq 6\omega(6\omega + \frac{1}{2})^k d^0$$

Thus if $\omega < \frac{1}{12}$ the functions $\{F^k\}$ is a Cauchy sequence and converge to a continuous limit set-valued function. \square

Remark 4.4. *Due to the loss of the exchange property in our proof for the convergence of the interpolatory subdivision scheme our result is weaker than that of the scalar ones which is proved in Dyn et al [4]. we are not sure whether the convergent condition of our set average can be exchanged or not which is an open problem.*

5. The Examples and Conclusions

By using of the new set average, we can overcome some problems that the metric average does not work. To show this we give the example as follow:

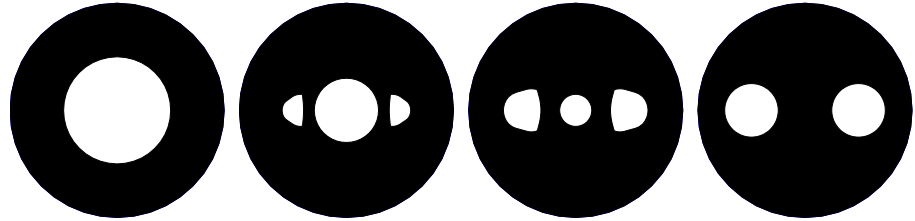


Figure 1: The metric set average of A and B

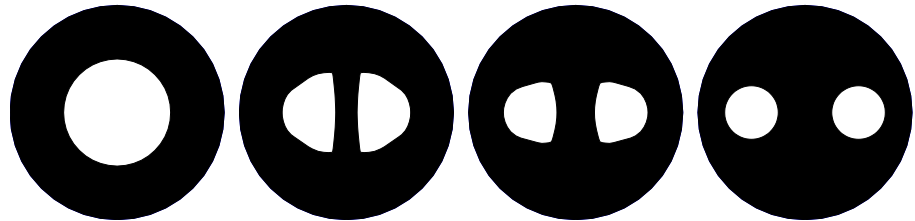


Figure 2: Our new set average of A and B

Note 1. *In this example we choose the initial sets A and B to be disks with one hole and two holes respectively, which can be viewed*

as a sectional picture of bifurcated blood vessel. the figure 1 shows metric average of A and B , for $A = A \oplus_0 B$, $A \oplus_{0.4} B$, $A \oplus_{0.7} B$, and $B = A \oplus_1 B$ in turn. To show the difference between the metric average and the new set average, the figure 2 gives the new set average of A and B , for $A = A \uplus_0 B$, $A \uplus_{0.4} B$, $A \uplus_{0.7} B$, and $B = A \uplus_1 B$ in turn, respectively.

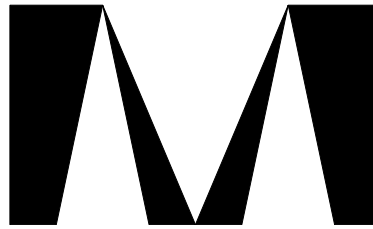


Figure 3: The longitudinal sectional drawing of the metric set average



Figure 4: The longitudinal sectional drawing of our new set average

We can observed that if the initial sets A and B are the cross-sections of a bifurcated blood vessel, then the blood vessel is blocked when we use the metric average which is not very satisfied for the application, while the new definition of set average works well. To verify this clearly we also give the longitudinal sectional pictures of the bifurcated vessel's reconstruction. The results with the metric average and the new definition of the set average are showed in figure 3 and figure 4 respectively.

Conclusions 1. *A new definition of compact set average is present and is applied in the spline subdivision and the interpolatory subdivision schemes. Compared with the metric average examples showed that the new definition of set average possesses more physical meanings in some senses. The new definition of set average is proved*

to have the analogously convergence properties with the old one in the spline subdivision schemes. Furthermore we have proved that the interpolatory subdivision scheme by using our new definition of the set average is convergent too, however sometimes stronger condition is required to compare with the result for interpolation of the scalar data .

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