Estimates of the spectral radius of refinement and subdivision operators with isotropic dilations

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Abstract. The paper presents lower bounds for the spectral radii of refinement and subdivision operators with continuous matrix symbols and with dilations from a class of isotropic matrices. This class contains main dilation matrices used in wavelet analysis. After obtaining general formulas, two kinds of estimate for the spectral radii are established: viz. – estimates using point values of the symbols and others involving integrals of different dimensions over designated subsets of the unit cube. For some symbol classes the exact value of the spectral radius of the refinement operator is found.

Introduction
Let X be a set, and let $s \geq 1$ be a natural number. As usual, the symbol X^s is used for the Cartesian product of s copies of X. If X is also a normed space, then the set X^s has the norm

$$||y||_s = ||y||_{X^s} := \left(\sum_{k=1}^{s} ||x_k||^2 \right)^{1/2}$$

where $y = (x_1, x_2, \ldots, x_s) \in X^s$ and $|| \cdot ||$ denotes the corresponding norm on X. Moreover, let $X^{s \times s}$ denote the set of all $s \times s$ matrices with entries from X.

Consider the unit circle $\mathbb{T} := \left\{ z \in \mathbb{C} : z = e^{ix},\, x \in \mathbb{R} \right\}$, and an essentially bounded measurable matrix-function $a : \mathbb{T}^s \to \mathbb{C}^{m \times m}$ with the Fourier representation $a(x) \sim \sum_{k \in \mathbb{Z}^s} a_k e^{ikx}, x \in \mathbb{R}^s$. If M is an $s \times s$ matrix with integer entries, then a and M generate two operators widely used in computer graphics and wavelet analysis : viz. – the operator $R_a^M : L_2^m(\mathbb{R}^s) \to L_2^m(\mathbb{R}^s)$ and the operator $S_a^M : l_2^m(\mathbb{Z}^s) \to l_2^m(\mathbb{Z}^s)$ defined by

$$R_a^M \varphi := \sum_{k \in \mathbb{Z}^s} a_k \varphi(M \cdot -k),$$

† Research supported in part by UBD Grant PNC2/2/RG/1(29).

$$\left(S_a^M \xi\right)_j := \sum_{k\in\mathbb{Z}^s} a_{j-Mk}\xi_k, \quad j \in \mathbb{Z}^s.$$

In passing, note that a non-singular integer matrix M is often called a dilation matrix if

$$\lim_{n\to\infty} M^{-n} = 0, \tag{1}$$

but our considerations here are not restricted to this condition. Thus let us assume that M is a non-singular matrix with integer entries, and for convenience call such matrices dilation matrices.

The operators R_a^M and S_a^M are referred to as refinement and subdivision operators, respectively. In wavelet literature the matrix-function a is called the mask of the corresponding operator, but throughout this paper it will also be called the symbol of the operator R_a^M or S_a^M. An important role of the refinement and subdivision operators lies in the construction and study of wavelet bases. Thus under some conditions the solutions of the equations

$$\varphi = R_a^M \varphi, \tag{2}$$

called refinable or M-refinable vector functions, produce wavelets. The operator S_a^M arises when one applies an iteration procedure to solve equation (2), [**6**]. Spectral radii $\rho(R_a^M)$ and $\rho(S_a^M)$ of these operators are used to establish the convergence of the iterative algorithm mentioned [**1**], [**11**], [**13**], [**19**], [**21**], [**22**] and to study the regularity of the refinable functions [**2**], [**3**], [**4**], [**5**], [**7**], [**8**], [**11**], [**14**], [**15**], [**21**], [**22**], [**23**], [**30**]. On the other hand, these two operators are closely associated with the transfer operator, also called the Ruelle operator or the Perron–Frobenius–Ruelle operator, which finds various applications in statistical mechanics, dynamical systems and ergodic theory [**18**], [**25**], [**26**], [**27**], [**28**], [**29**].

Despite this, the evaluation of $\rho(R_a^M)$ and $\rho(S_a^M)$ remains an open question. The most essential progress so far has been made for $m = 1$ in the univariate case $s = 1$. However, even in this relatively simple situation the evaluation of the spectral radius heavily depends upon the concrete form of the symbol, and usually the symbol of these operators is assumed to be a polynomial, so the corresponding spectral radius is evaluated using different limit characteristics of auxiliary finite matrices [**2**], [**7**], [**11**], [**31**], [**32**]. More detailed results have been obtained for the so-called continuous refinement operator – viz. the operator T_a^M defined by

$$T_a^M f := \int_{\mathbb{R}^s} a(\cdot - My) f(y) dy.$$

Thus it was shown in [**12**] that for a compactly supported non-negative function $a : \mathbb{R}^s \to \mathbb{C}$ and for a dilation matrix M satisfying condition (1) the spectral radius of the operator T_a^M can be found by the formula

$$\rho(T_a^M) = \frac{1}{\sqrt{|\det M|}} \int_{\mathbb{R}^s} a(y) dy. \tag{3}$$

A different approach to the evaluation of the spectral radius of the operator T_a^M has been employed in [**10**], where formula (3) was proved without assuming that a

is non-negative† and a compactly supported function but rather that $a \in L_1(\mathbb{R}^s)$. Moreover, it was shown that condition (1) can be dropped for some classes of matrices M. It is also worth mentioning that the approach of [**10**] leads to the same limit expression for $\rho(T_a^M)$ and $\rho(R_a^M)$. Thus one could expect that obtaining more convenient formulas for $\rho(R_a^M)$ would be similar to that problem for $\rho(T_a^M)$. However, in contrast to operator T_a^M the peculiarities of the symbol of the operator R_a^M preclude such an effective formula for $\rho(R_a^M)$ as (3) is for $\rho(T_a^M)$. It turns out that evaluation of the spectral radius of the operator R_a^M presents a problem similar to that for the operator S_a^M, so these two operators are considered here together.

Note that even for $m = 1$ and $s = 1$ there are only a few papers devoted to the evaluation of the spectral radius of the operator S_a^M for non-polynomial symbols. Thus for symbols whose Fourier coefficients are rapidly decreasing, two approximate algorithms have been proposed in [**3**]. Other results concerning continuous symbols can be found in [**16**], [**17**]. In [**9**], estimates for the spectral radii of the operators R_a^M and S_a^M with piecewise continuous symbol a are established. In some cases, the spectral radii of S_a^M and R_a^M have been estimated by using the supremum norm of the function a. As a rule, it is very difficult to extend the methods and results obtained in the univariate case $m = 1$ and $s = 1$ to the multivariate cases $m \geq 1$ and $s > 1$, and it appears that only the paper [**24**] and the recently published paper [**2**] contain some results concerning the approximate calculation of the spectral radius of S_a^M for the case $m > 1$ and $s > 1$. Note that the dilation matrix M in [**24**] is supposed to be isotropic, i.e. such that

$$MM^* = \lambda I,$$

where $\lambda > 1$ and I is the identity matrix.

The general case $s \geq 1$ and $m \geq 1$ for a class of dilation matrices is considered in the present paper. This class includes known isotropic matrices encountered in wavelet analysis. However, in contrast to [**24**], approximation methods are not used and the corresponding spectral radius is estimated from below. For $m = 1$, in some cases the estimates obtained yield

$$\rho(R_a^M) = \frac{1}{\sqrt{|\det M|}} ||a||_\infty, \tag{4}$$

and sufficient conditions for equality (4) are given below. Estimates for $\rho(S_a^M)$ are also presented in this paper.

Let us describe the set of dilations considered here, starting with an example. An isotropic dilation often employed in work on bivariate refinable functions corresponds to

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

with eigenvalues $\lambda_1 = 1 - i$ and $\lambda_2 = 1 + i$. Representing these eigenvalues in the polar form $\lambda_1 = r\exp(i\pi q_1)$ and $\lambda_2 = r\exp(i\pi q_2)$ one can pay attention to two

† In this case the integral in (3) has to be replaced by its modulus

remarkable properties they have: viz. – the indices $q_1(= -1/4)$ and $q_2(= 1/4)$ are rational, and there exists a positive integer $l(= 2)$ such that $|\lambda_1|^l = |\lambda_2|^l$ is an integer greater than one. It is easily seen that the second property is inherent for any isotropic dilation matrix, but the first is not. This motivates the introduction of a set $\mathfrak{M}^s$ of all isotropic dilation matrices M with the property that for any eigenvalue $\lambda_j = r\exp(i\pi q_j)$ of M, the index $q_j, j = 1, 2, \ldots, m$ is rational. Note that for $s = 2$ any dilation matrix M of the form

$$M = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad a^2 + b^2 \geq 2$$

is in $\mathfrak{M}^s$.

The aim of this paper is to obtain spectral radius estimates for the operators R_a^M and S_a^M with a continuous symbol a and isotropic dilation matrix $M \in \mathfrak{M}^s$. The paper is organized as follows. In §1 certain properties of the refinement and subdivision operators are presented, and lower estimates are established for spectral radii in terms of multiplicator norms of sequences of matrix-functions. In §2 the notion of μ-cyclic p-tuples is introduced and applied to obtain estimates of the spectral radii, using values of their symbols on designated sets of points. In §3 the above mentioned point value estimates are used to establish integral estimates for the spectral radius of R_a^M. Section 4 gives sufficient conditions when spectral radius of the refinement operator can be calculated exactly via formula (4).

1. *General estimates for the spectral radius of subdivision and refinement operators.*

Let $a \in L_\infty^{m\times m}(\mathbb{T}^s)$ and let $||a||_\infty$ denote the multiplicator norm on $L_\infty^{m\times m}(\mathbb{T}^s)$, i.e. the norm of the operator of multiplication by a on the space $L_2^m(\mathbb{T}^s)$. It is known that

$$||a||_\infty = \max_{1\leq j\leq m} \operatorname*{ess\,sup}_{x\in\mathbb{R}^s} \sqrt{\lambda_j(x)}, \quad t = e^{ix} \in \mathbb{T}^s \tag{5}$$

where $\lambda_j(x), j = 1, 2, \ldots, m$ are the eigenvalues of the matrix a^*a. Of course, if a is a constant matrix, then

$$||a||_\infty = \max_{1\leq j\leq m} \sqrt{\lambda_j}$$

and if $a \in \mathbf{C}^{m\times m}(\mathbb{T}^s)$, then for any $t_k \in \mathbb{T}^s$

$$||a(t_k)||_\infty \leq ||a||_\infty.$$

Let us start with the refinement operator R_a^M.

THEOREM 1.1. *Let $\{a_k\} \in l_1^{m\times m}(\mathbb{Z}^s)$, and let M be a non-singular integer matrix. Then*

$$\rho(R_a^M) = \frac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} || \prod_{k=0}^{n-1} a((M^T)^k \cdot)||_\infty^{1/n}. \tag{6}$$

Proof. Let $B_M : L_2^m(\mathbb{R}^s) \to L_2^m(\mathbb{R}^s)$ be the operator defined by

$$B_M f(x) = f(Mx), \quad x \in \mathbb{R}.$$

To derive formula (6) we represent the operator R_a^M as a product of a Fourier operator and the operator B_M. Denote by $\mathfrak{F}$ the Fourier transform on $L_2^m(\mathbb{R}^s)$, i.e.

$$\mathfrak{F}f = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} e^{-i\,(\cdot)\,y} f(y)dy.$$

It is easily seen that for any $b \in \mathbb{R}^s$

$$\mathfrak{F}e^{ib\cdot}\mathfrak{F}^{-1}f = f(\cdot - b).$$

Thus, if $\{a_k\} \in l_1^{m\times m}(\mathbb{Z}^s)$, the matrix a is continuous on $\mathbb{R}^s$ and

$$R_a^M = B_M \mathfrak{F} a \mathfrak{F}^{-1}. \tag{7}$$

Consider an $s \times s$ non-singular matrix C, and for any matrix-function $a : \mathbb{R}^s \to \mathbb{C}^{m\times m}$ define the matrix $a_C : \mathbb{R}^s \to \mathbb{C}^{m\times m}$ by

$$a_C(x) := a((C^T)^{-1}x).$$

There are two useful identities: viz. –

$$B_M \mathfrak{F} a \mathfrak{F}^{-1} = \mathfrak{F} a_M \mathfrak{F}^{-1} B_M; \tag{8}$$

and

$$B_M^* = \frac{1}{|\det M|} B_{M^{-1}}, \tag{9}$$

the last of which follows immediately from the definition of B_M. For (8) one has

$$\begin{aligned} B_M \mathfrak{F} a \mathfrak{F}^{-1} f(x) &= \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} e^{-iMxy} a(y) \int_{\mathbb{R}^s} e^{iyt} f(t)\, dtdy \\ &= \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} e^{-ixM^T y} a(y) \int_{\mathbb{R}^s} e^{iyt} f(t)\, dtdy \\ &= \frac{1}{(2\pi)^s |\det M|} \int_{\mathbb{R}^s} e^{-ixu} a((M^T)^{-1}u) \int_{\mathbb{R}^s} e^{i(M^T)^{-1}ut} f(t)\, dtdu \\ &= \frac{1}{(2\pi)^s |\det M|} \int_{\mathbb{R}^s} e^{-ixu} a((M^T)^{-1}u) \int_{\mathbb{R}^s} e^{iuM^{-1}t} f(t)\, dtdu \\ &= \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} e^{-ixu} a((M^T)^{-1}u) \int_{\mathbb{R}^s} e^{iuv} f(Mv)\, dvdu \\ &= \mathfrak{F} a_M \mathfrak{F}^{-1} B_M f(x) \end{aligned}.$$

From relations (8), (9) and induction one can show that the product of the operators $(R_a^M)^n$ and $((R_a^M)^n)^*$ admits the representation

$$(R_a^M)^n\, ((R_a^M)^n)^* = |\det M|^{-n} \mathfrak{F} \mathcal{A}_M^{(n)} \mathfrak{F}^{-1}, \quad n \in \mathbb{N} \tag{10}$$

where

$$\mathcal{A}_M^{(n)}(x) = \prod_{k=1}^{n} a_{M^k}(x) \prod_{k=0}^{n-1} a^*_{M^{n-k}}(x). \tag{11}$$

Recall that $\mathfrak{F} : L_2^m(\mathbb{R}^s) \to L_2^m(\mathbb{R}^s)$ is an isometrical isomorphism. Hence for any $a \in L_\infty^{m\times m}(\mathbb{R}^s)$ the norm of the Fourier operator $\mathfrak{F}a\mathfrak{F}^{-1}$ on the space $L_2^m(\mathbb{R}^s)$ is

$$||\mathfrak{F}a\mathfrak{F}^{-1}|| = ||a||_\infty. \tag{12}$$

Moreover, the norm of any linear bounded operator on $L_2^m(\mathbb{R}^s)$ satisfies the equation

$$||A||^2 = ||AA^*||. \tag{13}$$

Applying relation (13) to the operator $(R_a^M)^n$ and to the operator of multiplication by the matrix-function $\mathcal{A}_M^{(n)}$, and taking into account (10) and (12), one obtains

$$\begin{aligned} ||(R_a^M)^n||^2 &= ||(R_a^M)^n\,((R_a^M)^n)^*|| \\ &= |\det M|^{-n}||\mathcal{A}_M^{(n)}||_\infty \\ &= |\det M|^{-n}||\prod_{k=1}^n a_{M^k}||_\infty^2 \\ &= |\det M|^{-n}||\prod_{k=0}^{n-1} a((M^T)^k\cdot)||_\infty^2. \end{aligned}$$

This implies (6), since the spectral radius of any linear operator A can be represented in the form

$$\rho(A) = \lim_{n\to\infty} ||A^n||^{1/n}.$$

Relation (7) can be used to define and study refinement operators with general symbols from $L_\infty^{m\times m}(\mathbb{R})$. Then the symbol of the operator R_a^M considered earlier belongs to the class of almost periodic matrix-functions, whereas the operator T_a^M defined in the Introduction possesses a continuous symbol which vanishes at infinity.

In reference to the operator R_a^M below let us always assume that the sequence of the Fourier coefficients of the symbol a belongs to the space $l_1^{m\times m}(\mathbb{Z}^s)$.

COROLLARY 1.1. *Let a and M satisfy the assumptions of Theorem 1.1. Then*

$$\rho(R_a^M) \le \frac{1}{\sqrt{|\det M|}}||a||_\infty. \tag{14}$$

On the other hand, for any sequence $x_n \in \mathbb{R}^s$

$$\rho(R_a^M) \ge \frac{1}{\sqrt{|\det M|}} \limsup_{n\to\infty} ||\prod_{k=0}^{n-1} a((M^T)^k x_n)||_\infty^{1/n}. \tag{15}$$

COROLLARY 1.2. *Let a satisfy the assumptions of Theorem 1.1 and let $||a||_\infty = ||a(0)||_\infty$. If $a(0)$ is a normal matrix, then*

$$\rho(R_a^M) = \frac{||a(0)||_\infty}{\sqrt{|\det M|}}. \tag{16}$$

Proof. Inequality (15) implies that for any subsequence $\{n_j\} \subset \mathbb{N}$

$$\rho(R_a^M) \geq \frac{1}{\sqrt{|\det M|}} \lim_{n_j \to \infty} ||(a(0))^{n_j}||_\infty^{1/n_j}. \tag{17}$$

If $a(0)$ is a self-adjoint matrix, then

$$||(a(0))^2||_\infty = ||a(0)||_\infty^2$$

which yields

$$||(a(0))^{2^j}||_\infty = ||(a(0))||_\infty^{2^j}$$

for any $j \in \mathbb{N}$. Now let $a(0)$ be a normal matrix, when

$$||(a(0))^{2^j}||_\infty = ||(a(0))^{2^j}(a^*(0))^{2^j}||_\infty^{1/2} = ||(a(0)a^*(0))^{2^j}||_\infty^{1/2} = ||a(0)||_\infty^{2^j}.$$

Thus taking the subsequence $n_j = 2^j, j \in \mathbb{N}$, equality (16) follows from (14) and (17). □

Consider now the subdivision operator S_a^M. It is easily seen that the operator S_a^M is isometrically isomorphic to the operator $\widehat{S}_a^M : L_2^m(\mathbb{T}^s) \to L_2^m(\mathbb{T}^s)$,

$$\widehat{S}_a^M f(x) = a(x) f(M^T x), \quad t = e^{i2\pi x}, \quad x \in \mathbb{R}^s.$$

Let $q \geq 2$ denote the absolute value of the determinant of M. For a positive integer n, let $E_j^{(n)} + (M^T)^n \mathbb{Z}^s, j = 0, 1, \dots, q^n - 1$ be the distinct elements of the quotient space $\mathbb{Z}^s/(M^T)^n\mathbb{Z}^s$ such that $E_0^{(n)} = 0$. For $n = 1$ the corresponding elements $E_j^{(1)}$ are denoted by $E_j, j = 0, 1, \dots, q - 1$.

THEOREM 1.2. *Let $a \in L_\infty^{m\times m}(\mathbb{T}^s)$ and let M be a dilation matrix. Then*

$$\rho(S_a^M) = \frac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} || \sum_{j=0}^{q^n-1} \mathcal{A}_M^{(n)}(\cdot + E_j^{(n)})||_\infty^{1/2n} \tag{18}$$

where the matrix $\mathcal{A}_M^{(n)}$ is defined in (11).

Proof. Consider the operator $\widehat{T}_a^M : L_2^m(\mathbb{T}^s) \to L_2^m(\mathbb{T}^s)$ defined by

$$\widehat{T}_a^M g = \frac{1}{\sqrt{|\det M|}} \sum_{j=0}^{q-1} a_M(\cdot + E_j) g((M^T)^{-1}(\cdot + E_j)).$$

Then the adjoint for the operator $\widehat{T}_a^M$ has the form

$$(\widehat{T}_a^M)^* = \widehat{S}_{a^*}^M. \tag{19}$$

Relation (19) is widely used if $m = 1$ and $s \geq 1$. For $m > 1$ it can be derived from known results for the scalar case, and to obtain (19) one can exploit formula (3.5) of [**5**]. Taking into account (13) and (19) one obtains

$$||(\widehat{S}_a^M)^n||^2 = ||(\widehat{S}_{a^*}^M)^n||^2 = ||(\widehat{T}_a^M)^n(\widehat{S}_{a^*}^M)^n||.$$

Let $g \in L_2^m(\mathbb{T}^s)$. Then the function $(\widehat{T}_a^M)^n(\widehat{S}_{a^*}^M)^n g$ can be expressed in the form

$$
\begin{aligned}
&(\widehat{T}_a^M)^n(\widehat{S}_{a^*}^M)^n g(x) \\
&= \frac{1}{\sqrt{|\det M|}} \sum_{j_n=0}^{q-1} \sum_{j_{n-1}=0}^{q-1} \dots \sum_{j_1=0}^{q-1} \left(\prod_{k=1}^{n} a_{M^k}(x + \sum_{l=1}^{k} (M^T)^{l-1} E_{j_{n-l+1}}) \right. \\
&\qquad \left. \times \prod_{k=1}^{n} a^*_{M^{n-k+1}}(x + \sum_{l=1}^{n-k} (M^T)^l E_{j_{n-l+1}}) \right) g(x) \\
&= \frac{1}{\sqrt{|\det M|}} \sum_{j_n=0}^{q-1} \sum_{j_{n-1}=0}^{q-1} \dots \sum_{j_1=0}^{q-1} \left(\prod_{k=1}^{n} a_{M^k}(x + \sum_{l=1}^{n} (M^T)^{l-1} E_{j_{n-l+1}}) \right. \\
&\qquad \left. \times \prod_{k=1}^{n} a^*_{M^{n-k+1}}(x + \sum_{l=1}^{n} (M^T)^l E_{j_{n-l+1}}) \right) g(x) \\
&= \left(\frac{1}{\sqrt{|\det M|}} \sum_{j=0}^{q^n-1} \mathcal{A}_M^{(n)}(x + E_j^{(n)}) \right) g(x).
\end{aligned}
\tag{20}
$$

Note that the 1^s-periodicity of the matrix a is invoked, as well as the fact that the system

$$\mathcal{E} + M^T\mathcal{E} + \dots + (M^T)^{n-1}\mathcal{E}$$

where

$$\mathcal{E} := \{E_0, E_1, \dots, E_{q-1}\}$$

constitutes a basis in the coset space $\mathbb{Z}^s/(M^T)^n\mathbb{Z}^s$.

Equality (20) shows that $(\widehat{T}_a^M)^n(\widehat{S}_{a^*}^M)^n$ is just the operator of multiplication by the matrix $(1/|\det M|^n)\sum_{0=1}^{q^n-1} \mathcal{A}_M^{(n)}(x + E_j^{(n)})$. This implies formula (18). □

Corollary 1.3. *Let a and M satisfy the assumptions of Theorem 1.2. Then*

$$\rho(S_a^M) \le \lim_{n\to\infty} \|\prod_{k=0}^{n-1} a((M^T)^k \cdot)\|_\infty^{1/n}. \tag{21}$$

In addition, if $m = 1$ and the function a is continuous on $\mathbb{T}^s$, then for any sequence $x_n \in \mathbb{R}^s$

$$\rho(S_a^M) \ge \frac{1}{\sqrt{|\det M|}} \limsup_{n\to\infty} \|\prod_{k=0}^{n-1} a((M^T)^k x_n)\|_\infty^{1/n}. \tag{22}$$

2. *Estimates using point values of the symbol*

It is not an easy task to evaluate the spectral radii of the operators S_a^M and R_a^M using relations (6), (21), (22). However, for some classes of dilation matrices these formulas lead to estimates of $\rho(R_a^M)$ and $\rho(S_a^M)$ which do not contain a limit but only the spectral norms of certain constant matrices.

For any $a \in L_\infty^{m\times m}(\mathbb{T}^s)$ consider the quantity

$$S_n(a, M) := \|\prod_{k=0}^{n-1} a((M^T)^k \cdot)\|_\infty^{1/n}.$$

The following statement turns out to be useful in the problem under consideration. Its proof follows immediately from (5).

LEMMA 2.1. *Let $P = (p, p, \dots, p), p \in \mathbb{R}^+$ be a fixed vector and let*

$$\widetilde{a}_p(x) := a(Px), \quad x \in \mathbb{R}^s.$$

Then

$$S_n(a, M) = S_n(\widetilde{a}_p, M).$$

A consequence of this result is that while studying the limit in (6) one can always assume a to be a 1^s-periodic matrix-functions on $\mathbb{T}^s$, so henceforth the symbol a of the corresponding operators will be identified with matrix-functions on $\mathbb{R}^s/\mathbb{Z}^s$. Note that the 1^s-periodicity of the symbol a can be assumed from the very beginning, as was done earlier while studying the subdivision operator. Nevertheless, Lemma 2.1 allows the subdivision and refinement operators to be replaced by operators with the same spectral radii, and this property will be exploited below.

LEMMA 2.2. *Let $M \in \mathfrak{M}^s$. Then there exist numbers $\mu, q \in \mathbb{N}, \mu \geq 2$ and matrices $A_0, A_1, \dots, A_{q-1} \in \mathbb{Z}^{s\times s}$ such that for any number $n \in \mathbb{N}$ the matrix $(M^T)^n$ can be represented in the form*

$$(M^T)^n = A_r\,\mu^l I, \tag{23}$$

where $r, l \in \mathbb{N}$ are defined by the equation $n = lq + r, 0 \leq r \leq q - 1$.

Proof. By Schur's theorem [**20**], p.176, there exists a unitary matrix U and an upper triangular matrix T such that $M = UTU^*$. However, M is an isotropic dilation matrix, therefore

$$MM^* = \lambda I \tag{24}$$

with a $\lambda > 1$, and hence

$$TT^* = \lambda I.$$

Since the matrix T is upper triangular, the latter equation implies that T is a diagonal matrix with elements the eigenvalues of M. Let $\lambda_j = \lambda^{1/2} e^{i\pi(p_j/q_j)}, j = 1, 2, \dots, m$ be the eigenvalues of M, and let $u \in \mathbb{N}$ be the smallest number such that $\lambda^{u/2} \in \mathbb{N}$. Set

$$q := 2\,\mathrm{lcm}\,(q_1, q_2, \dots, q_m, u),$$

Then $\lambda^q \in \mathbb{N}$ and $qp_j/q_j = 2k_j$ with $k_j \in \mathbb{N}, j = 1, 2, \dots, m$. Set $\mu := \lambda^q$. Now one can use the representation $M = UTU^*$ to obtain

$$M^q = UT^qU^* = U\lambda^q\,\mathrm{diag}\,(e^{i2\pi k_1}, e^{i2\pi k_2}, \dots, e^{i2\pi k_m})U^* = \mu I,$$

so that writing $n \in \mathbb{N}$ in the form $n = ql + r, 0 \leq r \leq q - 1$ one has

$$M^n = M^{ql+r} = \mu^l M^r.$$

Thus, representation (23) is valid with $A_r = (M^T)^r, 0 \leq r \leq q - 1$, and the proof is complete. □

Let us now fix $\mu, p \in \mathbb{N}, \mu \geq 2$ and let $d_0, d_1, \ldots, d_{p-1} \in \mathbb{N} \cap [0, \mu-1]$. Consider a repeating fraction $x^{(p)} = 0.\overline{d_0, d_1, \ldots d_{p-1}}$ with the base μ, i.e.

$$x^{(p)} = \left(\frac{d_0}{\mu} + \frac{d_1}{\mu^2} + \ldots + \frac{d_{p-1}}{\mu^p}\right) + \left(\frac{d_0}{\mu^{p+1}} + \frac{d_1}{\mu^{p+2}} + \ldots + \frac{d_{p-1}}{\mu^{2p-1}}\right) + \ldots \quad .$$

Each such point may be associated with an ordered set $[x^{(p)}] = \{x_0^{(p)}, x_1^{(p)}, \ldots, x_{p-1}^{(p)}\}$ of p repeating fractions

$$\begin{aligned} x_0^{(p)} &= 0.\overline{d_0 d_1 \ldots d_{p-1}} \\ x_1^{(p)} &= 0.\overline{d_1 d_2 \ldots d_0} \\ \ldots & \;\;\ldots \\ x_{p-1}^{(p)} &= 0.\overline{d_{p-1} d_0 \ldots d_{p-2}}, \end{aligned}$$

with the base μ. The set $[x^{(p)}]$ is called μ-cyclic p-tuple corresponding to the point $x^{(p)}$ and is denoted by $\mathcal{C}_\mu^{(p)}$. The number p is referred to as the length of the corresponding tuple.

LEMMA 2.3. *Let a be a 1-periodic function and let $[x^{(p)}] = \{x_0^{(p)}, x_1^{(p)}, \ldots, x_{p-1}^{(p)}\} \in \mathcal{C}_\mu^{(p)}$. Then for any $l \in \mathbb{N}$ such that*

$$\begin{aligned} & l \equiv r \mod p \\ & 0 \leq r \leq p-1 \end{aligned}$$

one has

$$a(\mu^l x^{(p)}) = a(x_r^{(p)}).$$

The proof of this result is straightforward from the definition of $[x^{(p)}]$.

Now for any fixed number $p \in \mathbb{N}$ consider a system which consists of s of μ-cyclic p-tuples $[x_1^{(p)}], [x_2^{(p)}], \ldots, [x_s^{(p)}]$. Recall that

$$[x_j^{(p)}] = \{x_{j,0}^{(p)}, x_{j,1}^{(p)}, \ldots, x_{j,p-1}^{(p)}\}, \quad j = 1, 2, \ldots, s.$$

Combining the corresponding elements of these tuples, one can form vector-columns $\widetilde{x}_k^{(p)}, k = 0, 1, \ldots, p-1$ where

$$\begin{aligned} \widetilde{x}_0^{(p)} &= (\widetilde{x}_{1,0}^{(p)}, \widetilde{x}_{2,0}^{(p)}, \ldots, \widetilde{x}_{s,0}^{(p)})^T \\ \ldots&\ldots\ldots\ldots\ldots\ldots\ldots \quad . \\ \widetilde{x}_{p-1}^{(p)} &= (\widetilde{x}_{1,p-1}^{(p)}, \widetilde{x}_{2,p-1}^{(p)}, \ldots, \widetilde{x}_{s,p-1}^{(p)})^T \end{aligned}$$

Let $[\widetilde{x}^{(p)}]$ denote the family $\{\widetilde{x}_0^{(p)}, \widetilde{x}_1^{(p)}, \ldots, \widetilde{x}_{p-1}^{(p)}\}$, and the set of all such elements $[\widetilde{x}^{(p)}]$ is $\mathcal{C}_\mu^{(p),s}$. (Note that this notation differs from our previous agreement concerning the sets obtained via Cartesian products.) Let us also now always assume that a is a continuous matrix-function on $\mathbb{R}^s / \mathbb{Z}^s$, and proceed to estimate the spectral radius of the operator R_a^M. In passing, note that some of the results established also remain valid for the matrix-functions a of $L_\infty^{m\times m}(\mathbb{R}^s / \mathbb{Z}^s)$.

Let $M \in \mathfrak{M}^s$ be a dilation matrix and let $A_1, A_2, \ldots, A_{q-1}$ be the matrices introduced in Lemma 2.2. For any matrix-function a we define a matrix $b_{a,M}$ by

$$b_{a,M}(x) := a(x)a(A_1 x) \ldots a(A_{q-1}x).$$

The use of sequences generated by μ-cyclic p-tuples leads to the following general estimates for the spectral radius of the refinement operator.

THEOREM 2.1. *Let $a \in C^{m\times m}(\mathbb{R}^s/\mathbb{Z}^s)$. Then for any $l \in \mathbb{N}$ and for any $[\widetilde{x}^{(p)}] = \{\widetilde{x}_0^{(p)}, \widetilde{x}_1^{(p)}, \ldots, \widetilde{x}_{p-1}^{(p)}\} \in \mathcal{C}_\mu^{(p),s}$ the inequality*

$$\rho(R_a^M) \geq \frac{1}{\sqrt{|\det M|}} \limsup_{l\to\infty} ||(\prod_{r=0}^{p-1} b_{a,M}(\widetilde{x}_r^{(p)}))^l||_\infty^{1/lqp} \tag{25}$$

holds.

Proof. Fix $p \in \mathbb{N}$, and for the sequence $\{n_l\}_{l\in\mathbb{N}} = \{lpq\}_{l\in\mathbb{N}}$ set $x_{n_l} := \widetilde{x}_0^{(p)}$. Then by inequality (15) of Corollary 1.1 the spectral radius $\rho(R_a^M)$ satisfies the inequality

$$\rho(R_a^M) \geq \frac{1}{\sqrt{|\det M|}} \limsup_{l\to\infty} ||\prod_{j=0}^{n_l-1} a((M^T)^j x_{n_l})||_\infty^{1/n_l}.$$

Successive application of Lemmas 2.2 and 2.3 to the matrix $\prod_{j=0}^{n_l-1} a((M^T)^j x_{n_l})$ yields

$$\begin{aligned} &||\prod_{j=0}^{n_l-1} a((M^T)^j x_{n_l})||_\infty^{1/n_l} \\ &= ||\prod_{j=0}^{lp-1} b_{a,M}(\mu^j \widetilde{x}_0^{(p)})||_\infty^{1/lpq} \\ &= ||\prod_{j=0}^{l-1}\prod_{r=0}^{p-1} b_{a,M}(\widetilde{x}_r^{(p)})||_\infty^{1/lpq}, \end{aligned}$$

which implies inequality (25). □

Consider now some consequences of Theorem 2.1.

1) $m = 1, \quad s = 1$.

In this case, M is the operator of multiplication by an integer which implies that $q = 1, \mu = M$ if $M \geq 2$ and $q = 2, \mu = M^2$ if $M \leq -2$. Assume first that $M \geq 2$. Since all factors in the product (25) commute with each other, the following results are obtained (cf. also [**9**]):

COROLLARY 2.1. *Let $a : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be a continuous function and $M \geq 2$ be a positive integer. Then for any M-cyclic p-tuple $[x^{(p)}]$ the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geq \frac{1}{\sqrt{M}} \left(\prod_{r=0}^{p-1} |a(x_r^{(p)})|\right)^{1/p}. \tag{26}$$

Corollary 2.2. *Let $a : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be a continuous function and $M \geq 2$ be a positive integer. Then the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geq \frac{1}{\sqrt{M}} \sup_{p\in\mathbb{N}} \max_{[x^{(p)}]\in\mathcal{C}_\mu^{(p)}} \left(\prod_{r=0}^{p-1} |a(x_r^{(p)})| \right)^{1/p}.$$

An interesting result appears if one assumes that $M \leq -2$. Then $q = 2$ and $\mu = M^2$, so the estimates of the spectral radius of the operator R_a^M undergo a modification.

Corollary 2.3. *Let $a : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be a continuous function and $M \leq -2$ be a negative integer. Then for any M^2-cyclic p-tuple $[x^{(p)}]$ the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geq \frac{1}{\sqrt{|M|}} \left(\prod_{r=0}^{p-1} |a(x_r^{(p)})a(Mx_r^{(p)})| \right)^{1/2p}.$$

This result can be considered as a bridge between the multivariate case where matrix dilations with negative determinants are allowed and the univariate case, where so far only positive dilations has been considered. Moreover, it shows that representation (23) can be of interest even in the univariate case.

2) $m = 1, \quad s > 1.$

In this case the factors in (25) commute with each other, but the expression in the right-hand side of (25) becomes more complicated.

Corollary 2.4. *Let $a : \mathbb{R}^s/\mathbb{Z}^s \to \mathbb{C}$ be a continuous function and let $M \in \mathfrak{M}^s$. Then for any $[\widetilde{x}^{(p)}] \in \mathcal{C}_\mu^{(p),s}$ the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geq \frac{1}{\sqrt{|\det M|}} \left(\prod_{j=0}^{q-1} \prod_{r=0}^{p-1} |a(A_j\widetilde{x}_r^{(p)})| \right)^{1/qp} \tag{27}$$

where the matrices $A_0, A_1, \ldots, A_{q-1}$ are defined in Lemma 2.2

3) $m > 1, \quad s > 1.$

In this case the factors in (25) generally do not commute, but a simplification of relation (25) is sometimes possible.

Corollary 2.5. *Let $a : \mathbb{R}^s/\mathbb{Z}^s \to \mathbb{C}^{m\times m}$ be a continuous matrix-function and let $M \in \mathfrak{M}^s$. If there exists $[\widetilde{x}^{(p)}] \in \mathcal{C}_\mu^{(p),s}$ such that $\prod_{r=0}^{p-1} b_{a,M}(\widetilde{x}_r^{(p)})$ is a normal matrix, then the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geq \frac{1}{\sqrt{|\det M|}} || \prod_{r=0}^{p-1} b_{a,M}(\widetilde{x}_r^{(p)}) ||_\infty^{1/qp}. \tag{28}$$

The proof of inequality (28) mainly follows the arguments used in the proof of Corollary 1.2.

As we have seen, it is possible to get lower estimates for the spectral radius of the operator R_a^M by using the multiplicator norm of certain constant matrices.

However, from practical point of view it is preferable to work with supremum norms of functions instead of matrix multiplicator norms, and a simplification can be achieved in some cases where matrix a has special properties. For example, assume that a is unitary equivalent to a functional diagonal matrix, i.e. there exists a constant unitary matrix U such that $a(x) = Ud(x)U^*, x \in \mathbb{R}^s$ where $d(x) = \operatorname{diag}(\lambda_1(x), \lambda_2(x), \ldots, \lambda_s(x))$. Moreover, let us also assume that the matrix a possesses a dominant eigenvalue, i.e. an eigenvalue λ_{j_0} such that

$$|\lambda_{j_0}(x)| \geq |\lambda_j(x)|, \quad j \in \{1, 2, \ldots, m\}$$

for all $x \in \mathbb{R}^s$. In this case the spectral radius of R_a^M can be estimated by using values of the function λ_0 only, since under this condition, the matrices $a((M^T)^j \cdot)$ and $a((M^T)^k \cdot)$ commute for any non-negative integers j, k. Therefore for any $[\widetilde{x}^{(p)}] \in \mathcal{C}_\mu^{(p),s}$ the spectral radius of the operator R_a^M satisfies the inequality

$$\rho(R_a^M) \geq \frac{1}{\sqrt{|\det M|}} \left(\prod_{r=0}^{p-1} |\Lambda_{a,M}(\widetilde{x}_r^{(p)})| \right)^{1/qp} \tag{29}$$

where $\Lambda_{a,M}(x) = \lambda_{j_0}(x)\lambda_{j_0}(A_1 x) \ldots \lambda_{j_0}(A_{q-1}x)$.

Taking into account Corollary 1.3 one can get lower bounds for the spectral radius of the subdivision operator S_a^M which are analogous to the estimates for $\rho(R_a^M)$ established above.

3. *Integral estimates for* $\rho(R_a^M)$

Let the function $\ln|a| : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be Lebesgue integrable on $[0, 1]$ and let $M \geq 2$. Then the spectral radius of the subdivision operator S_a^M can be estimated as

$$\rho(S_a^M) \geq \exp\left(\int_0^1 \ln|a(x)| dx \right). \tag{30}$$

The proof of inequality (30) given in [**16**] essentially uses the Birkhoff ergodic theorem.

For the multivariate case $s > 1$, inequalities (26), (27)–(29) are a source of variety of new integral estimates for the spectral radii of the operators R_a^M and S_a^M, and some of these inequalities are established below. However, none of the proofs below invokes ergodic theorems.

From now on it is always assumed that $m = 1$, and first consider estimates of the spectral radius of the refinement operator R_a^M which use integrals over one-dimensional subsets. Let

$$E := \{x = (x_1, x_2, \ldots, x_s) \in [0, 1]^s : x_1 = x_2 = \ldots = x_s\},$$

be the main diagonal of the unit cube $[0, 1]^s$.

THEOREM 3.1. *Let $M \in \mathfrak{M}^s$ and let $a : \mathbb{R}^s/\mathbb{Z}^s \to \mathbb{C}$ be a continuous function such that the function $ln|a(\cdot)|$ is Lebesgue integrable on all subsets $E_j := (M^T)^j E, j = 0, 1, \ldots, q-1$. Then the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geq \frac{1}{\sqrt{|\det M|}} \exp\left(\frac{1}{q} \sum_{j=0}^{q-1} \int_0^1 \ln|a((M^T)^j x)| dx \right). \tag{31}$$

Proof. The proof of this Theorem and subsequent integral inequalities is based on the properties† of the sets $\mathcal{C}_\mu^{(p)}$. For convenience, let us now recall some of the results concerned.

1. Identifying the set $\mathcal{C}_\mu^{(p)}$ with the decimal representations of the real numbers comprising all the subsets $[x^{(p)}]$, then

$$\mathcal{C}_\mu^{(p)} = \left\{0, \frac{1}{\mu^p - 1}, \frac{2}{\mu^p - 1}, \ldots, \frac{\mu^p - 2}{\mu^p - 1}\right\}.$$

2. Let $[x^{(p)}], [y^{(p)}] \in \mathcal{C}_\mu^{(p)}$. If there are $j_0, r_0 \in \{0, 1, \ldots, p-1\}$ such that $x_{j_0}^{(p)} = y_{r_0}^{(p)}$, then the sets $[x^{(p)}]$ and $[y^{(p)}]$ coincide.
3. Let $p \in \mathbb{N}$ be a prime number. If there exist $j_0, r_0 \in \{0, 1, \ldots, p-1\}$ such that $x_{j_0}^{(p)} = x_{r_0}^{(p)}$ but $j_0 \neq r_0$, then

$$x^{(p)} = 0.\overline{d}$$

where $0 \leq d \leq \mu - 1$.

Let us first assume that the function $\ln |a(\cdot)|$ is Riemann integrable on the subsets $E_j, j = 0, 1, \ldots, q-1$. For a fixed prime number p, let us choose a μ-cyclic p-tuple $[x^{(p)}]$, and for any $j \in \{0, 1, \ldots, p-1\}$ consider the column $\widetilde{x}_j^{(p)} \in [0, 1)^s$ all s components of which are $x_j^{(p)}$. Thus any p-cyclic μ-tuple produces p vectors with equal components.

By the little Fermat theorem

$$\mu^p \equiv \mu \mod p.$$

Therefore there exists $l \in \mathbb{N}$ such that the difference

$$\mu^p - 1 = lp + (\mu - 1). \tag{32}$$

Taking into account the properties of μ-cyclic p-tuples, equality (32) can be viewed as follows:

1. $\mu - 1$ is the number of those distinct points in $\mathcal{C}_\mu^{(p)}$ which have the representation $0.\overline{d}, 0 \leq d \leq \mu - 2$.
2. l is the number of those μ-cyclic p-tuples $[x^{(p)}]$ which consist of entirely distinct elements $x_j^{(p)}, j = 0, 1, \ldots, p-1$.
3. $\mu^p - 1$ is the total number of points in $\mathcal{C}_\mu^{(p)}$.

Consider now inequality (27) for the tuples $[x_{(k)}^{(p)}], k = 1, 2, \ldots, l$ under 2. For a fixed k, the elements of such tuple are denoted by $x_{(k),r}^{(p)}, r = 0, 1, \ldots, p-1$. If one takes the logarithm from both sides of (27) and sum up all resulting inequalities, then

$$\frac{1}{p} \sum_{k=1}^{l} \sum_{r=0}^{p-1} \left(\frac{1}{q} \sum_{j=0}^{q-1} \ln |a(A_j \widetilde{x}_{(k),r}^{(p)})| \right) \leq l \ln(\sqrt{|\det M|} \rho(R_a^M)). \tag{33}$$

† See [**9**] for details

Recall that all components of any column $\widetilde{x}^{(p)}_{(k),r}$ coincide. If one divides inequality (33) by l and uses (32) then this inequality can we rewritten as

$$\frac{1}{pl}\sum_{m=0}^{\mu^p-1}\left(\frac{1}{q}\sum_{j=0}^{q-1}\ln|a(A_jt_m)|\right)-\frac{1}{pl}\sum_{m=0}^{\mu-2}\left(\frac{1}{q}\sum_{j=0}^{q-1}\ln|a(A_ju_m)|\right)$$
$$\le \ln(\sqrt{|\det M|}\,\rho(R_a^M))$$

or as

$$\begin{aligned}&\frac{\mu^p-1}{pl}\left[\frac{1}{\mu^p-1}\sum_{m=0}^{\mu^p-1}\left(\frac{1}{q}\sum_{j=0}^{q-1}\ln|a(A_jt_m)|\right)\right]\\&-\frac{1}{pl}\sum_{m=0}^{\mu-2}\left(\frac{1}{q}\sum_{j=0}^{q-1}\ln|a(A_ju_m)|\right)\le \ln(\sqrt{|\det M|}\,\rho(R_a^M))\end{aligned}\tag{34}$$

where $t_m, u_m \in E$,

$$t_m := \left(\frac{m}{\mu^p-1}, \frac{m}{\mu^p-1}, \dots, \frac{m}{\mu^p-1}\right)^T, \quad m = 0, 1, \dots, \mu^p-2,$$

and

$$u_m := \left(0.\overline{d}, 0.\overline{d}, \dots, 0.\overline{d}\right)^T, \quad d = 0, 1, \dots, \mu-2.$$

It remains to note that

$$\frac{1}{\mu^p-1}\sum_{m=0}^{\mu^p-1}\left(\frac{1}{q}\sum_{j=0}^{q-1}\ln|a(A_jt_m)|\right)$$

is a Riemann sum corresponding to the uniform partition of the interval $[0, 1]$ for the function

$$F(x) := \frac{1}{q}\sum_{j=0}^{q-1}\ln|a(A_jx)|, \quad x \in E.$$

Thus if p tends to infinity over the set of prime numbers, the first sum in the left-hand side of (34) tends to the integral in (31), while the second sum in (34) tends to zero. Extension to Lebesgue integrable symbols can be achieved as in [**9**], and is omitted here. □

Note that (31) is not the only possible lower estimate for $\rho(R_a^M)$ in terms of one-dimensional integrals. Other lower bounds based on one-dimensional integrals can be obtained by changing the corresponding vector-tuples appearing in the proof of Theorem 3.1. For example, let $k_1, k_2, \dots, k_s$ be positive integers and let D_K^μ denote the diagonal matrix

$$D_K^\mu = \operatorname{diag}(\mu^{k_1}, \mu^{k_2}, \dots, \mu^{k_s}),$$

when the following corollary provides another option for obtaining a lower estimate of the spectral radius $\rho(R_a^M)$ via one-dimensional integrals.

COROLLARY 3.1. *Let $M \in \mathfrak{M}^s$ and let $a : \mathbb{R}^s/\mathbb{Z}^s \to \mathbb{C}$ be a continuous function such that the function $ln|a(\cdot)|$ is Lebesgue integrable on all subsets $E_j^D := (M^T)^j D_K^\mu E, j = 0, 1, \ldots, q-1$. Then the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geq \frac{1}{\sqrt{|\det M|}} \exp\left(\frac{1}{q}\sum_{j=0}^{q-1}\int_0^1 \ln|a((M^T)^j D_K^\mu x)|dx\right). \tag{35}$$

Proof. The proof is closely related to that of Theorem 3.1. One only has to choose $p > \max(k_1, k_2, \ldots, k_s)$ and replace the vectors $\widetilde{x}_j^{(p)}$ with the equal components $x_j^{(p)}, j = 0, 1, \ldots, p-1$ by vectors with shifted components, i.e. by $\widehat{x}_j^{(p)} = \{x_{j+k_1}^{(p)}, x_{j+k_2}^{(p)}, \ldots, x_{j+k_s}^{(p)}\}$. Then the corresponding sum, obtained as the first sum in (34), is a Riemann sum for the function

$$F(x) := \frac{1}{q}\sum_{j=0}^{q-1} \ln|a(A_j D_K^\mu x)|, \quad x \in E.$$

Other steps in the proof are quite analogous to the proof of Theorem 3.1. □

In addition to the estimates of $\rho(R_a^M)$ represented by one-dimensional integrals, there exists a variety of estimates given by integrals of higher dimensions $l, 1 < l \leq s$. These lower bounds can be obtained in the same way as estimates (31), (35). For example, let us formulate a result similar to Theorem 3.1. Thus consider a partition Π of the set of the positive integers $I := \{1, 2, \ldots, s\}$ into l subsets $I_1, I_2, \ldots, I_l$ such that $I = \cup_{k=1}^l I_k$, and $I_{k_1} \cap I_{k_2} = \emptyset$ if $k_1 \neq k_2$. For each such partition Π let E_Π denote the subset of $[0,1]^s$ defined by

$$E_\Pi := \{(x_1, x_2, \ldots, x_s) \in [0,1]^s : x_{i_1} = x_{i_2} \quad \text{if} \quad i_1, i_2 \in I_k, k = 1, 2, \ldots, l\}.$$

In particular, if Π consists only of one set I, then $E_\Pi = E$ is the main diagonal of the cube $[0,1]^s$.

THEOREM 3.2. *Let $M \in \mathfrak{M}^s$ and let $a : \mathbb{R}^s/\mathbb{Z}^s \to \mathbb{C}$ be a continuous function such that the function $ln|a(\cdot)|$ is Lebesgue integrable on all subsets $E_j^\Pi := (M^T)^j E_\Pi$, $j = 0, 1, \ldots, q-1$. Then the spectral radius of the operator S_a^M satisfies the inequality*

$$\rho(R_a^M) \geq \frac{1}{\sqrt{|\det M|}} \exp\left(\frac{1}{q}\sum_{j=0}^{q-1}\int_{E_\Pi} \ln|a((M^T)^j x)|dx\right).$$

One can again use the same approach to prove this result, by choosing vector-tuples with equal coordinates for the variables x_j whose numbers j belong to the same group I_k of the partition Π. Note that the use of shifted vector-tuples leads to multi-dimensional integral estimates similar to inequality (35).

It is worth noting that none of the integral estimates above would give the best approximation of the spectral radius for all symbols a and for all matrices $M \in \mathfrak{M}^s$. Moreover, for each integral estimate above and for each matrix $M \in \mathfrak{M}^s$ one can find a symbol a such that the estimate used delivers the exact value of the spectral radius $\rho(R_a^M)$. For example, let A denote the maximum of the function $|a|$ on the cube $[0,1]^s$. Then immediately from Theorem 3.1 one can extract the following result.

COROLLARY 3.2. *Let $M \in \mathfrak{M}^s$ and let $a : \mathbb{R}^s/\mathbb{Z}^s \to \mathbb{C}$ be a continuous function such that $|a(x)| = A$ for all $x \in (M^T)^j E, j = 0, 1, \dots, q-1$. Then*

$$\rho(R_a^M) = \frac{A}{\sqrt{|\det M|}}. \tag{36}$$

Proof. The lower bound

$$\rho(R_a^M) \geq \frac{A}{\sqrt{|\det M|}}$$

for the spectral radius of R_a^M follows from Theorem 3.1. Comparing this estimate with (14) one obtains (36). □

Integral estimates of the spectral radius for the subdivision operator S_a^M have the same form as for the operator R_a^M, though in this case they usually give a less precise approximation for $\rho(S_a^M)$ than for $\rho(R_a^M)$.

4. *Exact values of the spectral radii for refinement operators*

Corollary 3.2 shows that for some classes of symbols the integral estimates from the previous paragraph can provide exact value of the spectral radius $\rho(R_a^M)$. Point value estimates for $\rho(R_a^M)$ are now considered. As before let A denote the maximum of the function $|a|$, and let $\mathcal{M}(a)$ be the set of points where $|a|$ attains its maximum.

THEOREM 4.1. *Let $a \in C(\mathbb{R}^s/\mathbb{Z}^s)$ and let $M \in \mathfrak{M}^s$. If there exists an $[\widetilde{x}^{(p)}] = \{\widetilde{x}_0^{(p)}, \widetilde{x}_1^{(p)}, \dots, \widetilde{x}_{p-1}^{(p)}\} \in \mathcal{C}_\mu^{(p),s}$ such that for all $k = 0, 1, \dots, q-1$ and for all $j = 0, 1, \dots, p-1$*

$$(M^T)^k \widetilde{x}_j^{(p)} \in \mathcal{M}(a), \tag{37}$$

then

$$\rho(R_a^M) = \frac{A}{\sqrt{|\det M|}} \tag{38}$$

and

$$\frac{A}{\sqrt{|\det M|}} \leq \rho(S_a^M) \leq A. \tag{39}$$

Proof. Equality (37) follows from (14) and (27). For inequality (39) one has to invoke relations (21), (22) and the proof of Theorem 2.1 where the limit $\limsup_{n\to\infty} || \prod_{k=0}^{n-1} a((M^T)^k x_n)||_\infty^{1/n}$ has been estimated.

Note that the estimates in (39) are sharp. The upper bound is achieved for any constant symbol a whereas the lower bound appears in examples considered in [**11**], [**13**]. □

COROLLARY 4.1. *Let the symbol a of the refinement operator R_a^M have non-negative Fourier coefficients. Then*

$$\rho(R_a^M) = \frac{|a(0)|}{\sqrt{|\det M|}}.$$

Proof. If the Fourier coefficients of a are non-negative, then

$$|a(x)| \leq \sum_{k \in \mathbb{Z}^s} a_k = |a(0)|$$

and the results follows from the fact that 0 is a μ-cyclic 0-tuple. □

To study a more general situation, let us fix an $\varepsilon > 0$ and consider the set

$$E_a^\varepsilon := \{x \in [0,1)^s : |a(x)| \geq A - \varepsilon\},$$

where A means the modulus maximum of the function a on $[0,1]^s$ as before. Having defined the set E_a^ε, for each positive integer p let us introduce a set $\mathcal{N}_p^M(E_a^\varepsilon)$ by

$$\mathcal{N}_p^M(E_a^\varepsilon) := \{j \in \mathbb{Z}^s : j/(\mu^p - 1) \in E_a^\varepsilon\},$$

and recall that any μ-cyclic p-tuple $[x^{(p)}] = \{x_0^{(p)}, x_1^{(p)}\}, \ldots, x_{p-1}^{(p)}\}\} \in \mathcal{C}_\mu^{(p)}$ can be represented as

$$[x^{(p)}] = \left\{ \frac{r_0}{\mu^p - 1}, \frac{r_1}{\mu^p - 1}, \ldots, \frac{r_{p-1}}{\mu^p - 1} \right\} \tag{40}$$

where $r_0, r_1, \ldots, r_{p-1}$ are non-negative integers which do not exceed $\mu^p - 2$. Moreover, the integer r_0 defines the number $r_1, r_2, \ldots, r_{p-1}$ and their order. Thus, if a column of s non-negative integers $\mathbf{r} = (r_0^{(1)}, r_0^{(2)}, \ldots, r_0^{(s)})^T, 0 \leq r_0^{(i)} \leq \mu^p - 2$ is given, then p columns $\mathbf{r}_l, l = 0, 1, \ldots, p-1$ consisting of non-negative integers which appear in the representations (40) of the corresponding μ-cyclic p-tuples, are defined – viz. $\mathbf{r}_0 := \mathbf{r}$ and $\mathbf{r}_l := (r_l^{(1)}), r_l^{(2)}), \ldots, r_l^{(s)})^T, l = 1, 2, \ldots, p-1$. Recall that $0 \leq r_l^{(i)} \leq \mu^p - 2$ for all $l = 0, 1, \ldots, p-1$ and for all $i = 1, 2, \ldots, s$. For convenience, the set of all columns $\mathbf{r}$ satisfying the conditions mentioned is denoted by $\mathcal{R}$.

Consider now a system of congruences

$$\begin{array}{l} (M^T)^l \mathbf{r}_k u = \mathbf{n}_{lk} \mod \mu^p - 1 \\ l = 0, 1, \ldots, q-1;\ k = 0, 1, \ldots, p-1 \end{array} \tag{41}$$

where $u \in \mathbb{N}$ is an unknown positive integer and $\mathbf{n}_{lk} = (n_{lk}^{(1)}, n_{lk}^{(2)}, \ldots, n_{lk}^{(s)})^T$ with $\mathbf{n}_{lk} \in (\mathcal{N}_p^M(E_a^\varepsilon))^s$ for all $l = 0, 1, \ldots, p-1$ and $k = 0, 1, \ldots, p-1$.

THEOREM 4.2. *Let $a \in C(\mathbb{R}^s / \mathbb{Z}^s), M \in \mathfrak{M}^s$ and μ, q, ε be as above. If there exists $p \in \mathbb{N}$ such that system (41) is solvable for at least one $\boldsymbol{r} \in \mathcal{R}$ and for at least one choice of right-hand sides $\boldsymbol{n}_{lk} \in (\mathcal{N}_p^M(E_a^\varepsilon))^s$, then the spectral radii of the operators S_a^M and R_a^M satisfy the inequalities*

$$\rho(S_a^M) \geq \frac{A}{\sqrt{|\det M|}} - \varepsilon, \quad \rho(R_a^M) \geq \frac{A}{\sqrt{|\det M|}} - \varepsilon. \tag{42}$$

Proof. Let $\mathbf{n}^*_{lk}, l = 0,1,\dots,p-1; k = 0,1,\dots,p-1$ and $\mathbf{r}^*$ be right-hand sides from $\mathcal{N}_p^M(E_a^\varepsilon)$ and the corresponding columns of non-negative numbers such that system of congruences (41) is solvable. For any solution u^* of such system define a function a_{u^*} by

$$a_{u^*} := a(u^* \cdot).$$

Then a_{u^*} is a 1^s-periodic function, and Lemma 2.1 implies

$$\rho(R_a^M) = \rho(R_{a_{u^*}}^M).$$

However, by Corollary 2.4 the spectral radius of the operator $R_{a_{u^*}}^M$ can be estimated as follows

$$\begin{aligned} \rho(R_{a_{u^*}}^M) &\geq \frac{1}{\sqrt{|\det M|}} \left(\prod_{l=0}^{q-1} \prod_{k=0}^{p-1} \left| a\left(\frac{u^*(M^T)^j \mathbf{r}_k}{\mu^p - 1} \right) \right| \right)^{1/qp} \\ &\geq \frac{1}{\sqrt{|\det M|}} \left(\prod_{l=0}^{q-1} \prod_{k=0}^{p-1} \left| a\left(\frac{\mathbf{n}_{lk}}{\mu^p - 1} \right) \right| \right)^{1/qp} \\ &\geq \frac{A}{\sqrt{|\det M|}} - \varepsilon. \end{aligned}$$

This completes the proof for the operator R_a^M. The estimates for the spectral radius of the operator S_a^M can be obtained similarly. □

Combining the last result with Corollary 1.1 leads to another sufficient condition for the equality (4).

THEOREM 4.3. *Let $a \in C(\mathbb{R}^s/\mathbb{Z}^s)$ and $M \in \mathfrak{M}^s$. If for any $\varepsilon > 0$ system (41) is solvable in the sense of Theorem 4.2, then*

$$\rho(R_a^M) = \frac{A}{\sqrt{|\det M|}}.$$

Thus, knowledge of the maximum of the symbol often allows us to obtain the exact value of the spectral radius for the refinement operator. A natural question to ask is whether the spectral radius of the refinement operator can always be given by formula (4). The answer to this question is no. A counterexample can be constructed using results of [**31**].

As far as the subdivision operators S_a^M are concerned, the location of exact values of their spectral radii is more complicated, cf. (39) and the remark in the proof of Theorem 4.1. Thus to improve estimate (39) one has to study expression (18) in more detail.

Acknowledgement. The work was partially done while the author was visiting the Institute for Mathematical Sciences, National University of Singapore in 2003. The visit was supported by the Institute.

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