Wavelet Oriented Anisotropic Diffusion in Image Enhancement

Jianzhong Wang[1]
Department of Mathematics and Statistics
Sam Houston State University, Huntsville, TX 77341
(mth_jxw@shsu.edu)

March 2004

[1]Research was supported by EGR Grant-2003 at Sam Houston State University, and by CIM of National University of Singapore when the author visited there in December 2003.

Abstract

In this paper, we introduce a new anisotropic diffusion partial differential equation, in which the gradients is adjusted by the wavelet transform of the solution. The new model adopts wavelet transform theory so that the numerical implementation is more effective and accurate. The existence and uniqueness of the solutions are proved. The equation can formulate image enhancement and the numerical implementation shows that the new model gives a better result in image noise removal and edge detection.

0.1 Introduction

In the past decade, the use of partial differential equations (PDE) in image processing became a raising research area. This approach models image in a continuous spatial domain so that it takes the advantages of effective treatments from the PDE's theory and obtains high accuracy and stability of the processing with help of the rich results of numerical analysis. One of most influential work in this aspect is the anisotropic diffusion model introduced by Perona and Malik in 1990 [28]. Assume a target image $\hat{u}$ is destroyed (by noise, blurring, or other reasons). Then only a destroyed image, say u_0, is observed. To recover $\hat{u}$ from u_0 , Perona and Malik proposed a directional diffusion that preserves edges. They formulated the processing to solving the following nonlinear anisotropic diffusion equation (P-M model):

$$\frac{\partial u}{\partial t} = \nabla \cdot (c(|\nabla u|)\nabla u), \quad x \in \Omega(\subset \mathbb{R}^2), \tag{1}$$

$$u(x,0) = u_0(x),$$

$$\frac{\partial u}{\partial \vec{n}} = 0, \quad x \in \partial\Omega, \tag{2}$$

where $c(p)$ is a positive decreasing function with $\lim_{s\to\infty} c(s) = 0$. The initial function $u_0(x)$ standards for the observed image. The model tries to recover the target image $\hat{u}$ after a certain time duration $\hat{t}$. That is, the solution of the equation (1) at time $\hat{t}$, say $u(x,\hat{t})$, provides an approximation of $\hat{u}$. The strategy of the anisotropic diffusion process is to force strong forward diffusion to occur in the regions where the image is smooth while to force backward diffusion to be created in the regions where edges are present. Since this kind of anisotropic diffusion avoids blurring edges in the diffusion processing, it results images that gradually produce cleaning edges of the target image in the t-spaces.

Several important models in image processing have been derived from the P-M model (1). For example, if we select $(pc(p))' = 0$, i.e., $c(p) = \frac{1}{p}, p = |\nabla u|,$. then the P-M model (1) becomes

$$\frac{\partial u}{\partial t} = \mathrm{div}(\frac{\nabla u}{|\nabla u|}), \tag{3}$$

which leads to the variational problem

$$\min E(u),$$

where the energy functional

$$E(u) = \int_\Omega |\nabla u| dxdy \tag{4a}$$

is the total variation of the image u ([7], [8], [9], [10], and [34]).

Adding a motion control term to the diffusion term in (3), we obtain the mean curvature motion equation ([14], [25]):

$$\frac{\partial u}{\partial t} = |\nabla u| \operatorname{div}(\frac{\nabla u}{|\nabla u|}), \tag{5}$$

which provides the geodesic active contour model [3], [27], [30], [31] , and links to the level-set method [15], [24], [29].

However, the P-M model meets several serious practical and theoretical difficulties [16]. The first difficulty is that it is very sensitive to noise. Assume an image carries strong noise. The P-M model will conserve the noise in the processing. Another difficulty arises from the existence of the local backward diffusion in the area where $(pc(p))' < 0$. There is no existent theory supports the uniqueness of the solutions of the equation (1). Examples show that (1) is unstable in the sense that very close images could produce divergent solutions [37]. To overcome the instability of P-M model, Catte, Lions, Morel, and Coll [6], introduced a regularization of the model. They substitute ∇u inside the divergence term by its smooth version $G_\sigma * \nabla u$ (called the regularization of ∇u), where

$$G_\sigma(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{|x|^2}{2\sigma^2}}, \quad x \in \mathbb{R}^2.$$

is the Gaussian kernel. Thus, the regularized model they proposed is

$$\begin{aligned} \frac{\partial u}{\partial t} &= \operatorname{div}(c(|\nabla (G_\sigma * u)|)\nabla u), \\ u(0,x) &= u_0(x), \quad x \in \bar{\Omega}, \\ \frac{\partial u}{\partial \vec{n}} &= 0, \quad x \in \partial\Omega, t \in [0,\infty). \end{aligned} \tag{6}$$

In this model, the Gaussian filter reduces the influence of noise on the image u in processing. Meanwhile, this slight change in (6) also avoid the inconsistency caused by P-M model. Other methods of regularization have been also adopted. Torkamani-Azar and Tait (1996) [32] suggested using $G^1_\sigma(x) = \frac{1}{c_\sigma} e^{-\frac{|x_1|+|x_2|}{2\sigma^2}}$ instead of G_σ in (**??**), Acton (1998) [1] proposed a more complicate replacement $(\nabla u \circ \zeta) \cdot \zeta$ for ∇u, where $v \to v \circ \zeta$ is the morphological opening operator and $v \to v \cdot \zeta$ is the morphological closing operator.

All of these models try to regularize ∇u to reduce the influence of noise. The effectiveness of a regularization depends on the type of noise on the image. For instance, if the noise does not obey Gaussian distribution, then $G_\sigma * \nabla u$ does not provide a good regulation. The motivation of this paper is to introduce a new regularization of ∇u oriented by wavelet transforms. It provides a more efficient and accurate formulation for edge-preserving diffusion. The paper is organized as follows. In Section 2, we introduce the wavelet regularization of P-M model. In Section 3, we discuss the consistency of the new regularized model. In Section 4, we develop an adaptive numerical method for its solutions. Finally, we present some examples in Section 5.

0.2 Wavelet Regularization of Anisotropic Diffusion

As we discussed in the previous section, if an image $\hat{u}$ is destroyed by noise so that only the noisy image u is observed. Therefore, the gradient ∇u diverges from $\nabla \hat{u}$. To obtain a good edge-preserving diffusion model for the image processing, we need to regularize ∇u. Wavelet theory provides a good framework for the regularization. As usual, a vector-valued function $\psi = (\psi_1(x), \cdots, \psi_l(x))^T \in L^2 \cap L^1(\mathbb{R}^2)$ is called a wavelet if

$$\int_{\mathbb{R}^2} \psi(x)dx = 0, \quad x = (x_1, x_2)^T.$$

The continuous wavelet transform of u with respect to ψ in $\mathbb{R}^2$ is defined by

$$W_\psi^s u(x) := W_\psi(u, x, s) = \int_{\mathbb{R}^2} \psi_s(v - x)u(v)dv = (\psi_s * u)(x), \quad s > 0,$$

where $\psi_s(\cdot) = \frac{1}{s^2}\psi(\frac{\cdot}{s})$. (See [11], [13], [18], [19].) To use wavelet transform of u to estimate $\nabla \hat{u}$, we construct the wavelet ψ as follows. Let $\phi_1, \phi_2 \subset L^1 \cap C(\mathbb{R})$ be two one-dimensional non-negative r-regular ($r \geq 1$) functions [22], which exponentially decay and satisfy $||\phi_1||_{L^1} = ||\phi_2||_{L^1} = 1$. Defining $\phi(x) = \phi_1(x_1)\phi_2(x_2)$, we have

$$\lim_{s \to 0^+} ||\frac{1}{s}\int_{\mathbb{R}^2} \phi(\frac{\cdot - y}{s})u(y)dy - u(\cdot)||_{L^p} = 0, \quad u \in L^p(\mathbb{R}^2), 1 \leq p < \infty,$$
$$\lim_{s \to 0^+} ||\frac{1}{s}\int_{\mathbb{R}^2} \phi(\frac{\cdot - y}{s})u(y)dy - u(\cdot)||_{C} = 0, \quad u \in C(\mathbb{R}^2).$$

If u is also differentiable, then

$$\lim_{s \to 0^+} \frac{1}{s^2}\int_{\mathbb{R}^2} (\nabla\phi)(\frac{x - y}{s})u(y)dy - \nabla u(x) = 0, \quad x \in \mathbb{R}^2, \tag{7}$$

which yields, by setting $\psi = \nabla\phi$,

$$\lim_{s\to 0^+} W_{\psi,s}u(x) = \nabla u(x). \tag{8}$$

In image processing, the image function u is usually not differentiable everywhere. Hence, $W_{\psi.s}u(x)$ provides a regularization of ∇u. Thus, we obtain a wavelet regularization of the anisotropic diffusion model (1).

- **Wavelet regularization of Type 1:**

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mathrm{div}(c(|W_{\psi,s}(u)|)\nabla u), \qquad (9)\\ u(0,x) &= u_0(x), \quad x \in \bar{\Omega},\\ \frac{\partial u}{\partial \vec{n}} &= 0, \quad x \in \partial\Omega, t \in [0,\infty),\end{aligned}$$

where s is a fixed scale.

Remark 1 *If ∇u does not exist everywhere, the equation (9) has to be understood in a generalized sense. That is, we shall seek for the viscosity solutions of the equation (9). The details in this aspect are referred to [**?**] and [**?**].*

The model (9) generalizes the equation (6) in the sense that, if we choose $\phi = e^{-|x|^2}$ and $s = \sigma$, then (9) becomes (6). The new model enables us to select the regularization kernels for u in a wide scope. However, the regularization of this type is uniform on the whole spatial domain. If the noise does not evenly distribute on the whole image region, the regularization does not model the desired procession well. To obtain a finer regularization, we need to model the procession based on a local analysis of the observed images, which will enable us to distinguish the edge and noise. There are several ways to do local analysis. In this paper, we adopt the method suggested in [20] and [21]. As [20] and [21] pointed out, the local Lipschitz regularity of u provides a characterization of edge as well as of noise. According to [21], the Lipschitz regularity of a function f at a point x_0 is defined as follows.

Definition 2 *Let $0 \le \alpha < 1$. A function $f(x)$ is called uniformly Lipschitz α over an open set $O \subset \mathbb{R}^2$, denoted by $f \in Lip_\alpha(O)$, if there exists a constant K such that for all $x_1, x_2 \in O$,*

$$|f(x_1) - f(x_2)| \le K|x_1 - x_2|^\alpha.$$

The superior bound α of all $\bar{\alpha}$ such that $f(x)$ is uniformly Lipschitz $\bar{\alpha}$:

$$\alpha = \sup\{\bar{\alpha} | \quad f \in Lip_{\bar{\alpha}}(O)\}$$

is called the Lipschitz regularity of $f(x)$ over O, and f is said to have Lipschitz regularity α at x_0 if and only if there is an open set $O_x \subset \mathbb{R}^2$ such that $x_0 \in O_x$ and f has Lipschitz regularity α over O_x.

It is known that an image has Lipschitz regularity 0 at the discontinuous edge and has Lipschitz regularity $\alpha \in (0, 1)$ at other (continuous) edges. The authors of [20] proved that the local Lipschitz regularity of an image can be detected by its wavelet transforms at multiscales, which is stated in the following theorem.

Theorem 3 *[Mallat and Hwang [20]] Let $0 \leq \alpha < 1$. Let $\theta(x)$ be a twice differentiable scaling function on $\mathbb{R}^2$ such that $\int_{\mathbb{R}^2} \theta(x)dx = 1$ and $\theta(x)$ exponentially decays as $|x| \to \infty$. Let $\psi^1 = \frac{\partial\theta}{\partial x_1}, \psi^2 = \frac{\partial\theta}{\partial x_2}, \psi^1_s = \frac{1}{s^2}\psi^1(\frac{\cdot}{s})$, and $\psi^2_s = \frac{1}{s^2}\psi^2(\frac{\cdot}{s})$. For a function $f \in L^2(\mathbb{R}^2)$, let its wavelet transforms be $W^1_{\psi,s}f = f * \psi^1_s$ and $W^2_{\psi,s}f = f * \psi^2_s$. Let the wavelet transform modulus is denoted by*

$$M_{\psi,s}f(x) = \sqrt{|W^1_{\psi,s}f(x)|^2 + |W^2_{\psi,s}f(x)|^2}.$$

Then f is uniformly Lipschitz α over an open set $O \subset \mathbb{R}^2$ if and only if there exists a constant K such that for all points $x \in O$

$$M_{\psi,s}f(x) \leq K_\psi s^\alpha.$$

By the theorem, f has Lipschitz regularity α at x if and only if there is an open set $O_x, x \in O_x$, such that for all $\bar{x} \in O_x$, $M_{\psi,s}f(\bar{x}) = \bigcirc(s^\alpha)$. Thus, the Lipschitz regularity of an image can be measured from the evaluation of the wavelet maxima across scales. Particularly, when $\bar{x}$ is a point on the discontinuous edge, then $M_{\psi,s}f(\bar{x}) = K_{\bar{x}}(s)$, where $K_{\bar{x}}(s)$ is a bounded function of s.

Recall that the Gaussian noise as a distribution is singular everywhere, which can be characterized by negative Lipschitz orders (see [20]). Let $n(x)$ be a stationary, white noise random field of variance σ^2. Let $M_{\psi,s}n(x)$ be the modulus of the wavelet transform of $n(x)$ at the scale s, and let $E(X)$ be the expected value of a random variable X. The authors of [20] proved that

$$E((M_{\psi,s}n(x))^2) = \frac{\sigma^2(||\psi^1||^2 + ||\psi^2||^2)}{s},$$

where $\psi^1$and ψ^2 are the wavelets defined in Theorem 3. Thus, we can discriminate the image singularity (which occurs at edge) from the noise singularity by their wavelet transform modulus across scales: As the scale s increases, the wavelet transform modulus of edge points increase while the modulus created by noise decrease .

Based on the different behaviors of wavelet transforms of image singularity and noise singularity across scales, we suggest the following wavelet regularization. Let s_1 and s_2 be two positive scales and $s_2 > s_1$. Let

$$M_{\psi,s_1,s_2}f(x) = \min\{c_{s_1}M_{\psi,s_1}f(x), c_{s_2}M_{\psi,s_2}f(x)\}, \tag{10}$$

where c_s is chosen so that $c_s M_{\psi,s}f(x)$ is a constant for each s when f has Lipschitz singularity 0. We mow use $M_{\psi,s_1,s_2}f(x)$ to introduce the wavelet regularization of Type 2.

- **Wavelet Regularization of Type 2.**

$$\begin{aligned} \frac{\partial u}{\partial t} &= \operatorname{div}(c(M_{\psi,s_1,s_2}u)\nabla u), \\ u(0,x) &= u_0(x), \quad x \in \bar{\Omega}, \\ \frac{\partial u}{\partial \vec{n}} &= 0, \quad x \in \partial\Omega, t \in [0,\infty), \end{aligned} \tag{11}$$

Note that, in the region dominated by image content $M_{\psi,s_1,s_2}f(x) = M_{\psi,s_1}f(x)$, while in the region dominated by noise $M_{\psi,s_1,s_2}f(x) = M_{\psi,s_2}f(x)$. Hence, this regularization automatically adjusts the wavelet estimates of $|\nabla u|$ according to the local analysis of the image function u when u carries noise.

In applications, spline wavelets are powerful tools. Recall that the one-dimensional, central, cardinal B-spline is inductively defined by

$$\begin{aligned} N_1^c(x) &= \chi_{[-\frac{1}{2},\frac{1}{2})}(x), \\ N_m^c(x) &= N_{m-1}^c * N_1^c(x), \quad m = 2,3,\cdots. \end{aligned} \tag{12}$$

We often use the tenser product to construct bivariate B-splines of order m:

$$N_m^c(x) = N_m^c(x_1)N_m^c(x_2), \quad x = (x_1,x_2)^T \in \mathbb{R}^2.$$

B-splines have several important properties. First, for a fixed integer m, It is an identical approximation kernel (see [?] and [11]), i.e., $N_m^c(x) \geq 0, \int_{\mathbb{R}^2} N_m^c(x)dx = 1$, and

$$\begin{aligned} &\lim_{s\to 0^+} \frac{1}{s^2}\int_{\mathbb{R}} N_m^c(\frac{x-y}{s})f(y)dy - f(x) = 0, \quad x \in \mathbb{R}^2, \quad f \in C(\mathbb{R}^2), \\ &\lim_{s\to 0^+} ||\frac{1}{s^2}\int_{\mathbb{R}} N_m^c(\frac{\cdot - y}{s})f(y)dy - f||_{L^2} = 0, \quad f \in L^2(\mathbb{R}^2). \end{aligned}$$

Second, it approximates Gaussian as the order m tends to infinity. More precisely, we have (see [12] and [33])

$$\lim_{m\to\infty} ||\sqrt{\frac{m}{12}} N_m^c(x) - \frac{1}{2\pi} e^{-\frac{|x|^2}{2(m/12)}}||_{L^2} = 0.$$

This property is extremely useful when we want to discretize Gaussian kernel. We now adopt the gradient of B-splines to construct spline wavelets for our purpose. For a given integer m and a fixed $h > 0$, we define

$$\psi_m^h(x) = \frac{1}{h^2}\sqrt{\frac{m}{12}}\frac{1}{h^2}(\nabla N_m^c)(\frac{x}{h}), \quad W_{B_m} f = \psi_m^c * f$$

and

$$M_{B,m_1,m_2} f(x) = \min(c_{m_1}|\psi_{m_1}^c * f(x)|, c_{m_2}|\psi_{m_2}^c * f(x)|),$$

where c_m is chosen so that $c_m|\psi_m^c * f(x)|$ is a constant for each integer m when f has Lipschitz singularity 0.at x. In practice, h is the distance vector of two successive sample points in $x_1$and $x_2$directions respectively. We now introduce the wavelet regularization of Type 3.

- **Wavelet Regularization of Type 3**

$$\begin{aligned} \frac{\partial u}{\partial t} &= \operatorname{div}(c(M_{B,m_1,m_2} u)\nabla u), \qquad (13) \\ u(0,x) &= u_0(x), \quad x \in \bar{\Omega}, \\ \frac{\partial u}{\partial \vec{n}} &= 0, \quad x \in \partial\Omega, t \in [0,\infty), \end{aligned}$$

0.3 Consistence of Wavelet regularization Models

In this section we prove the existence and uniqueness of the solution of the wavelet regularization models in (9), (11), and (13). The proofs for these three types are similar. Hence, we only present the proof for Type 2, which follows the idea in [6]. We start from the following lemma.

Lemma 4 *Let $w = w(x,t) \geq 0, x \in \Omega, t \in [0,\infty)$, and $u = u(x,t)$ be the solution of the equation*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \operatorname{div}(w\nabla u), \qquad (14) \\ u(x,t) &= u_0(x), \quad x \in \bar{\Omega}, \\ \frac{\partial u}{\partial \vec{n}} &= 0, \quad (x,t) \in \partial\Omega \times [0,\infty). \end{aligned}$$

Then $||u(\cdot,t)||_{L^2}$ *is a decreasing function of* t.

Proof.

$$
\begin{aligned}
\frac{\partial}{\partial t}||u(\cdot,t)||_{L^2}^2 &= \frac{\partial}{\partial t}\int_\Omega u^2(x,t)dx = 2\int_\Omega u(x,t)\frac{\partial u(x,t)}{\partial t}dx \\
&= 2\int_\Omega u(x,t)\,\mathrm{div}(w(x,t)\nabla_x u(x,t))dx \\
&= -2\int_\Omega w\,(\nabla u\cdot\nabla u)\,dx + 2\oint_{\partial\Omega} wu\frac{\partial u}{\partial\vec{n}}ds \\
&= -2\int_\Omega w\,(\nabla u\cdot\nabla u)\,dx \le 0,
\end{aligned}
$$

which implies that $||u(\cdot,t)||_{L^2}$ is a decreasing function of t.

The following lemma gives an upper bound of $|W_\psi^s u(x,t)|$.

Lemma 5 **??***Let* ϕ *be a* r*-regular scaling function* ($r>1$) *and* $\psi=\nabla\phi$. *Let* $u_t(x) := u(x,t)$ *be the solution of the equation (14) and* $W_\psi^s u_t$ *be the wavelet function of* u_t *with respect to* ψ *and* $s>0$. *Then*

$$
|W_\psi^s u_t(x)| \le \frac{1}{s}||\psi||_{L^2}||u_0||_{L^2}, \quad x\in\Omega,
$$

where $||\psi||_{L^2} = \sqrt{||\psi_1||_{L^2}^2 + ||\psi_2||_{L^2}^2}$.

Proof. We have

$$
\begin{aligned}
|W_\psi^s u_t(x)| &= \frac{1}{s}\left|\frac{1}{s}\left(\psi(\frac{\cdot}{s}) * u_t\right)(x)\right| \\
&\le \frac{1}{s}\left(||\frac{1}{s}\psi_1(\frac{\cdot - x}{s})||_{L^2}^2||u_t||_{L^2}^2 + ||\frac{1}{s}\psi_2(\frac{\cdot - x}{s})||_{L^2}^2||u_t||_{L^2}^2\right)^{1/2} \\
&= \frac{1}{s}||\psi||_{L^2}||u_t||_{L^2}.
\end{aligned}
$$

By Lemma 4, $||u_t||_{L^2} \le ||u_0||_{L^2}$ for $t>0$, which yields

$$
|W_\psi^s u_t(x)| \le \frac{1}{s}||\psi||_{L^2}||u_0||_{L^2}, \quad x\in\Omega.
$$

We now prove the existence of the solution of (9) (or (11), (13)).

Theorem 6 (Existence) *Assume that* $c'(p) < 0$. *The equation (9) (or (11), (13)) has a solution* $u(x,t)$ *such that*

$$
||u(\cdot,t)||_{L^2} \le ||u_0||_{L^2}, \quad t>0.
$$

Proof. We prove the theorem for (11). The proofs for (9) and (13) are similar. By Lemma **??**, for each $s > 0$, we have

$$M_s u_t(x) \le \frac{1}{s} ||\psi||_{L^2} ||u_0||_{L^2}, \tag{15}$$

which yields

$$M_{s_1,s_2} u_t(x) \le \frac{1}{\min(s_1, s_2)} ||\psi||_{L^2} ||u_0||_{L^2}, \quad x \in \Omega \tag{16}$$

Without loss of generality, we assume $s_1 < s_2$ and write $M = \frac{1}{s_1} ||\psi||_{L^2} ||u_0||_{L^2}$. Since $c(p) > 0$ is decreasing. Then

$$c\left(M_{s_1,s_2} u_t(x)\right) \ge c(M) > 0, \quad \forall (x,t) \in \Omega \times [0, \infty),$$

which implies that the equation (11) is uniformly elliptic. Hence is has a solution. By Lemma 4, we have $||u(\cdot,t)||_{L^2} \le ||u_0||_{L^2}, \quad t > 0$.

To obtain the uniqueness of the equation (9) (or (11), (13)), we need Gronwall's inequality.

Lemma 7 *[Gronwall's inequality of integral form] Let $v(t)$ be a nonnegative, integrable function on $[0, T]$, satisfying*

$$v(t) \le C_1 \int_0^t v(s) ds + C_2, \tag{17}$$

where $C_1$and C_2 are nonnegative. Then

$$v(t) \le C_2 \left(1 + C_1 t e^{C_1 t}\right), \quad t \in [0, T]. \tag{18}$$

In particular, if $C_2 = 0$, then $v(t) = 0$ on $[0, T]$.

We now prove the uniqueness of the solution of (9) (or (11), (13))

Theorem 8 *Let $c(p)$ be a positive function satisfying the Lipschitz condition*

$$|c(p) - c(q)| \le L|p - q|, \quad \forall p, q \in \mathbb{R}.$$

Then solution of the equation of (9) (or (11), (13)) is unique.

Proof. Let $u(x,t)$ and $v(x,t)$ be the solution of (11). We have

$$\frac{\partial u}{\partial t} = \operatorname{div}(c(M_{s_1,s_2} u) \nabla u),$$
$$\frac{\partial v}{\partial t} = \operatorname{div}(c(M_{s_1,s_2} v) \nabla v)$$

and

$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial v}{\partial \vec{n}} = 0, \quad x \in \partial\Omega,$$
$$u(x,0) = v(x,0) = u_0(x), \quad x \in \Omega.$$

For convenience, we write $a(x,t) = c(M_{s_1,s_2}u)$ and $\bar{a}(x,t) = c(M_{s_1,s_2}v)$. Hence, we have

$$2(u-v)\frac{\partial(u-v)}{\partial t} = 2(u-v)\left(\operatorname{div}\left[(a\nabla(u-v)\right] - 2\operatorname{div}\left[(a-\bar{a})\nabla v\right]\right). \quad (19)$$

Taking integral (over Ω) on the both sides of the equation above, then the integral of the left-hand side of (19) yields

$$\int_\Omega 2(u-v)\frac{\partial(u-v)}{\partial t}d\Omega = \frac{\partial}{\partial t}\int_\Omega (u-v)^2 d\Omega = \frac{\partial}{\partial t}||u(\cdot,t) - v(\cdot,t)||_{L^2}.$$

By the Green's formula, the first term on the right-hand side is

$$\begin{aligned}&2\int_\Omega (u-v)\operatorname{div}\left[(a\nabla(u-v)\right]d\Omega \\ &= -2\int_\Omega a\nabla(u-v)\cdot\nabla(u-v)d\Omega + 2\oint_{\partial\Omega} a(u-v)\frac{\partial(u-v)}{\partial\vec{n}}ds,\end{aligned}$$

where the second integral is 0 due to $\frac{\partial(u-v)}{\partial\vec{n}} = 0$ on $\partial\Omega$. Thus, we have

$$2\int_\Omega (u-v)\operatorname{div}\left[(a\nabla(u-v)\right]d\Omega = -2\int_\Omega a\nabla(u-v)\cdot\nabla(u-v)d\Omega.$$

Similarly,

$$\begin{aligned}&-2\int_\Omega (u-v)\operatorname{div}\left((a-\bar{a})\nabla v\right)d\Omega \\ &= 2\int_\Omega (a-\bar{a})\nabla v\cdot\nabla(u-v)d\Omega - 2\oint_{\partial\Omega}(a-\bar{a})\nabla v\frac{\partial(u-v)}{\partial\vec{n}}ds \\ &= 2\int_\Omega (a-\bar{a})\nabla v\cdot\nabla(u-v)d\Omega.\end{aligned}$$

Combining them together, we have

$$\begin{aligned}&\frac{\partial}{\partial t}||u(\cdot,t) - v(\cdot,t))||^2_{L^2} \\ &= -2\int_\Omega a\nabla(u-v)\cdot\nabla(u-v)d\Omega + 2\int_\Omega (a-\bar{a})\nabla v\cdot\nabla(u-v)d\Omega\end{aligned}$$

We now estimate the right part of the equation above. Recall that

$$a(x,t) = c(M_{s_1,s_2}u) \geq c(M) > 0.$$

Hence the first integral satisfies the inequality

$$-2\int_\Omega a\nabla(u-v)\cdot\nabla(u-v)d\Omega \leq -2c(M)||\nabla(u(\cdot,t)-v(\cdot,t))||_{L^2}^2$$

From $|c(p)-c(q)| \leq L|p-q|$, we derive

$$\begin{aligned}|a(x,t)-\bar{a}(x,t)| &= |c(M_{s_1,s_2}u)-c(M_{s_1,s_2}v)| \\ &\leq L\,|M_{s_1,s_2}u - M_{s_1,s_2}v| \\ &\leq \max\{|M_{s_1}\,(u-v)\,|, |M_{s_2}\,(u-v)\,|\} \\ &\leq C||u-v||_{L^2}\end{aligned}$$

where $C = 1/\max(s_1,s_2)||\psi||_{L^2}$. Thus,

$$\begin{aligned}&2\int_\Omega (a-\bar{a})\nabla v\cdot\nabla(u-v)d\Omega \\ &\leq 2C||u-v||_{L^2}\int_\Omega |\nabla v\cdot\nabla(u-v)|\,d\Omega \\ &\leq 2C||u(\cdot,t)-v(\cdot,t)||_{L^2}||\nabla v||_{L^2}||\nabla(u-v)||_{L^2}.\end{aligned}$$

Finally, we have

$$\begin{aligned}&\frac{\partial}{\partial t}||u(\cdot,t)-v(\cdot,t))||_{L^2}^2 \\ &\leq -2C(M)||\nabla(u(\cdot,t)-v(\cdot,t))||_{L^2}^2 + 2C||u(\cdot,t)-v(\cdot,t)||_{L^2}||\nabla v||_{L^2}||\nabla(u-v)||_{L^2}.\end{aligned}$$

For any $\mu > 0$,

$$\begin{aligned}&2C||u(\cdot,t)-v(\cdot,t)||_{L^2}||\nabla v||_{L^2}||\nabla(u-v)||_{L^2} \\ &\leq \left(\frac{C}{\mu}||u(\cdot,t)-v(\cdot,t)||_{L^2}||\nabla v||_{L^2}\right)^2 + \mu^2||\nabla(u(\cdot,t)-v(\cdot,t))||_{L^2}^2.\end{aligned}$$

Choosing $\mu = \sqrt{2c(M)}$, we have

$$\frac{\partial}{\partial t}||u(\cdot,t)-v(\cdot,t))||_{L^2}^2 \leq \frac{C^2}{2c(M)}||u(\cdot,t)-v(\cdot,t)||_{L^2}^2||\nabla v||_{L^2}^2. \tag{20}$$

Applying Gronwall's inequality to $||u(\cdot,t)-v(\cdot,t))||_{L^2}^2$ (as a function of t), we have $||u(\cdot,t)-v(\cdot,t))||_{L^2}^2 = 0$. Hence, the solution of (6) is unique.

0.4 Numerical Solutions and Adaptive Algorithms

We now discuss the numerical solutions of the equations (9), (11), and (13). Let h be the space step, Δt be the time step, and $u_{i,j}^n \doteq u(ih, jh, n\Delta t)$ be the discrete representation of the solution $u(x,t)$ at the time $t = n\Delta t$. Let $g(u)$ be the wavelet regularization of $|\nabla u|$, which is defined by $g(u) = \left|W_{\psi,s}u\right|$ in Type1, $g(u) = M_{\psi,s_1s_2}u$ in Type 2, and $g(u) = M_{B,m_1,m_2}u$ in Type 3 respectively. We also write

$$\alpha_{i,j}^n \doteq c\left(g\left(u_{i,j}^n\right)\right), \quad \alpha_{i+1/2,j} = \frac{1}{2}(\alpha_{i,j} + \alpha_{i+1,j}), \quad \alpha_{i,j+1/2} = \frac{1}{2}(\alpha_{i,j} + \alpha_{i,j+1}).$$

The Euler explicit method for the equation is

$$\begin{aligned} u_{i,j}^{n+1} = u_{i,j}^n + \frac{\Delta t}{h^2}[\alpha_{i-1/2,j}^n u_{i-1,j}^n + \alpha_{i,j-1/2}^n u_{i,j-1}^n + \alpha_{i+1/2,j}^n u_{i+1,j}^n + \alpha_{i,j+1/2}^n u_{i,j+1}^n \\ - (\alpha_{i,j-1/2}^n + \alpha_{i-1/2,j}^n + \alpha_{i+1/2,j}^n + \alpha_{i,j+1/2}^n)u_{i,j}^n] \end{aligned} \tag{21}$$

with the initial conditions

$$\begin{aligned} &u_{i,j}^0 = u_0(ih, jh), \quad 1 \le i \le N_1, 1 \le j \le N_2, \\ &u_{i,0}^n = u_{i,1}^n, \quad u_{i,N}^n = u_{i,N+1}^n, \quad 0 \le i \le N_1 + 1, \\ &u_{0,j}^n = u_{1,j}^n, \quad u_{N,j}^n = u_{N+1,j}^n, \quad 0 \le j \le N_2 + 1. \end{aligned}$$

For writing the iterative formula (21) is the matrix form, we define $U^n = \left(u_{i,j}^{n+1}\right)_{N_1 \times N_2}$. Then (21) becomes

$$U^{n+1} = (I - \Delta t A_{h,U^n})U^n.$$

Assume $c(0) = 1$. It is known that the algorithm is convergent when $\Delta t < \frac{1}{4}h^2$ To remove the restriction on Δt, we can employ the implicit scheme

$$U^{n+1} = U^n - \Delta t A_{h,U^n} U^{n+1},$$

which yields

$$\left(I + \Delta t A_{h,U^n}\right) U^{n+1} = U^n. \tag{22}$$

Since $I + \Delta t A_{h,U^n}$ is invertible for each $\Delta t > 0$, we have

$$U^{n+1} = \left(I + \Delta t A_{h,U^n}\right)^{-1} U^n,$$

where Δt can be any positive number.

In the algorithms (21) and (22), the size of matrix U^n usually is very large. We now develop adaptive method for accelerate the algorithm. Denote

$$\delta_E u_{i,j} = \frac{1}{h}(u_{i+1,j} - u_{i,j}), \quad \delta_W u_{i,j} = \frac{1}{h}(u_{i-1,j} - u_{i,j}),$$
$$\delta_S u_{i,j} = \frac{1}{h}(u_{i,j+1} - u_{i,j}), \quad \delta_N u_{i,j} = \frac{1}{h}(u_{i,j-1} - u_{i,j}).$$

Then (21) can be written as

$$u_{i,j}^{n+1} = u_{i,j}^n + \frac{\Delta t}{h}[\alpha_{i+1/2,j}^n \delta_E u_{i,j}^n + \alpha_{i,j-1/2}^n \delta_N u_{i,j}^n + \alpha_{i-1/2,j}^n \delta_W u_{i,j}^n + \alpha_{i,j+1/2}^n \delta_S u_{i,j}^n]. \tag{23}$$

We set

$$p_{i,j}^n = \left|\delta_E u_{i,j}^n\right| + \left|\delta_N u_{i,j}^n\right| + \left|\delta_W u_{i,j}^n\right| + \left|\delta_S u_{i,j}^n\right|$$

and define the matrix $P_{u^n} = \left(p_{i,j}^n\right)_{N_1 \times N_2}$. Usually the matrix P_{u^n} is a sparse one in the numerical sense. By (23), we suggest the following adaptive algorithm.

Algorithm 9 *INPUT: image matrix U_0, time-step Δt, threshold δ, iteration n, conduct function c, edge-stop parameter K, low-pass filter ϕ, scales s_1 and s_2.*

SET index-set $S = \emptyset$;
FOR $k = 1$ TO n
FOR $i = 1$ TO N_1
FOR $j = 1$ TO N_2
IF (i, j) NOT IN S
COMPUTE $de = (\delta_E u_{i,j})$; $dn = (\delta_N u_{i,j})$; $dw = (\delta_W u_{i,j})$; $ds = (\delta_S u_{i,j})$;
$p = |de| + |dn| + |dw| + |ds|$;
IF $p < \delta$
PUT (i, j) IN S;
ELSE
CALL ALGORITHM (USING $c(p), K, \phi, s_1, s_2$) TO COMPUTE $\alpha_{i+1/2,j}$; $\alpha_{i,j-1/2}$; $\alpha_{i-1/2,j}$; $\alpha_{i,j+1/2}$;
SET $u_{i,j} = u_{i,j} + \frac{\Delta t}{h}[\alpha_{i+1/2,j}\delta e + \alpha_{i,j-1/2}\delta n + \alpha_{i-1/2,j}\delta w + \alpha_{i,j+1/2}\delta s]$;
END (IF p)
END (IF (i, j))
END (FOR j)
END (FOR i)

END (FOR k)
$Y = U$;
OUTPUT image Y.

Bibliography

[1] S. Acton, *Locally monotonic diffusion,* IEEE Trans. on Signal Processing, **48** no. 5 (2000) pp. 1379-139.

[2] F. T. Azar and K. E. Tait, *Image recovery using anisotropic diffusion equation*, IEEE Trans. on Image Proc. **5** no.11 (1996) pp. 1573-1578.

[3] A. Black and M. Isard, Active Contours, Springer-Verlag, 1998.

[4] M. Black, G. Sapiro, D. Marimont, and D. Heeger, *Robust anisotropic diffusion,* IEEE Trans. on Image Processing, **7** no.3 (1998) pp. 421-432.

[5] P.L. Butzer and R. J. Nessel, Fourier Analysis and Approximation, **Vol 1**, Birkhäuser, 1971.

[6] F. Catte, P-L. Lions, J-M. More, and T. Coll, *Image selective smoothing and edge detection by nonlinear diffusion,* SIAM J. Numer. Anal. **29** no.1 (1992) pp. 182-193.

[7] T. F. Chan, G. H. Golub, and P. Mulet, *A nonlinear primal -dual method for total variantion based image restoration*, UCLA Math Department CAM report, 95-43, September 1995.

[8] T. F. Chan, P. Mulet, *Iterative methods for total variantion image restoration*, UCLA Math Department CAM report, 96-38, October 1996.

[9] T. Chan, B.Y. Sandberg, and L. Vese, *Active contours without edges for vector-valued images,* Journal of Visual Communicstion and image Reconstruction, **11**: 130-141, 2000.

[10] T. Chan and L. Vese. *Active contours without edges,* IEEE Transactions on Image Processing, **10**(2): 266-277, 2001.

[11] C. K. Chui, *An introduction to Wavelets,* Academic Press, Boston, 1992.

[12] C. K. Chui and J. Z. Wang, *High-order orthonormal scaling functions and wavelets give poor time-frequency localization*, *J. Fourier Anal. and Appl.*

[13] I. Daubechies, *"Ten Lectures on Wavelets",* SIAM Publications, 1992.

[14] A.I. El-Fallah and G.E. Ford, On mean curvature diffusion in nonlinear image filtering, *Pattern Recognition Letters,* **19** 433-437, 1998.

[15] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature I,* J. Diff. Geo. **33** (1991) pp. 635-681.

[16] S. Kichenassamy, The Perona-Malik paradox, *SIAM J. Appl. Math.,* **57** 1328-1342, 1997.

[17] J. Koenderink, The structure of images, *Biol. Cybern.,* **54** (1984) pp. 363-370.

[18] S. Mallat, A Wavelet Tour of Singnal Processing, *Academic Press, Sam Diego,* 1998.

[19] S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbb{R})$, Trans. Amer. Math. Soc. **315** (1989), 69–87.

[20] Mallat, S. and W.L. Hwang, *Singularity detection and processing with wavelets*, IEEE Trans. Information Theory **38** (1992) 617–643.

[21] S. Mallat and S. Zhong, *Characterization of Signals from Multiscale edges*, IEEE Trans. Patt. Recog. and Mach. Intell., **14** (7) (1992) 710-732.

[22] Y. Meyer, *"Ondelettes et Opérateurs I: Ondelettes"*, Hermann, Paris, France, 1990.

[23] D. Mumford and J. Shah, *Optimal approximations by piesewise smooth functions and associated variational problems*, Comm. on Pure and Appl. Math. **XLII** no.5 (1989).

[24] S. Osher and R.P. Fedkiw, Level set methods, *Technical Report 00-08, UCLA CAM Report,* 2000.

[25] S. Osher and L. Rudin, *Feature oriented image enhancement using shock filters*, SIAM J. Appl. Math. **27** (1990) 919-940.

[26] S. Osher, B. Engquist, S. Zhong, *Fast wavelet based algorithms for linear evolution equations*, SIAM J. Appl. Math. **15** no.4 (1994) 755-775.

[27] N. Paragios and R. Deriche, Geodesic active contours and level-sets for the detection and tracking of moving objects, *IEEE Transactions on Pattern Analysis and Machine Intellegence,* **22** 266-280, 2000.

[28] P. Perona and J. Malik, Scale-space and edge detection using anisotropic diffusion, *IEEE PAML,* **12** (1990) pp. 629-639.

[29] C. Samson, L. Blanc-Féraud, G. Aubert, and J. Zerubia, A level-set model in image classification, in Mads Nielsen, P. Johansen, O.F. Olsen, and J. Weickert, editors, *Scale-Space Theories in Computer Vision,* **1682** of *Lecture Notes in Computer Science,* 306-317, Springer-Verlag, 1999.

[30] G. Sapiro, *From active contours to anisotropic diffusion: connections between basic PDE's in image processing, IEEE,* 1996.

[31] G. Sapiro, *Geometric Partial Differental Equations and Image Analysis,* Cambridge University Press, 2001.

[32] F. Torkamani-Azar and K. E. Tait, *Image recovery using the anisotropic diffusion equation,* IEEE Trans. on Image Processing, **10** no. 11 (1996), 1573-1578.

[33] M. Unser, A. Aldroubi, and M. Eden, *On the asymptotic convergence of B-spline wavelets to Gabor functions*, IEEE Trans. Inform. Theory **38** (1992), 864–871.

[34] L. Vese, *A study in the BV space of a denoising-deblurring variational problem,* Applied Mathematics and Optimization, **44** 131-161, 2001.

[35] R. T. Whitaker and S. Pizer, *A mutiscale approach to nonuniform diffusion*, CVIP: Image Understanding, **57** no.1 (1993) 99-110.

[36] A. Witkin, *Scale-space filtering*, in "Int. Joint. Conf. Artificial Intelligence", Karlsruhe, West Germany, (1983) 1019-1021.

[37] Y. You, W. Xu, A. Tannenbaum, and M. Kaveh, *Behavioral analysis of anisotropic diffusion in image processing,* IEEE Trans. Image Processing, **5** (1996) pp 1539-1553.